Markets Quantitative Analysis

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Models for pricing quanto products in finance

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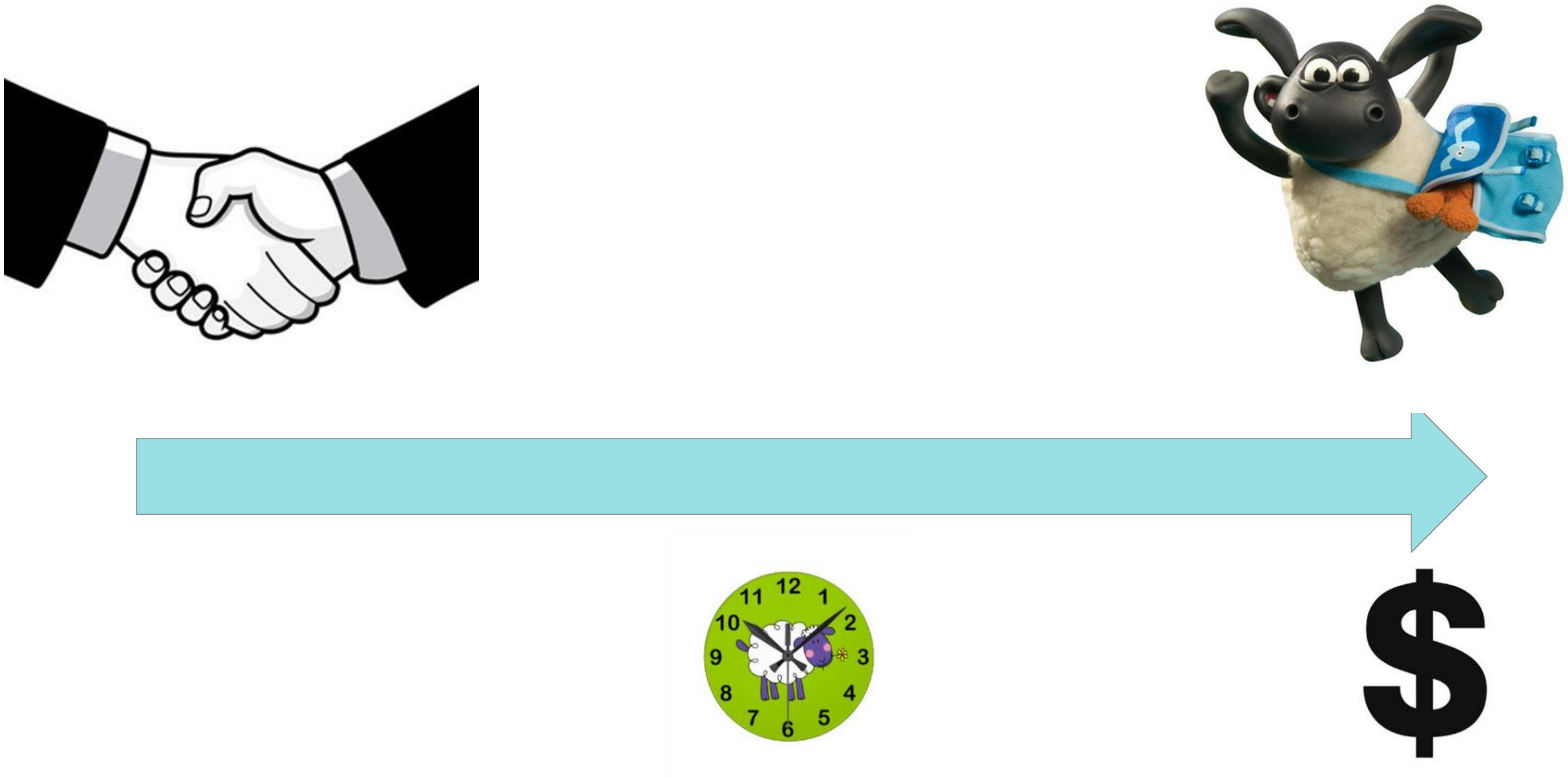
Talk Summary

- Introducing the basic financial concepts:
  - What are forward contracts?
  - Examples:
    - Vanilla equity forward
    - FX forward

- Putting exchange rates and equity/credit products together
  - Introducing quantos, usage and some examples.
    - Simple models for pricing quanto equity forwards. Why the market does not fit predictions from the most simplistic model.
      - Underlying equity and FX models.
    - Introducing vanilla CDS, a kind of life insurance for default of an entity. When do we use quanto CDS.
      - Introducing a Jump-diffusion FX model.

Forward Contract

- Agreement to deliver a position at maturity for an agreed delivery price: (for example an airline needs kerosene one year from now and is looking to fix the price)
- Agree on the (fixed) price today for an asset  to be delivered at expiry: $



Example: Pricing an Equity Forward Contract

You are looking to buy a stock at a future date T, the price to be paid at delivery.

Let's say you buy the GOOG forward

What is the payout at maturity (time T)?

S_T USD, where S_T is the price of GOOG at time T

How much does it cost you to guarantee this payout? (In other words, imagine you start from zero money, you borrow money to construct this payout, what will be your debt at time T?)

At what price are you willing to sell this obligation?

Pricing a Simple (Vanilla) Forward Contract

- A perfect replicating strategy: buy the stock now hold until maturity. My replicating cost is the initial stock price + interest I have to pay on the borrowed funds.

