# Diszkrét kétszintű optimalizálás

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BME Modellalkotási Szeminárium, 2012. ősz

# Outline

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# Problem formulation

- Two decision makers, Leader and Follower, who make decisions sequentially, in this order
- General form: optimistic case

$$\begin{aligned} \min_{x,y} \quad & f(x,y) && (1) \\ \text{subject to} \quad & && \\ & L(x,y) && (2) \\ & y \in \arg\min_{y'}(g(x,y') \mid F(x,y')) && (3) \end{aligned}$$

- General form: pessimistic case

$$\begin{aligned} \min_{x} \max_{y} \quad & f(x,y) && (4) \\ \text{subject to} \quad & && \\ & L(x,y) && (5) \\ & y \in \arg\min_{y'}(g(x,y') \mid F(x,y')). && (6) \end{aligned}$$

# Linear bilevel optimization

- 2 levels
- Variables: $x = (x^1, x^2) \in \mathbb{R}_+^{n_1+n_2}$
- Variables $x^i$ are exclusively controlled by player $i$
- Formulation:

$$\begin{aligned}
\min_x \quad & c^{11}x^1 + c^{12}x^2 && (7)\\
\text{subject to} \quad & \\
& A^{11}x^1 + A^{12}x^2 \geq b^1 && (8)\\
& x^2 \in \arg\min_y (c^{22}y \mid A^{21}x^1 + A^{22}y \geq b^2). && (9)
\end{aligned}$$

# The polynomial hierarchy

## Definition (Polynomial hierarchy by Karp)

Let $L \subset S^+$ be a language over a finite alphabet. For any $k \geq 1$,

- $L \in \sum_k^p$ if and only if $\exists p_1, \ldots, p_k$ polynomials and $L' \in \mathcal{P}$ such that for any $x \in S^+$
  $x \in L$ iff $(\exists y_1)_{p_1} (\forall y_2)_{p_2} \ldots (Q y_k)_{p_k} [(x, y_1, \ldots, y_k) \in L']$
- $L \in \Pi_k^p$ if and only if $\exists p_1, \ldots, p_k$ polynomials and $L' \in \mathcal{P}$ such that for any $x \in S^+$:
  $x \in L$ iff $(\forall y_1)_{p_1} (\exists y_2)_{p_2} \ldots (Q y_k)_{p_k} [(x, y_1, \ldots, y_k) \in L']$

## Definition (Polynomial hierarchy by Stockmeyer)

$\sum_k^p = \mathcal{NP}(\sum_{k-1}^p)$ with $\sum_0^p = \mathcal{P}$
$\Pi_k^p = co - \mathcal{NP}(\sum_{k-1}^p)$

## Theorem (Wrathal)

*The two definitions are equivalent.*

# Complexity of multi-level programming

### Theorem (Jeroslow)

*Bilevel linear programming is NP-hard.*

### Theorem (Jeroslow)

$(k+1)$*-level linear programming is* $\Sigma_k^p$*-hard.*

### Theorem (Jeroslow)

$k$*-level integer (binary) programming is* $\Sigma_k^p$*-hard.*

### Corollary

*Unless the polynomial hierarchy collapses at level 1, integer (binary)* $k$*-level programs cannot be modeled by mixed integer programs of size polynomial in the size of the input, for any* $k \geq 2$*.*

### Corollary

*Unless the polynomial hierarchy collapses at level 1, linear* $k$*-level programs cannot be modeled by mixed integer programs of size polynomial in the size of the input, for any* $k \geq 3$*.*

# The bilevel total weighted completion time problem

- $n$ jobs and $m$ parallel, identical machines, no preemption
- Leader: assigns jobs to machines ($J = J_1 \cup J_2 \cup \cdots \cup J_m$)
  - Optimistic objective: $\min \sum_{j \in J} w_j^1 C_j$
  - Pessimistic objective: $\min \max \sum_{j \in J} w_j^1 C_j$
- Follower: sequences the assigned jobs on each machine

$$\min \sum_{i=1}^{m} \sum_{j \in J_i} w_j^2 C_j$$

- For a given machine assignment $J_1, \ldots, J_m$, Follower solves $m$ single machine problems $1||\sum_j w_j^2 C_j$ by Smith's rule (WSPT order)
- Leader has to find the best assignment knowing the strategy of the Follower

