Projekt tervezési probléma megoldása vágósíkos algoritmussal

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Outline of the Talk

- Motivation
- Problem data and MILP formulation
- Computational complexity
- Polyhedral results
- Branch-and-Cut
- Computational Results
- Research possibilities

Motivation

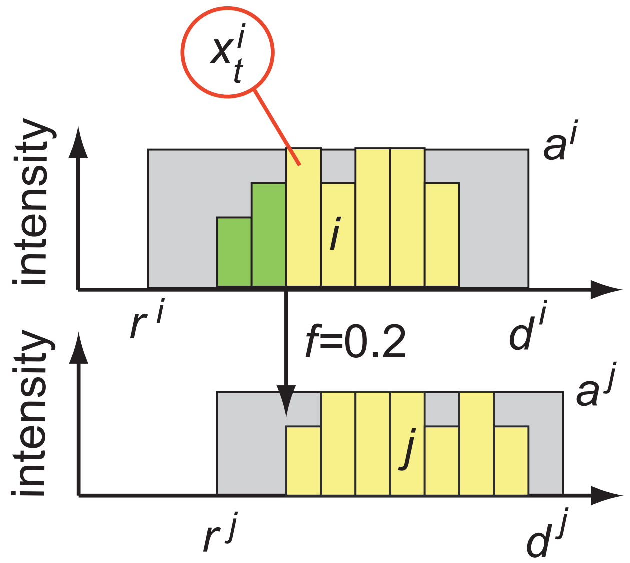
- Industrial long term planning problem (one year)
- Each customer order is a project:
  - Main activities: *Design, Machining, Assembly Electrical Design, Electrical Assembly, Testing*
  - Time horizon: 12-52 weeks
  - Release time & due date
  - The intensity of each activity may vary over time
  - The activities may overlap in time
  - Each activity requires one or more resources
- The resources have finite capacities, but external resources can be subcontracted

Problem Data (1)

- Discrete time horizon: $t = 1, \ldots, T$
- Continuously divisible renewable resources: $k \in R$
  - normal capacity: $b_t^k \geq 0$
  - extra capacity: $\bar{b}_t^k \geq 0$
  - cost of using extra capacity: $c_t^k \geq 0$
- Variable intensity activities: $i \in N$
  - maximum intensity: $0 < a^i \leq 1$
  - time window: $\{r^i, \ldots, d^i\}$
  - resource requirements: $q_k^i \geq 0$, $k \in R$

Problem Data (2)

- Feeding Precedence Constraints: $(i, j, f) \in A$
  - At least f fraction of activity i must be completed before j may start
  - The fraction of activity j completed up to any period τ never exceeds the fraction of activity i completed up to period τ, i.e., $\sum_{t=r^i}^{\tau} x_t^i \geq \sum_{t=r^j}^{\tau} x_t^j$, $\forall\ \tau$



The MILP formulation (1)

Variables:

$$x_t^i \;=\; \textit{intensity} \text{ of activity } i \text{ in period } t$$

$$z_t^{i,f} \;=\; f\textit{-fraction mask} \text{ of activity } i \text{ in period } t,\ z_t^{i,f} \in \{0,1\}$$

$$y_t^k \;=\; \text{external capacity of resource } k \text{ used in period } t$$

Objective:

$$\min \sum_t \sum_{k \in R} c_t^k y_t^k$$

The MILP formulation (2)

Constraints:

$$\begin{aligned}
\sum_{t=r^i}^{d^i} x_t^i &= 1, \quad i \in N && (1)\\
0 \le x_t^i &\le a^i, \quad i \in N,\ t \in \{r^i, \ldots, d^i\} && (2)\\
\sum_{i \in N,\ t \ni \{r^i, \ldots, d^i\}} q_k^i x_t^i &\le b_t^k + y_t^k, \quad k \in R, t \in \{1, \ldots, T\} && (3)\\
0 \le y_t^k &\le \bar{b}_t^k, \quad k \in R,\ t \in \{1, \ldots, T\} && (4)
\end{aligned}$$