	0	T
Cash:	$-S_0$	$-S_0e^{rT}$
Asset:	1	1

Fair strike of a vanilla forward (amount to be paid at delivery at time T):

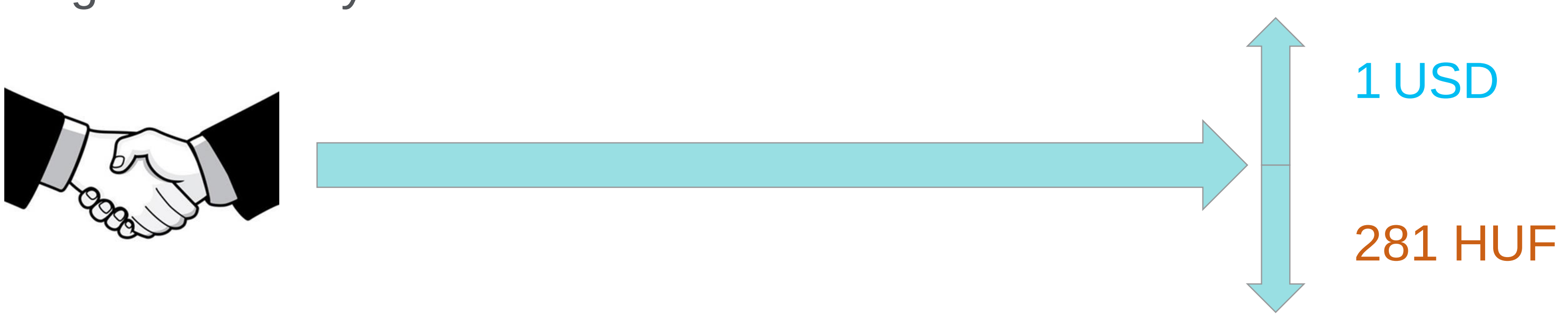
$$F_T = S_0 e^{rT}$$

No dynamic hedging is needed until expiry.

FX Forwards

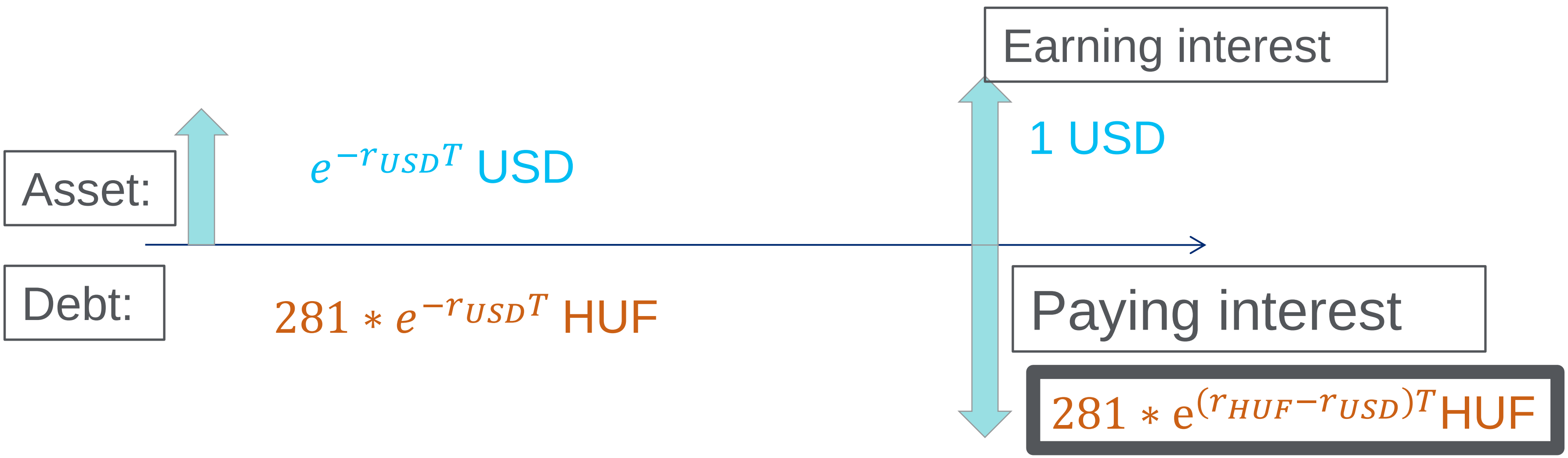
- An FX forward is a contract that guarantees me a fixed exchange rate at expiration: USD/HUF, for instance:
  – Currently: USD/HUF is 281, this is my spot price (i.e. current market price).
  – A one-year FX forward struck at 281 will involve exchange of cash-flows:
    - Buyer will receive 281 HUF
    - Seller will receive 1 USD

- Used when I am exposed to exchange rate risk: e.g., I am a Hungarian company, expecting 100 Dollars 1 year from now, but want to fix the exchange rate today.



How to Price FX Forwards?

- What is the fair price of such a contract? How many HUF does 1 USD is worth 1 year from? Surprisingly, this more of an interest rate question.
  - Let's assume that the USD interest rate is r(USD), the Hungarian rate is r(HUF).
  - Borrow $281 * e^{-r_{USD}T}$ HUF
  - Buy $\mathrm{e}^{-r_{USD}T}$ USD, invest it in the USD money market and wait until maturity until it becomes 1 USD.
  - Deliver 1 USD in exchange for F_T forints.