# Results on bilevel total weighted completion time problem

| Restriction | Complexity |
| --- | --- |
| no restriction | decision version is NP-complete |
| $w^1 \equiv 1$, $w^2$ induces an increasing proc. time order | equivalent to $P\|\| \sum_j C_j$ |
| $w^1 \equiv 1$, $w^2$ induces A decreasing proc. time order | reduces to a special MAX $m$-CUT problem |
| $w^1 \equiv 1$, $w^2$ arbitrary $m$ constant | FPTAS of Sahni for $Pm\|\| \sum_j w_j C_j$ can be generalized |

# The structure of optimal solutions

### Lemma

*There is a global ordering of jobs such that in an optimal solution on each machine the job sequence respects the global order.*
*In the optimistic case the global order is WSPT with respect to $w^2$ and in case of ties WSPT w.r.t. $w^1$.*
*In the pessimistic case the global order is WSPT with respect to $w^2$ and in case of ties reverse WSPT w.r.t. $w^1$.*

# Reduction to the MAX $m$-CUT problem

MAX $m$-CUT (optimization version)
**input**: the number of vertices (of a complete graph) $n$, edge weights $c_e$ for all the $n(n-1)/2$ edges, a number $m$ with $m \leq n$ (all data in $\mathbb{Z}_+$)
**output**: a partitioning of the vertices into $m$ disjoint classes $V_1, \ldots, V_m$ such that the total weight of edges between the classes is maximized, i.e.,

$$\max_{(V_1,\ldots,V_m)} \sum_{k=1}^{m-1} \sum_{\ell=k+1}^{m} \sum_{i \in V_k, j \in V_\ell} c_{ij}$$

where the maximum is over all $m$-partitions of the $n$ nodes
**Reduction**: the nodes are identified with the $n$ tasks, and

$$c_{jk} = p_j w_k^1 \quad \text{if } \frac{w_j^2}{p_j} > \frac{w_k^2}{p_k}; \text{ or } \frac{w_j^2}{p_j} = \frac{w_k^2}{p_k} \text{ and } \frac{w_j^1}{p_j} \geq \frac{w_k^1}{p_k}$$

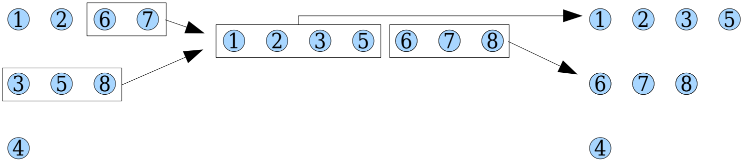
# A special MAX $m$-CUT problem

**Special weights**

If $w^1 \equiv 1$ and $\frac{w_j^2}{p_j} > \frac{w_k^2}{p_k}$ iff $p_j > p_k$, then $c_{jk} = \max\{p_j, p_k\}$

## Theorem

*There exists an optimal solution to MAX m-CUT such that* $V_1 = \{1, \dots, k_1\}, V_2 = \{k_1 + 1, \dots, k_2\}, \dots, V_m = \{k_{m-1} + 1, \dots, m\}$, *where* $p_j \geq p_k$ *for* $j < k$.




## Corollary

*The MAX m-CUT problem with the above weights can be solved by dynamic programming in polynomial time*

# The bilevel order acceptance problem

- There are $n$ jobs with processing times $p_j$, due-dates $d_j$, and job-weights $w_j^1$, $w_j^2$; and a single machine
- Leader: selects a subset of jobs $A$ (accepted jobs) to maximize $\sum_{j \in A} w_j^1$
- Follower: sequences the jobs non-preemptively to minimize $\sum_{j \in A} w_j^2 C_j$
- The solution is feasible iff the optimal solution chosen by the Follower meets the due-dates of all jobs in $A$
- If the Leader is optimistic, it selects $A$ such that at least one optimal solution of the Follower meets all the due-dates
- If the Leader is pessimistic, it selects $A$ such that all the optimal solutions of the Follower with respect to $A$ meets all the due-dates

# Results on the bilevel order acceptance problem

| **Restriction** | **Complexity** |
|---|---|
| no restriction | decision version is NP-complete<br>solvable in pseudo-poly time |
| $w^1 \equiv 1$ | Polynomial (generalized Moore-Hodgson alg.) |

# A polynomial algorithm for the $w^1 \equiv 1$ case

**The Moore-Hodgson algorithm for** $1||\sum U_j$

1. Order the jobs in EDD order: $d_1 \leq \ldots \leq d_n$
2. Starting with the first job, process the jobs one-by-one. If all jobs can be completed on time, stop. Otherwise, let $k_1$ be the first job such that $\sum_{j=1}^{k_1} p_j > d_{k_1}$. Remove from the first $k_1$ jobs the one with largest $p_j$ value, and proceed with the next job.