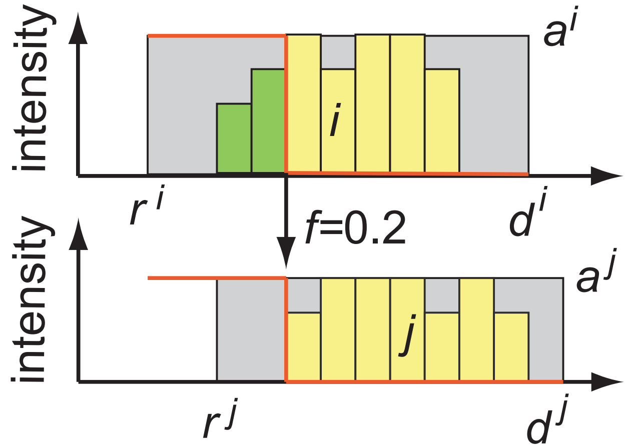
The MILP formulation (3)

$$\sum_{t=r^j}^{\tau} x_t^j - \sum_{t=r^i}^{\tau} x_t^i \leq 0, \quad (i,j,f) \in A,\ \tau \in \{r^i, \ldots, d^i\} \tag{5}$$

$$\sum_{t=r^i}^{\tau-1} x_t^i + f \cdot z_\tau^{i,f} \geq f,\ (i,j,f) \in A,\ \tau \in \{r^i + p^{i,f}, \ldots, d^i\} \tag{6}$$

$$\sum_{t=\tau+1}^{d^j} x_t^j - z_\tau^{i,f} \geq 0, \quad (i,j,f) \in A,\ \forall \tau \tag{7}$$

$$z_t^{i,f} - z_{t+1}^{i,f} \geq 0, \quad \forall i, f, t \tag{8}$$



Other objective functions

Below we assume that there are no extra capacities: $\bar{b}_t^k = 0$

- Min Makespan: by dichotomic search find the smallest T such that a feasible solution exists with $d^i = T$.
- Min Maximum Tardiness: $d^i = \tilde{d}^i + T_{\max}$ and dichotomic search on $T_{\max}$, where $\tilde{d}^i$ is the due-date of $i \in N$
- Min Weighted Total Tardiness: define weights $w_t^i = 0$ if $t \in \{r^i, \ldots, \tilde{d}^i\}$ and $w_t^i = w^i$ if $t \in \{\tilde{d}^i + 1, \ldots, T\}$. The objective is $\min \sum_{i \in N} \sum_{t \in \{r^i, \ldots, T\}} w_t^i z_t^i$.

Computational complexity

- If extra capacity is unlimited, a feasible solution can be found in polynomial time.
  - Heuristics often assume and exploit that a feasible solution can be found quickly
- If extra capacity is bounded, then feasible solution existence is NP-complete in the strong sense.

Branch and cut

- Exact algorithm for solving mixed integer programs with linear constraints
- Branch and Bound + automatic generation of cutting planes in search-tree nodes
- Each node is processed as follows:
  1. Solve the linear program corresponding to the node
  2. Try to add valid inequalities and resolve
  3. Try to find a feasible solution with heuristics
  4. If a feasible solution is found, then fathom the node
  5. Otherwise create child nodes by some branching rule

Cutting planes from precedence constraints

- Relaxation: drop all resource constraints (R-MILP)
- Feasible solutions of R-MILP = all intensity assignments to activities respecting the precedence constraints
- A valid inequality for R-MILP is valid for the original MILP
- LP-R-MILP, the LP relaxation of R-MILP, has vertices with fractional z coordinates
- LP-R-MILP can be tightened by automatically generating valid inequalities

The polytope K^{ijf} (1)

Consider $(i,j,f) \in A$ and let the polytope K^{ijf} be the convex hull of those $(x^i, x^j, z^{i,f})$ satisfying the following:

$$\begin{aligned}
\sum_{t=r^i}^{d^i} x_t^i &= 1, && (9)\\
0 \le x_t^i &\le a^i, \quad t \in \{r^i, \ldots, d^i\} && (10)\\
\sum_{t=r^i}^{\tau-1} x_t^i + f \cdot z_\tau^{i,f} &\ge f, \quad \tau \in \{r^i + p^{i,f}, \ldots, d^i\} && (11)\\
z_t^{i,f} - z_{t+1}^{i,f} &\ge 0, \quad t \in \{r^i + p^{i,f}, \ldots, d^i - 1\} && (12)\\
z_t^{i,f} &\in \{0, 1\} && (13)
\end{aligned}$$