$$\boxed{FX(T) = FX_0 * \mathrm{e}^{(r_{HUF}-r_{USD})T}}$$

Quanto Contracts

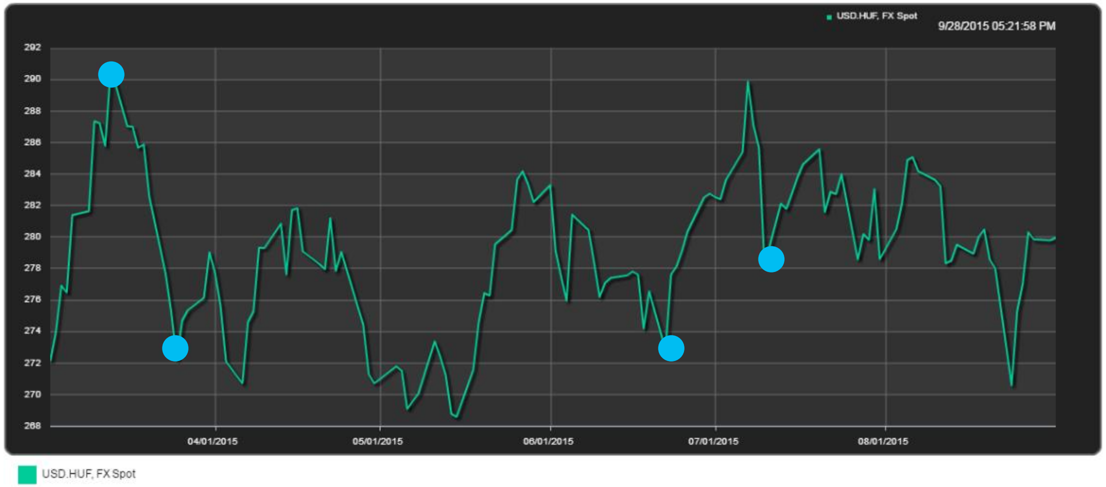
- Quantos are cross-currency derivatives that have an underlier denominated in one “foreign” currency (USD for USA stocks), but settles in another “domestic” currency (Hungarian Forints). Quantos are popular choice for investors, because then can eliminate cross-currency risk. For example:

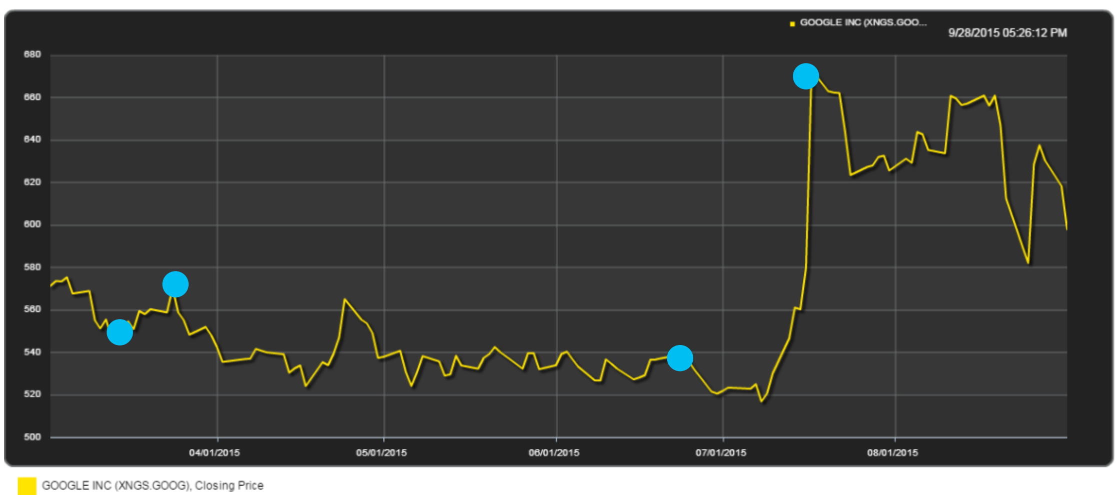
Trade:

- Imagine you want exposure to GOOG, you want to hold the stock for a year.
- Your home currency is Hungarian Forint and you do not want to be exposed to the USDHUF FX-rate

(By convention USD/HUF FX is the value of 1 USD in Hungarian Forints)

Stock Price and FX Rate Exposure





1USD=290Huf
GOOG= 550Usd
GOOG=159.5K HUF

1USD=272Huf
GOOG= 570Usd
GOOG=155.05K HUF

1USD=273Huf
GOOG= 540Usd
GOOG=147.42K HUF

1USD=278Huf
GOOG= 670Usd
GOOG=186.26K HUF

I don’t want exposure to USD! 1st solution: Buy GOOG and Long the HUFUSDFX rate

- Imagine you want to put 10000HUF notional in this trade
- Buy GOOG for 10.000HUF (buy 10.000/(S*USDHUFFX) piece of GOOG shares)

- Buy the HUFUSDFX rate for 10.000HUF

Manual FX Risk Elimination

- How to buy the HUFUSDFX in 10.000HUF notional?
- Borrow 10.000/(USDHUFFX) USD and convert it to HUF

Problems:

- Throughout your trade, you need to keep the same notional in GOOG and in HUFUSDFX, which means plenty of trades -> transaction cost
- You are not protected against second order effect-> big simultaneous moves in GOOG and HUFUSDFX

Quanto Forward Contract

- 2nd solution: Buy a Quanto forward on GOOG

Payout at maturity (T):

$$S_T * FX_0 \text{ Huf}$$

Where S_T is the spot price of GOOG at time T, FX_0 is the FX rate also specified in the contract at inception (time 0).