Modification for the bilevel order acceptance problem:

1. Order the jobs in the Follower's WSPT order: $j < k$ iff $\frac{w_j^2}{p_j} > \frac{w_k^2}{p_k}$ and in case of ties if $d_j < d_k$ $(d_j > d_k)$

# Uncapacitated lot-sizing with backlogging (ULSB)

$$\min\left\{\sum_{t=1}^{n}(p_t x_t + f_t y_t + h_t s_t + g_t r_t) \mid (11) - (15)\right\} \tag{10}$$

where

$$x_t + (s_{t-1} - r_{t-1}) = d_t + (s_t - r_t), \quad t = 1, \ldots, n \tag{11}$$
$$x_t \leq M y_t, \quad t = 1, \ldots, n \tag{12}$$
$$s_0 = s_n = r_0 = r_n = 0, \tag{13}$$
$$x_t, s_t, r_t, \geq 0, \quad t = 1, \ldots, n \tag{14}$$
$$y_t \in \{0, 1\}, \quad t = 1, \ldots, n \tag{15}$$

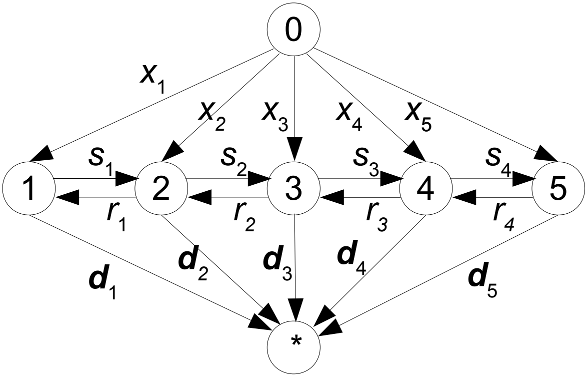
where

- The $p_t$, $f_t$, $h_t$, $g_t$ are the cost parameters, the $d_t$ are the demands
- The $x_t$, $y_t$, $s_t$, $r_t$ are the variables

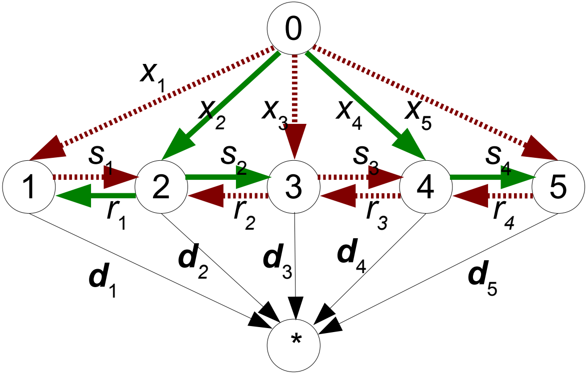
# Some related work


- W. I. Zangwill, A backlogging model and a multi-echelon model of a dynamic economic lot size production system – A network approach. Management Science, 15(9):506–527, 1969.
- A. Federgruen, M. Tzur, The dynamic lot-sizing model with backlogging: A simple $O(n \log n)$ algorithm and minimal forecast horizon procedure. Naval Res. Logitics 40, 459–478, 1993.
- Y. Pochet and L. A. Wolsey. Lot-size models with backlogging: Strong reformulations and cutting planes. Mathematical Programming, 40:317–335, 1988.
- S. Kucukyavuz and Y. Pochet. Uncapacitated lot-sizing with backlogging: the convex hull. Mathematical Programming, Ser. A, 118:151–175, 2009.