The polytope K^{ijf} (2)

$$\sum_{t=r^j}^{d^j} x_t^j \quad = \quad 1, \tag{14}$$

$$0 \leq x_t^j \quad \leq \quad a^j, \qquad t \in \{r^j, \ldots, d^j\} \tag{15}$$

$$\sum_{t=\tau+1}^{d^j} x_t^j - z_\tau^{i,f} \quad \geq \quad 0, \qquad \tau \in \{r^i + p^{i,f}, \ldots, d^i\} \tag{16}$$

(To simplify notation, assume $r^i + p^{i,f} = r^j$ and $d^i + p^j \leq d^j$.)

Results for K^{ijf}

Decompose K^{ijf} into two smaller dimensional polytopes:

$$
\begin{aligned}
K_i^{ijf} &= \{(x^i, z^{i,f}) \mid (x^i, z^{i,f}) \text{ satisfies } (9)-(13)\} \\
K_j^{ijf} &= \{(x^j, z^{i,f}) \mid (x^j, z^{i,f}) \text{ satisfies } (12)-(16)\}
\end{aligned}
$$

Lemma $(x^i, x^j, z^{i,f}) \in K^{ijf}$ if and only if $(x^i, z^{i,f}) \in K_i^{ijf}$ and $(x^j, z^{i,f}) \in K_j^{ijf}$

Observation Both K_i^{ijf} and K_j^{ijf} are instances of the polytope $K^{f'}$:

$$
K^{f'} = \operatorname{conv}\{(x, z) \in \mathbb{R}^n \times \mathbb{B}^m \mid \sum_{t=1}^{n} x_t = 1,\ \forall t : 0 \le x_t \le a
$$

$$
\forall \tau : \sum_{t=\tau+1}^{m} x_t - f' \cdot z_\tau \ge 0,\ \forall t : z_t \ge z_{t+1}\}
$$

Results for K^f (1)

- If $f = 1$, a minimal linear representation of K^f is known. The non-trivial inequalities are of the form:

$$a_{\mathrm{rem}} z_{t_1} + \sum_{t \in S_1 \setminus \{t_1\}} a z_t \leq \sum_{t \in \{t_1, \ldots, n\} \setminus (S_1 \cup S_2)} x_t,$$

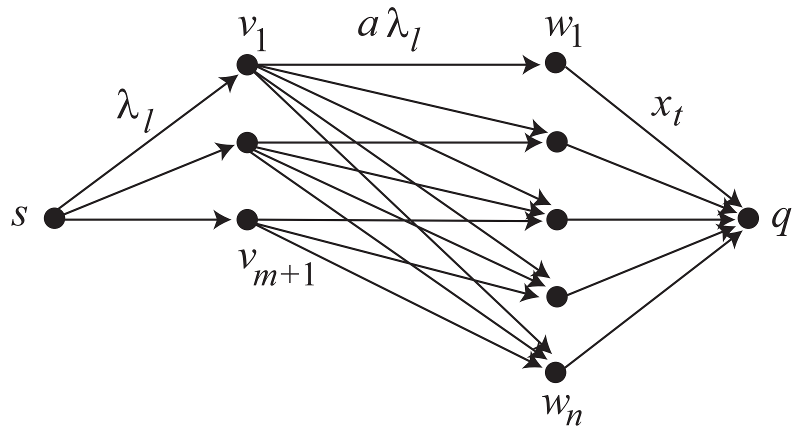
where $(p-1)a < 1 \leq pa$, $a_{\mathrm{rem}} = 1 - (p-1)a$, $t_1 = \min S_1$, $S_1 \subseteq \{1, \ldots, m\}$, $S_2 \subset \{m+1, \ldots, n\}$ and $|S_1| + |S_2| = p$,

$$\begin{aligned} x_t + a z_t &\leq a, \text{ for all } t = 1, \ldots, n, \\ z_t &\geq z_{t+1}, \text{ for all } t = p, \ldots, n-1. \end{aligned}$$