Note that the payout is directly in HUF, hence no FX sensitivity

Fundamental Theorem of Asset Pricing

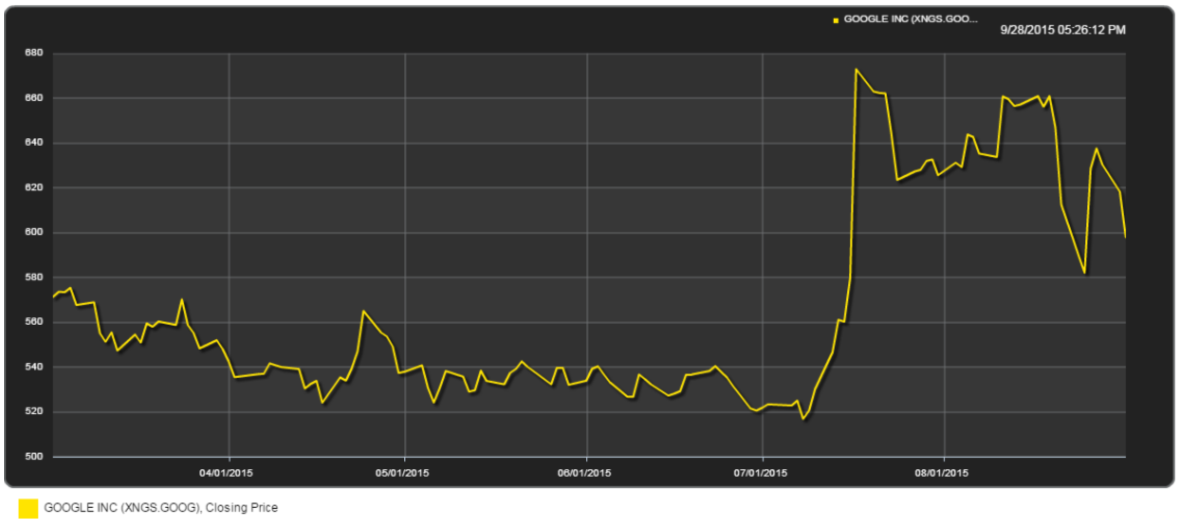
- From the Fundamental Theory of Asset Pricing (FTAP), there exists a measure under which we can compute derivative prices by taking discounted expectation i.e.

$$P_0 = e^{-rT} E(P_T(S_T))$$

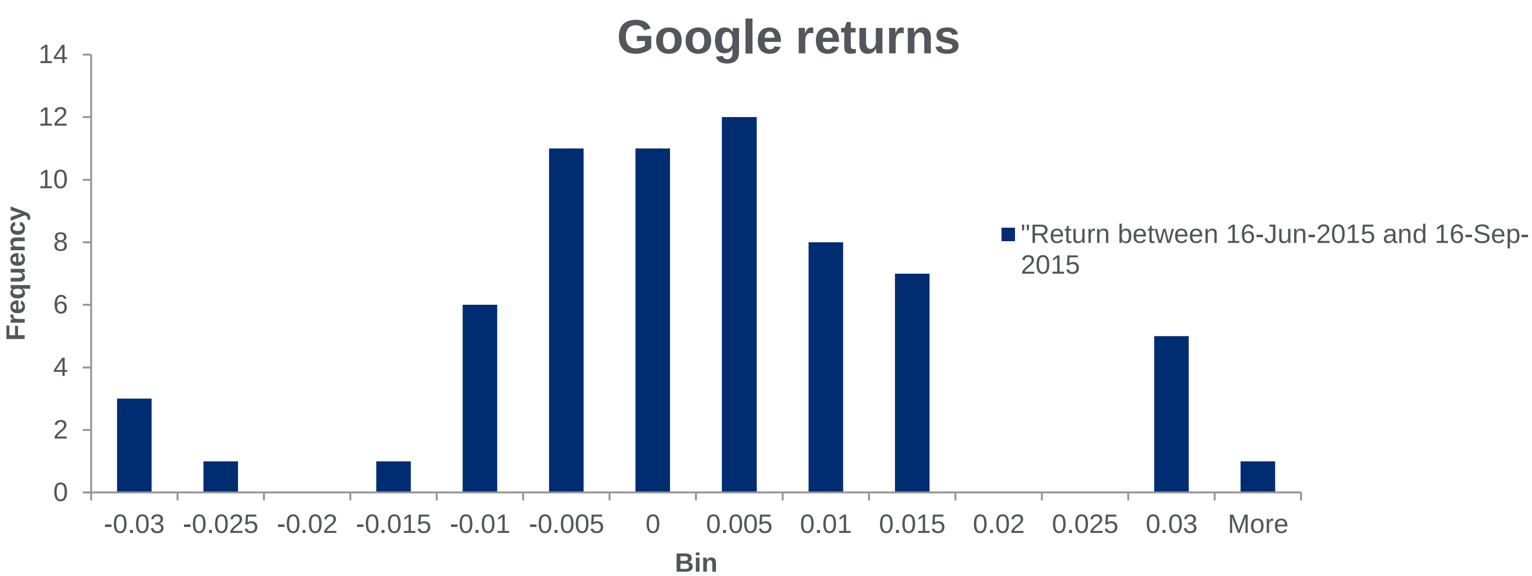
where $P_T(S_T)$ is the payout (e.g. identity function in case of a Forward Contract), r is the interest rate in the economy where the stock is listed (e.g. for GOOG it is r_{USD})

- FTAP price gives back the price corresponding to replication

Google Returns



$$return_i = \ln(S_i \,/\, S_{i-1})$$



Stock Price Distribution

- Let us assume a lognormal stock price distribution, with know volatility σ_S
- We will reverse engineer the drift, by applying FTAP to the forward contract