# Network representation of ULSB



# Extreme point solutions for ULSB



# Bilevel lot-sizing

Rules of the game

- Both decision makers solve an uncapacitated lot-sizing problem with backlogging
- The Leader has external demand ($d_t^1$)
- The Leader's production ($x_t^1$) equals the supply received from the Follower
- The Follower's demand ($\delta_t$) is set by the Leader
- Both the Leader and the Follower may backlog some of its demand
- The Follower pays the backlogging cost to the Leader as penalty for late delivery
- In those periods when the Follower backlogs, there is no delivery to the Leader ($r_t^2 x_t^1 = 0$)
- If the Follower does not backlog in some period $t$, then it supplies all the demands from the last supply point, i.e., $\sum_{u=t'+1}^{t} \delta_t$, where $t'$ is the last supply point ($x_{t'}^1 > 0$) or $t' = 0$

# Formulation

$$\text{Minimize} \sum_{t=1}^{n} \left(p_t^1 x_t^1 + f_t^1 y_t^1 + h_t^1 s_t^1 + g_t^1 r_t^1 - g_t^2 r_t^2\right) \tag{16}$$

subject to

$$x_t^1 + s_{t-1}^1 - r_{t-1}^1 = d_t^1 + s_t^1 - r_t^1, \quad t = 1, \ldots, n \tag{17}$$

$$\boxed{r_t^2 = \textstyle\sum_{\tau=1}^{t} (\delta_\tau - x_\tau^1),} \quad t = 1, \ldots, n \tag{18}$$

$$x_t^1 \leq M y_t^1, \quad t = 1, \ldots, n \tag{19}$$

$$\boxed{x_t^1 \leq M(1 - \beta_t^2),} \quad t = 1, \ldots, n-1 \tag{20}$$

$$s_0^1 = s_n^1 = r_0^1 = r_n^1 = 0, \tag{21}$$

$$x_t^1, r_t^1, s_t^1, \delta_t \geq 0, \quad t = 1, \ldots, n \tag{22}$$

$$y_t^1 \in \{0, 1\}, \quad t = 1, \ldots, n \tag{23}$$

# Formulation (cont.d)

$$\begin{pmatrix} y^2 \\ x^2 \\ s^2 \\ r^2 \\ \beta^2 \end{pmatrix} \in \arg\min \left\{ \sum_{t=1}^{n} \left( p_t^2 x_t^2 + f_t^2 y_t^2 + h_t^2 s_t^2 + g_t^2 r_t^2 \right) \;\middle|\; (25) - (31) \right\} \tag{24}$$

where

$$\begin{aligned}
& x_t^2 + (s_{t-1}^2 - r_{t-1}^2) = \delta_t + (s_t^2 - r_t^2), && t = 1, \ldots, n && (25) \\
& x_t^2 \leq M y_t^2, && t = 1, \ldots, n && (26) \\
& s_0^2 = s_n^2 = r_0^2 = r_n^2 = 0, && && (27) \\
& x_t^2, s_t^2, r_t^2 \geq 0, && t = 1, \ldots, n && (28) \\
& y_t^2 \in \{0, 1\}, && t = 1, \ldots, n && (29) \\
& \boxed{r_t^2 \leq M \beta_t^2,} && t = 1, \ldots, n-1 && (30) \\
& \boxed{\beta_t^2 \in \{0, 1\}} && t = 1, \ldots, n-1. && (31)
\end{aligned}$$

# Example

Optimal solution of a sample problem

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $d_t^1$ | 71 | 84 | 43 | 21 | 4 | 81 | 59 | 44 | 32 | 46 |
| $x_t^1$ | 82 | 73 | 68 | | | 82.49 | 57.51 | 55.46 | 21.93 | 44.61 |
| $s_t^1$ | 11 | | 25 | 4 | | 1.49 | | 11.46 | 1.39 | |
| $r_t^1$ | | | | | | | | | | |
| $\delta_t$ | 82 | 73 | 68 | | 42.72 | 39.77 | 57.51 | 55.46 | 21.93 | 44.61 |
| $x_t^2$ | 82 | 141 | | | | 140.00 | | 122.00 | | |
| $s_t^2$ | | 68 | | | | 57.51 | | 66.54 | 44.61 | |
| $r_t^2$ | | | | | 42.72 | | | | | |

$$
\begin{array}{llll}
f^1 = 100 & p^1 = 1 & h^1 = 6 & g^1 = 18 \\
f^2 = 492 & p^2 = 1 & h^2 = 2 & g^2 = 6
\end{array}
$$