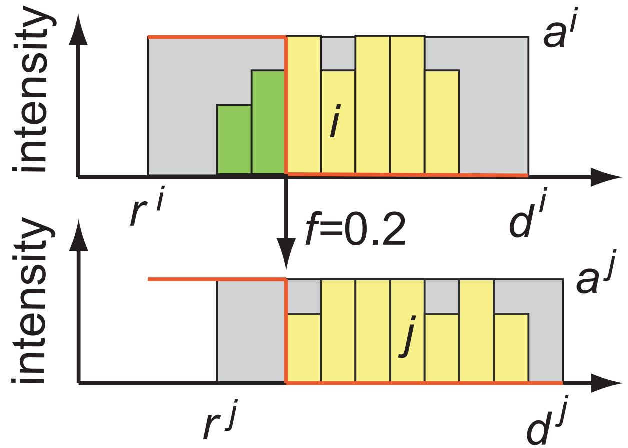
- There is an $O(n \log n)$ algorithm for separating (S_1, S_2) inequalities

Construction

Illustration $(f = 1)$:

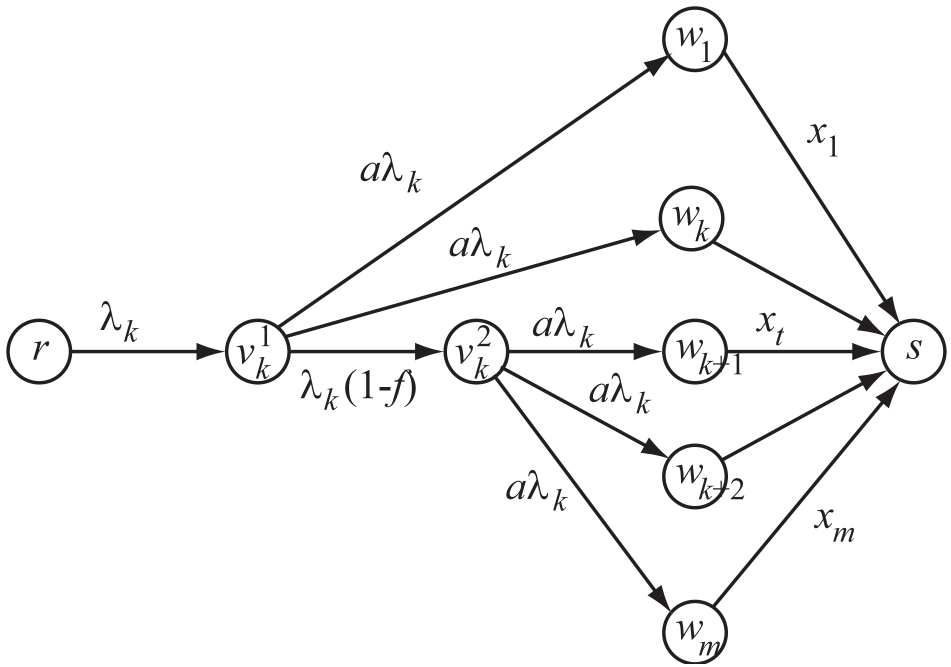


where $\lambda_1 = 1 - z_1$, $\lambda_\ell = z_{\ell-1} - z_\ell$ and $\lambda_{m+1} = z_m$.
For any $s \in S \subseteq V - \{q\}$: $c(\delta(S)) \geq 1$ is valid for K^f



Results for K^f (2)

- When $0 < f < 1$, then (only) a linear representation of K^f is known. The facet representing inequalities correspond to a subset of $s-q$ cuts in an appropriately defined network:



Test Instances

- De Boer instances:
  - $f = 1$
  - randomly generated precedence constraints
  - $1/a^i$ between 1 and 5
  - $n = 10, 20$ or 50 activities
  - $r = 3, 5$ or 20 resources
  - average $d^i - r^i - p^i + 1$ value: $s = 2, 5, 10, 15$ or 20 (over the activities of the instance)
  - extra capacity is unlimited ($\bar{b}_t^k = \infty$)
  - 10 instances with the same n, r and s parameters
- My computational environment: PC, Pentium 4, 1.6GHz
- Algorithm B: Branch-and-Cut truncated after 675 seconds

Comparison to Branch-and-Price

Algorithm H:
E. Hans, *Resource loading by branch-and-price techniques*, Ph. D. thesis, Twente University Press, The Netherlands, 2001.