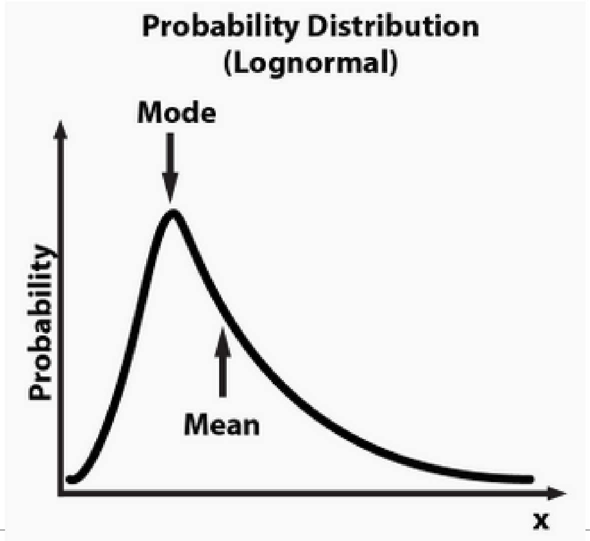
We previously calculated the discounted Forward price: S_0

FTAP has to be consistent with this price $e^{-rT}E(P_T(S_T))= e^{-rT}E(S_T)= S_0$

$$S_T = S_0 e^{rT} e^{N(0,T)\sigma_S - \frac{{\sigma_S}^2 T}{2}}$$

Where N(0,T) is normal

(E($e^{N(0,T)} = e^{T/2}$))



Quanto Forward Pricing

- Going back to the quanto forward pricing
- The payout in Hungarian forint: $S_T * FX_0$
- Is identical to a contract which pays: $S_T * \frac{FX_0}{FX_T}$ in USD at maturity
- So the discounted fair price of the contract in USD is
- $\mathrm{P}_{quanto} = e^{-r_{USD}T} E \quad [S_T \frac{FX_0}{FX_T}]$

Quanto Forward Pricing

- To evaluate the quanto forward price, $\mathrm{P}_{quanto} = e^{-r_{USD}T} E \quad [S_T \frac{FX_0}{FX_T}]$, we need the FX distribution as well
- Let us assume similar log normal distribution with volatility σ_{FX}
- The can calculate the drift to match to FX Forward, hence

$$FX_T = FX_0 e^{Q(0,T)\sigma_{FX} - \frac{\sigma_{FX}{}^2 T}{2} + (r_{HUF} - r_{USD})T}$$

with normal variable $Q(0,T)$

Quanto Forward Pricing

$$\mathrm{P}_{quanto} = e^{-r_{USD}T} E \left[S_T \frac{FX_0}{FX_T} \right] = e^{-r_{HUF}T} S_0 e^{r_{USD}T} e^{-\sigma_S \sigma_{FX} \rho T}$$

Where ρ is the correlation of $Q(0,T)$ and $N(0,T)$, σ_S and σ_{FX} are the volatility of the stock and FX respectively

(see Appendix A for the derivation)

The term $e^{-r_{HUF}T}$ stands for discounting

$S_0 e^{r_{USD}T}$ is the vanilla forward

$e^{-\sigma_S \sigma_{FX} \rho T}$ is the quanto adjustment

The same quanto adjustment can be used for options

Summary

- Vanilla Equity Forward Contract: $F_T = S_0 e^{rT}$
- FX Forward Contract: $FX_T = FX_0 e^{(r_{HUF} - r_{USD})T}$
- Investment in foreign markets
- Asset price distribution
- Risk neutral expectation
- Quanto adjustment: $e^{-\sigma_S \sigma_{FX} \rho T}$
- Quanto forward: $F_{quanto} = S_0 e^{r_{USD}T} e^{-\sigma_S \sigma_{FX} \rho T}$

Appendix A – Quanto Forward Pricing

$$\mathrm{P}_{quanto} = e^{-r_{USD}T} E \left[S_T \frac{FX_0}{FX_T} \right] =$$

$$= e^{-r_{USD}T} S_0 \; E(e^{N(0,T)\sigma_S - \frac{{\sigma_S}^2 T}{2} + r_{USD}T} e^{-Q(0,T)\sigma_{FX} + \frac{{\sigma_{FX}}^2 T}{2} - (r_{HUF} - r_{USD})T})$$

$$= e^{-r_{USD}T} S_0 \; E(e^{N(0,T)\sigma_S - \frac{{\sigma_S}^2 T}{2} + r_{USD}T} e^{-Q(0,T)\sigma_{FX} + \frac{{\sigma_{FX}}^2 T}{2} - (r_{HUF} - r_{USD})T})$$

$$= e^{(-r_{HUF} + r_{USD})T} S_0 \; E(e^{N(0,T)\sigma_S - \frac{{\sigma_S}^2 T}{2} - Q(0,T)\sigma_{FX} + \frac{{\sigma_{FX}}^2 T}{2}})$$

- Using that $Var(\sigma_S N(0,T) - \sigma_{FX} Q(0,T)) = \sigma_S^2 T + \sigma_{FX}^2 T - 2\sigma_S \sigma_{FX} \rho$, where ρ is the correlation of the normals, we arrive at

$$\mathrm{P}_{quanto} = \; e^{-r_{HUF}T} S_0 e^{r_{USD}T} e^{-\sigma_S \sigma_{FX} \rho T}$$