# Example

Optimal solution of a sample problem

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $d_t^1$ | 71 | 84 | 43 | 21 | 4 | 81 | 59 | 44 | 32 | 46 |
| $x_t^1$ | 82 | 73 | 68 | | | 82.49 | 57.51 | 55.46 | 21.93 | 44.61 |
| $s_t^1$ | 11 | | 25 | 4 | | 1.49 | | 11.46 | 1.39 | |
| $r_t^1$ | | | | | | | | | | |
| $\delta_t$ | 82 | 73 | 68 | | 42.72 | 39.77 | 57.51 | 55.46 | 21.93 | 44.61 |
| $x_t^2$ | 82 | 141 | | | | 140.00 | | 122.00 | | |
| $s_t^2$ | | 68 | | | | 57.51 | | 66.54 | 44.61 | |
| $r_t^2$ | | | | | 42.72 | | | | | |

$$f^1 = 100 \quad p^1 = 1 \quad h^1 = 6 \quad g^1 = 18$$
$$f^2 = 492 \quad p^2 = 1 \quad h^2 = 2 \quad g^2 = 6$$

# Formulation MIP-1

**Definition**

*Let $OP^2$ be the set of those $(\bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2, \bar{\delta})$ vectors such that $\sum_{t=1}^{n} \bar{\delta}_t = \sum_{t=1}^{n} d_t^1$, and $(\bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2)$ is an <u>optimal</u> solution for the ULSB of the Follower w.r.t. demand $\bar{\delta}$.*

Let $Z^{ULSB}(\delta)$ denote the optimum value of ULSB for fixed $\delta > 0$

**Question**

*Does $OP^2$ admit a compact (extended) mixed integer formulation?*

**Answer**

*YES! Idea: use an extended formulation for ULSB with $\delta$ in the objective function only.*

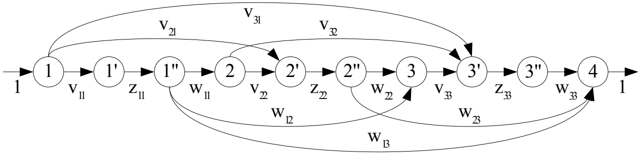
# Formulation MIP-1 (cont.d)

## Lemma

*(Pochet and Wolsey (1988)) The optimum value of ULSB equals the optimum value of the following mathematical program*

$$L^{SP}(\delta) = \min \sum_{k=1}^{n} \left( \sum_{\ell=1}^{k-1} a_{k\ell} v_{k\ell} + p_k \delta_k z_{kk} + \sum_{\ell=k+1}^{n} b_{k\ell} w_{k\ell} \right) + \sum_{t=1}^{n} f_t z_{tt}$$

*subject to a shortest path formulation in the network below, where* $a_{k\ell} = p_k \delta_{\ell,k-1} + \sum_{t=\ell}^{k-1} g_t \delta_{\ell,t}$ *for* $1 \leq \ell < k \leq n$, *and* $b_{k\ell} = p_k \delta_{k+1,\ell} + \sum_{t=k}^{\ell-1} h_t \delta_{t+1,\ell}$ *for* $1 \leq k < \ell \leq n$, *and* $\delta_{k,\ell} = \sum_{t=k}^{\ell} \delta_t$ *for* $1 \leq k \leq \ell \leq n$.



# Formulation MIP-1 (cont.d)

The dual of the shortest path formulation is

$$D^{SP}(\delta) = \max \phi_1^2 \tag{32}$$

subject to

$$\left.\begin{array}{lr} \phi_t^2 - \phi_{k'}^2 \leq a_{k,t}, & k = t, \ldots, n \\ \phi_{t'}^2 - \phi_{t''}^2 \leq p_t^2 \delta_t + f_t^2, & \\ \phi_{t''}^2 - \phi_{k+1}^2 \leq b_{t,k}, & k = t, \ldots, n \end{array}\right\} \text{ for all } t = 1, \ldots, n. \tag{33}$$

By the strong duality of linear programming $Z^{ULSB}(\delta) = D^{SP}(\delta)$ for any fixed $\delta \geq 0$.