The average of $ub(B)/ub(H)$

s	$n = 10$			$n = 20$			$n = 50$		
	$r = 3$	10	20	3	10	20	3	10	20
2	1	1	1	1	1	1	1	1	1
5	1	1	1	0.99	0.99	0.99	0.95	0.96	0.99
10	1	1	0.99	0.86	0.97	0.98	0.73	0.94	0.97
15	0.88	0.98	0.98	0.63	0.94	0.97	0.63	0.92	1.03
20	0.89	0.98	0.98	0.71	0.91	0.97	0.56	0.92	0.96

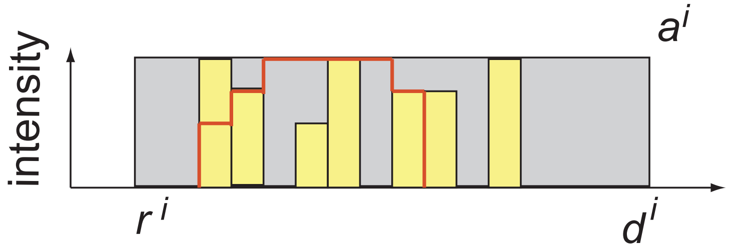
Results on instances with $f = 0.5$

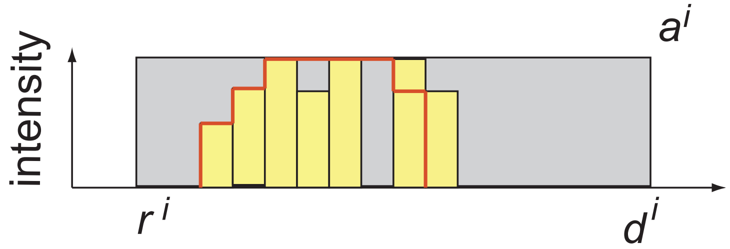
Average ub^+/ub^- (first row), and lb^+/lb^- (second row)

s	$n = 10$			$n = 20$			$n = 50$		
	$r = 3$	10	20	3	10	20	3	10	20
2	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1
15	1	1	1	1	1	1	0.98	0.99	0.99
	1	1	1	1	1	1	1	1	1
20	1	1	1	1	0.99	1	0.99	0.95	1.01
	1	1	1	1	0.99	0.99	1	1	0.99

Application to Production Planning

- The first optimal solution with cost $C^* = \sum_{t,k} c_t^k y_t^k$ is usually unsatisfactory

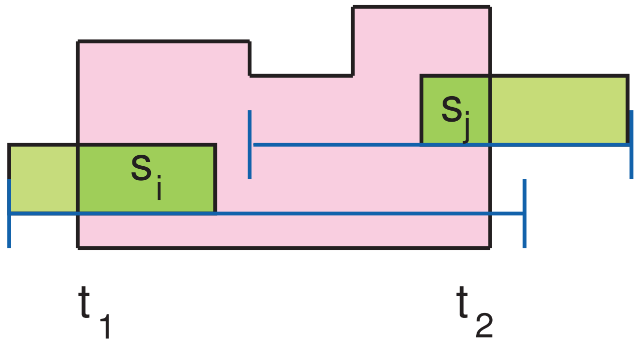




- Push the activities to the left (or to the right) by minimising $\sum_{i,t} w_t^i z_t^i$ with appropriate weights w_t^i under the additional constraint $\sum_{t,k} c_t^k y_t^k \leq C^*$
- Transform the solution to this problem by e.g., solving a linear program with a linear objective function after fixing the $z_t^{i,f}$ to the values found previously.

Future work (1)

- New valid inequalities



Valid inequalities from kanpsack sets with a single continuous variable:

$$Y = \left\{ (z, y) \in \mathbb{B}^n \times \mathbb{R} \mid \sum_{j=1}^{n} z_j \leq b + y \right\}$$

Future work (2)

- New Models
  - The maximum intensity of activities determine their minimum length
  - It is possible to limit their maximum length by generalizing our results: VUB flow models with allowed configurations

$$X = \{(x, z) \in \mathbb{R}^n \times Z \mid \sum_{j=1}^{n} x_j = 1, 0 \leqslant x_j \leqslant u_j z_j, \forall j\},$$

where Z is the set of allowed configurations.