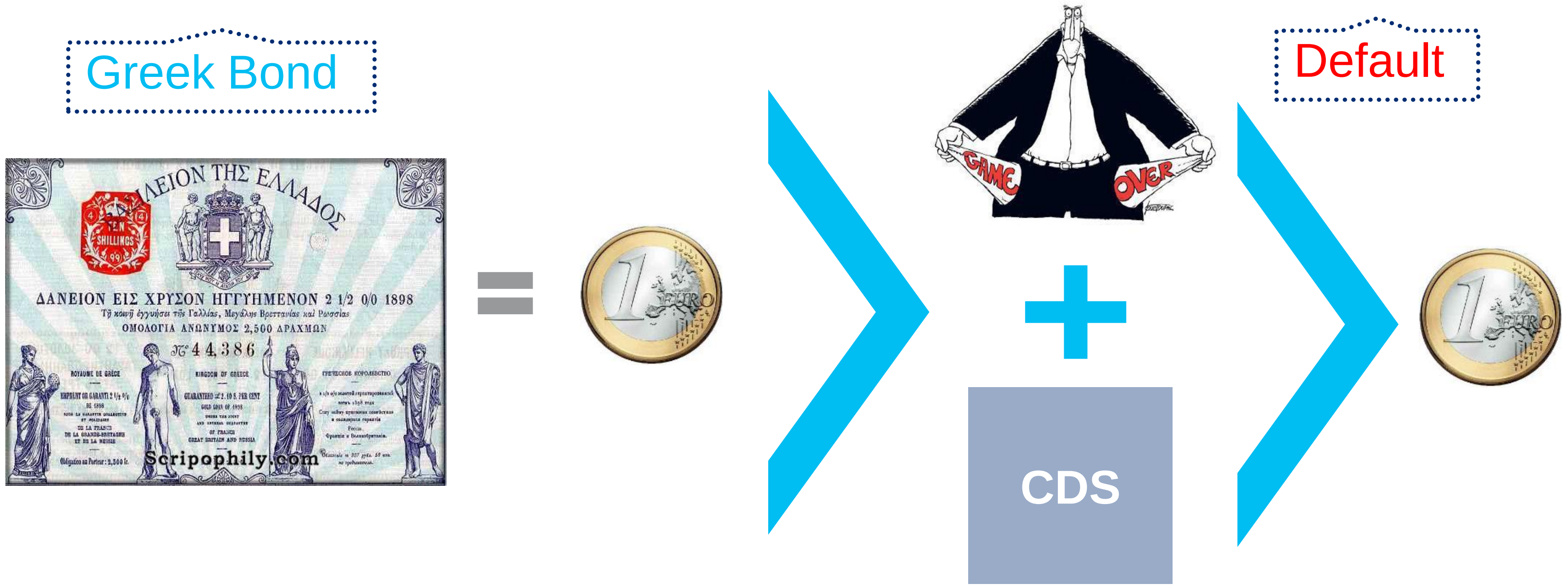
2. ‘Life insurances’ in the corporate world

What are credit default swaps and how they are used?

Credit Default Swaps

How to protect my investment against a sovereign/corporate default?

- A credit default swap is similar to an insurance: e.g., death insurance, home insurance, or car insurance.
- Credit default swaps provide you protection against the default of a corporation or government by compensating you for your losses.



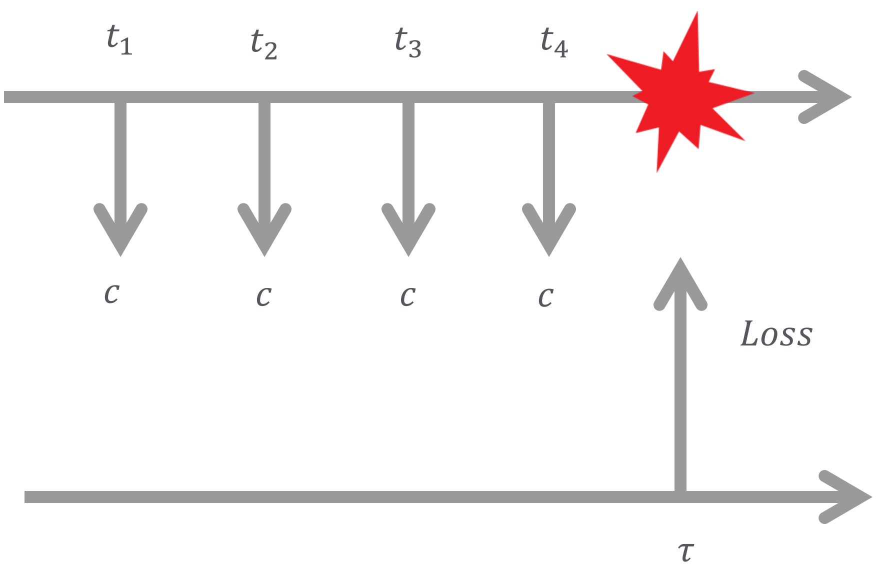
How to price a vanilla ‘plain’ CDS?

- One participant pays coupons (insurance premium) for protection, until the death (bankruptcy) of the company. Coupons are paid as long as the company is viable.

- $SP(t)$: survival probability of the company until time t.

Protection buyer cash-flows

Fair value of cash-flows:



$$= SP(t_1) * c + SP(t_2) * c + \cdots$$

c: fair coupon

$$= \left(1 - SP(T)\right) * Loss$$

Protection seller cash-flows

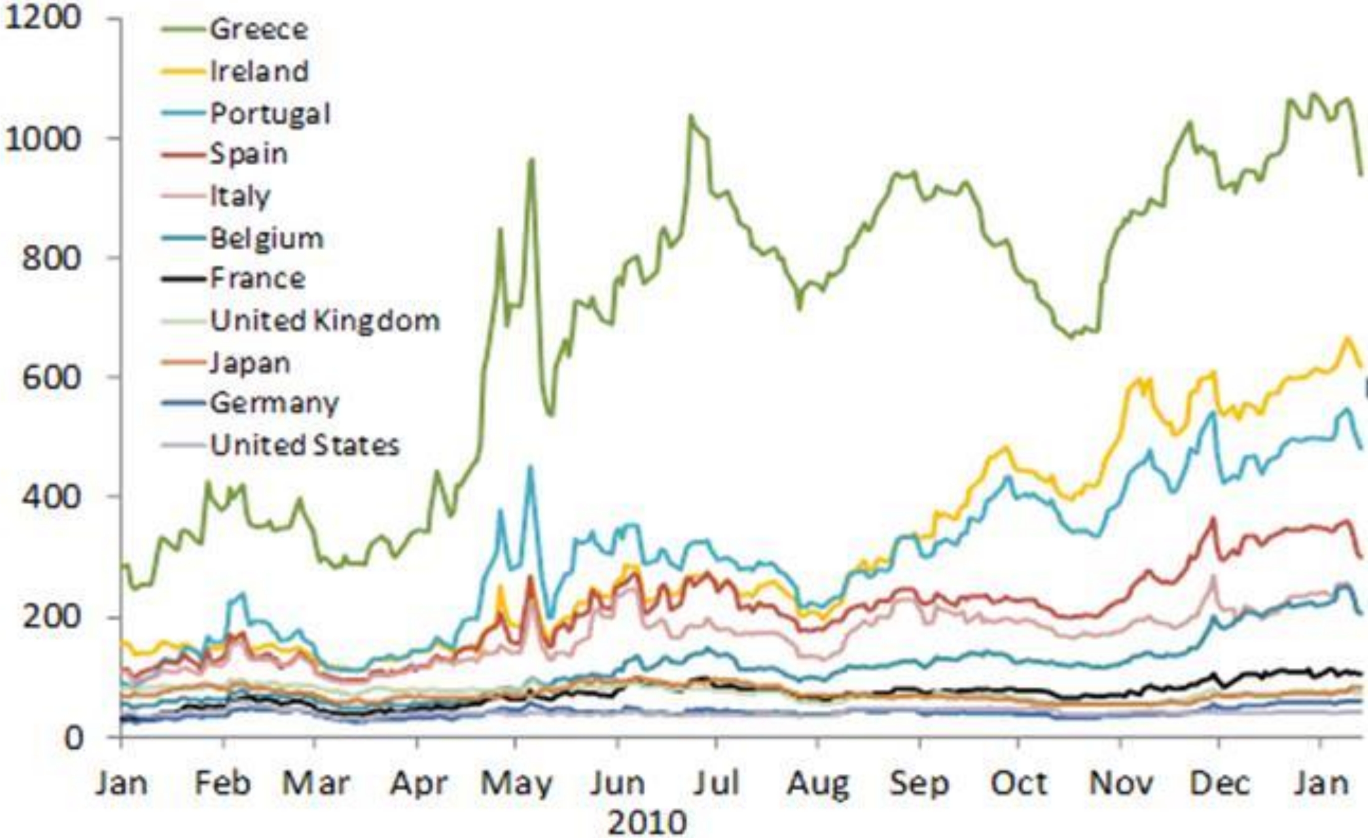
Simplifying assumption: forget about discounting for now.

Market Quotes

Fair spread reflects the riskiness of an investment as observed by the market participants.

1 bps = 0.01 %

Figure 1. Sovereign Credit Default Swap Spreads
(In basis points)