## Lemma

*$(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2, \hat{\delta}) \in OP^2$ if and only if $\sum_{t=1}^n \delta_t = \sum_{t=1}^n d_t^1$, and there exists $\hat{\phi}^2$ such that $(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2, \hat{\beta}, \hat{\delta}, \hat{\phi}^2)$ satisfies the constraints (25)-(31), (33), and the equation*

$$\sum_{t=1}^n (p_t^2 x_t^2 + f_t^2 y_t^2 + h_t^2 s_t^2 + g_t^2 r_t^2) = \phi_1^2. \tag{34}$$

# Formulation MIP-1 (cont.d)

The complete formulation:

$$\text{MIP-1}: \ \min \left\{ \sum_{t=1}^{n} \left( p_t^1 x_t^1 + f_t^1 y_t^1 + h_t^1 s_t^1 + g_t^1 r_t^1 - g_t^2 r_t^2 \right) \ \middle| \ \begin{array}{l} (17)\text{-}(19), \\ (21)\text{-}(23), \\ (25)\text{-}(31), \\ (33),(34) \end{array} \right\}.$$

## Lemma

*We have the following correspondence between the feasible solutions of the bilevel lot-sizing problem and that of MIP-1:*

(i) *Any feasible solution of MIP-1 can be projected onto a feasible solution of the bilevel lot-sizing problem of the same value.*

(ii) *Conversely, any feasible solution of the bilevel lot-sizing problem can be extended to a feasible solution of MIP-1 of the same value.*

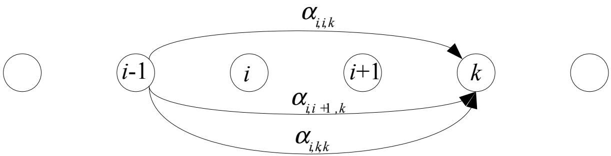
# Formulation MIP-2

- Again, based on a shortest path formulation

$$\alpha_{ijk} = \begin{cases} 1 & \text{the requests } \delta_i, \ldots, \delta_k \text{ are produced in } j \in \{i, \ldots, k\} \\ 0 & \text{otherwise} \end{cases}$$

- If $\alpha_{ijk} = 1$, then $s^2_{i-1} = s^2_k = 0$, and $r^2_{i-1} = r^2_k = 0$.
- Cost associated with $\alpha_{ijk}$:

$$c_{ijk} = a_{j,i} + f_j + p_j \delta_j + b_{j,k}$$



# Formulation MIP-2 (cont.d)

$$\text{MIP-2}: \quad \min \sum_{t=1}^{n} \left(p_t^1 x_t^1 + f_t^1 y_t^1 + h_t^1 s_t^1 + g_t^1 r_t^1 - g_t^2 r_t^2\right)$$

subject to the constraints of the Leader, and

$$r_t^2 \leq M(1 - \beta_t^2), \quad t = 1, \ldots, n-1$$

$$\beta_t^2 = \sum_{i \leq t < j \leq k} \alpha_{i,j,k}, \quad t = 1, \ldots, n-1$$

$$\sum_{i \leq t \leq k} \sum_{i \leq j \leq k} \alpha_{i,j,k} = 1, \quad t = 1, \ldots, n$$

$$a_{j,i} + f_j + p_j \delta_j + b_{j,k} + \phi_{i-1} \geq \phi_k, \quad 1 \leq i \leq j \leq k \leq n$$

$$a_{j,i} + f_j + p_j \delta_j + b_{j,k} + \phi_{i-1} \leq \phi_k - M'(1 - \alpha_{i,j,k}), \quad 1 \leq i \leq j \leq k \leq n$$

$$\phi_0 = 0,$$

$$\alpha_{i,j,k} \in \{0, 1\}, \quad 1 \leq i \leq j \leq k \leq n.$$

# Extreme Point Solutions

## Definition

*A solution to the bilevel lot-sizing problem is an* extreme point solution *if the Follower's part is an extreme point solution of ULSB with demands $\delta_t$.*

**Assumption** $g_t^2 + h_t^2 > 0$ for all $t = 1, \ldots, n-1$.
This assumption excludes that a solution with $r_t^2 s_t^2 > 0$ is optimal for the Follower.

## Lemma

*Under the assumption, if the bilevel optimization problem admits an optimal solution, then it admits an extreme point optimal solution.*

# Strengthening the formulations

- Bounds on variables
  $Z_t = \min_{u \geq t+1}(p_u + \sum_{v=t}^{u-1} g_v)$ is the minimum cost incurred by backlogging a unit of production from period $t$ to a later period.
  $S_t = \min_{1 \leq u < t}(p_u + \sum_{v=u}^{t-1} h_v)$ is the minimum cost of stocking a unit production from an earlier period to $t$.

## Lemma

*The backlogged quantities $r_t$ and the stock levels $s_t$ in any extreme point optimal solution of ULSB satisfy*

$$(Z_t - p_t)r_t \leq f_t, \text{ for } t = 1, \ldots, n-1 \tag{B}$$

$$(S_t - p_t)s_{t-1} \leq f_t, \text{ for } t = 1, \ldots, n-1 \tag{B}$$

- Cuts

## Lemma

*Extreme point solutions satisfy*

$$s_{t-1}^2 \leq M(1 - y_t^2 - \beta_t^2), \quad t = 2, \ldots, n \tag{C}$$

# Computational experiments

- For each $n \in \{10, 15, 20, 25, 30, 40, 50\}$, 100 random instances with parameters

$$
\begin{array}{llll}
f_t^1 \leftarrow U[100, 200] & p_t^1 \leftarrow U[1, 5] & h_t^1 \leftarrow U[2, 20] & g_t^1 \leftarrow U[4, 40] \\
f_t^2 \leftarrow U[250, 1000] & p_t^2 \leftarrow U[2, 10] & h_t^2 \leftarrow U[1, 10] & g_t^2 \leftarrow U[2, 20] \\
d_t^1 \leftarrow U[0, 100] & & &
\end{array}
$$

- Implementation in FICO XPRESS Mosel environment
- Tests performed on a workstation with Intel Xeon CPU (2.5 GHz), Linux operating system

# Results

| | | opt | LB gap (%) | | UB gap (%) | | time (sec) | |
|---|---|---|---|---|---|---|---|---|
| | | | max | avg | max | avg | max | avg |
| MIP-1 | $n = 10$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 0.49 | 0.16 |
| | $n = 20$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 9.91 | 1.14 |
| | $n = 30$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 188.00 | 16.75 |
| | $n = 40$ | 88 | 17.36 | 0.89 | 16.99 | 0.47 | 1200.44 | 329.86 |
| | $n = 50$ | 53 | 15.09 | 2.91 | 12.38 | 1.88 | 1200.90 | 749.97 |
| MIP-1B | $n = 10$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 0.53 | 0.16 |
| | $n = 20$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 13.83 | 1.19 |
| | $n = 30$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 357.02 | 22.00 |
| | $n = 40$ | 90 | 19.68 | 0.79 | 16.99 | 0.46 | 1200.30 | 322.78 |
| | $n = 50$ | 59 | 14.41 | 2.51 | 11.85 | 1.94 | 1200.58 | 714.31 |
| MIP-1C | $n = 10$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 0.61 | 0.18 |
| | $n = 20$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 8.41 | 1.20 |
| | $n = 30$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 248.62 | 16.07 |
| | $n = 40$ | 92 | 16.99 | 0.60 | 17.82 | 0.46 | 1200.30 | 248.81 |
| | $n = 50$ | 51 | 15.19 | 2.77 | 11.81 | 1.88 | 1200.65 | 730.48 |
| MIP-1CB | $n = 10$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 0.76 | 0.27 |
| | $n = 20$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 8.82 | 1.54 |
| | $n = 30$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 158.35 | 14.74 |
| | $n = 40$ | 96 | 17.02 | 0.46 | 16.99 | 0.44 | 1200.26 | 227.71 |
| | $n = 50$ | 65 | 12.11 | 1.97 | 11.51 | 1.83 | 1200.55 | 645.94 |
| MIP-2 | $n = 10$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 35.65 | 17.93 |
| | $n = 20$ | 0 | 70.73 | 43.06 | 2974.59 | 1515.28 | 1200.00 | 1200.00 |
| MIP-2B | $n = 10$ | 100 | 0.00 | 0.00 | 0.00 | 0.00 | 46.83 | 10.57 |
| | $n = 20$ | 0 | 59.32 | 17.79 | 192.80 | 108.28 | 1200.00 | 1200.00 |