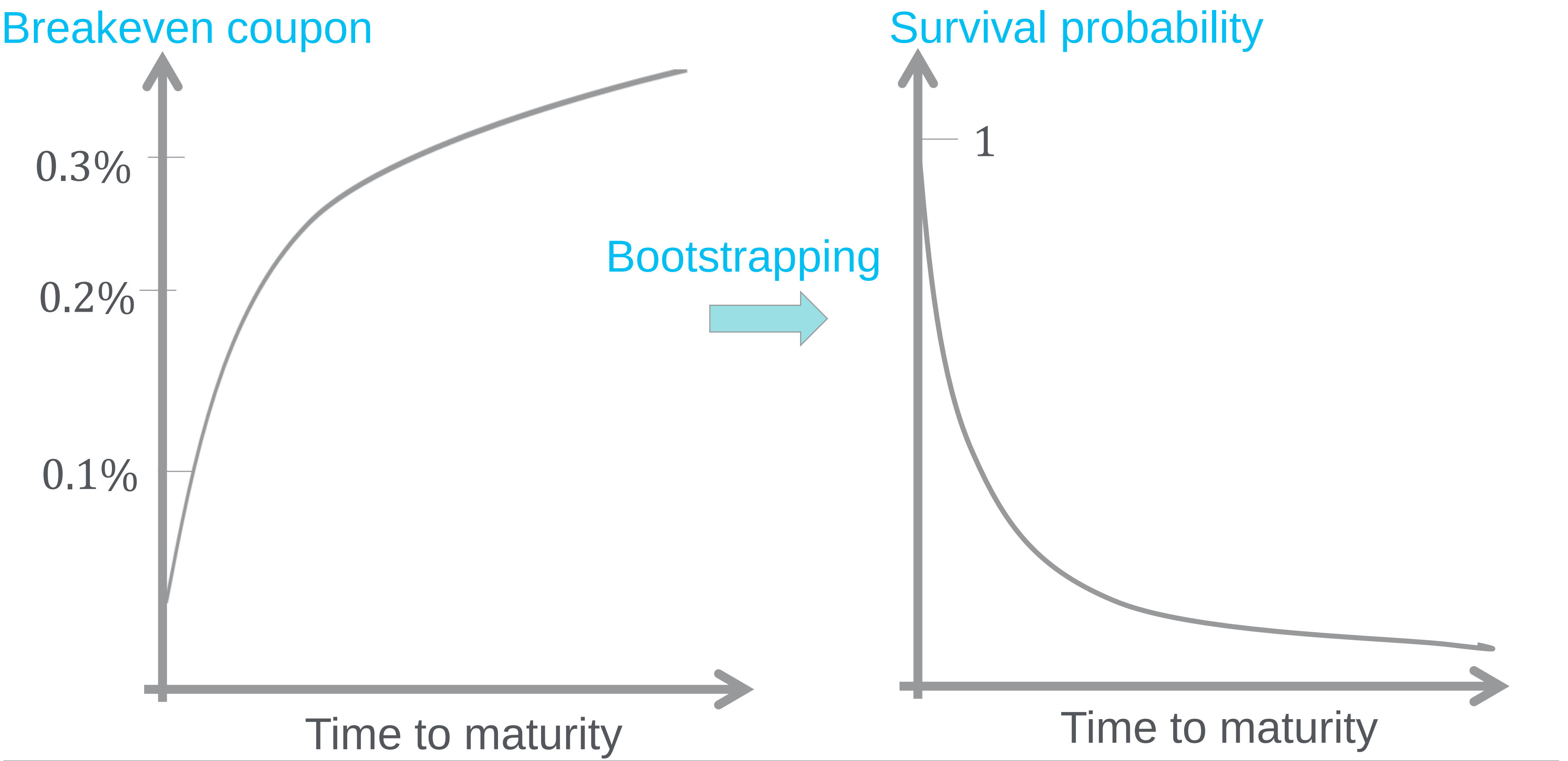


Source: Bloomberg L.P.

Modelling default probabilities

- Default probability is modelled as a **Poisson-process**, with time-dependent intensity $h(s)$:

$$survival\ probability(t_1 < \tau < t_2 | survival\ up\ to\ t_1) = e^{-\int_{t_1}^{t_2} h(s)ds}$$



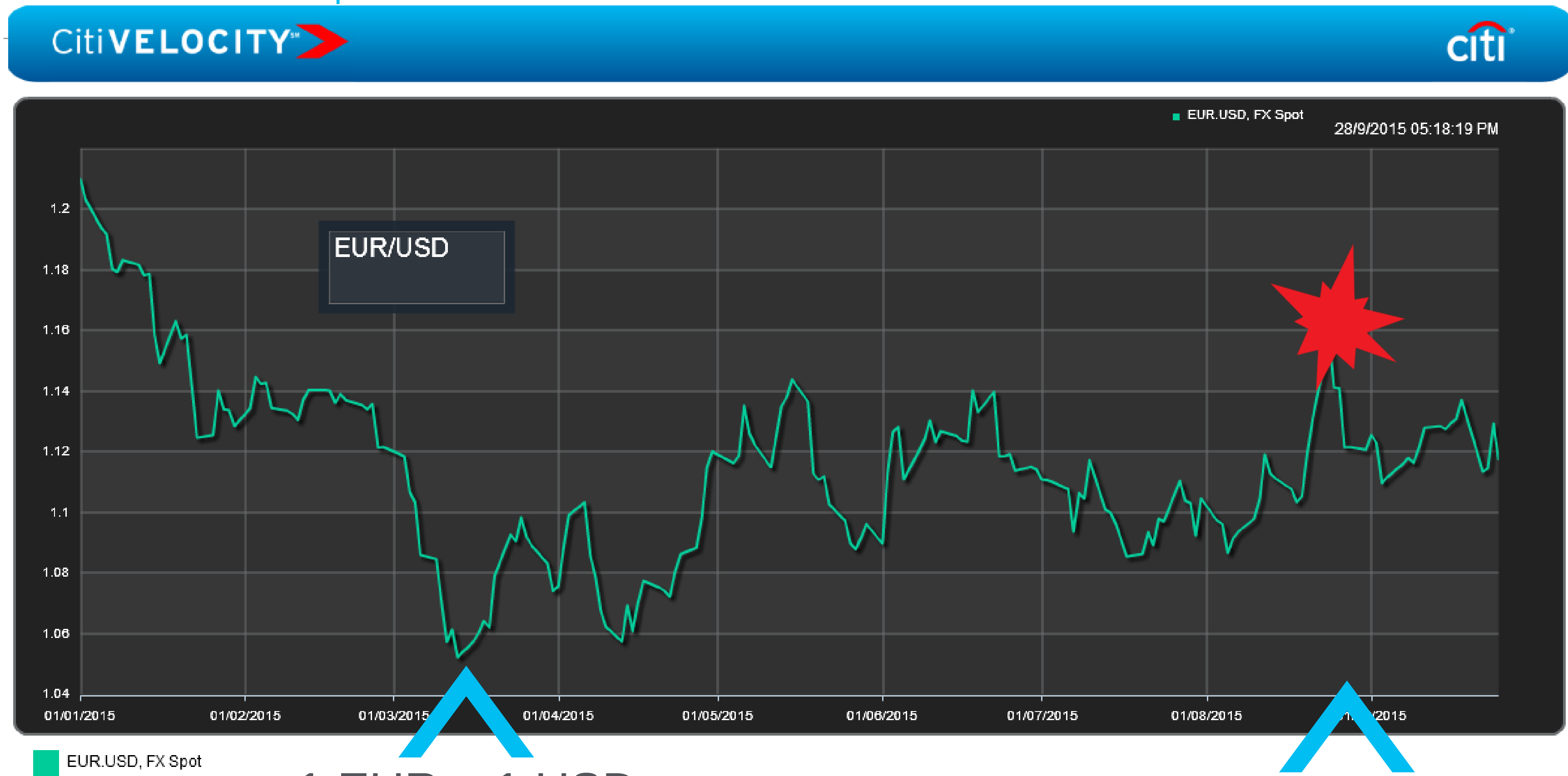
Vanilla CDS continued

Some facts about the CDS market.

- Once we have the survival probability we can price credit risky instruments other than CDS.
- CDS are typically quoted is USA dollars, this is the most liquid market.
- Among different durations the 5 year CDS is the most frequently traded one, so quotes for this are the most reliable.

Quantos motivation

In real life we need to protect investments in currencies other than USD.



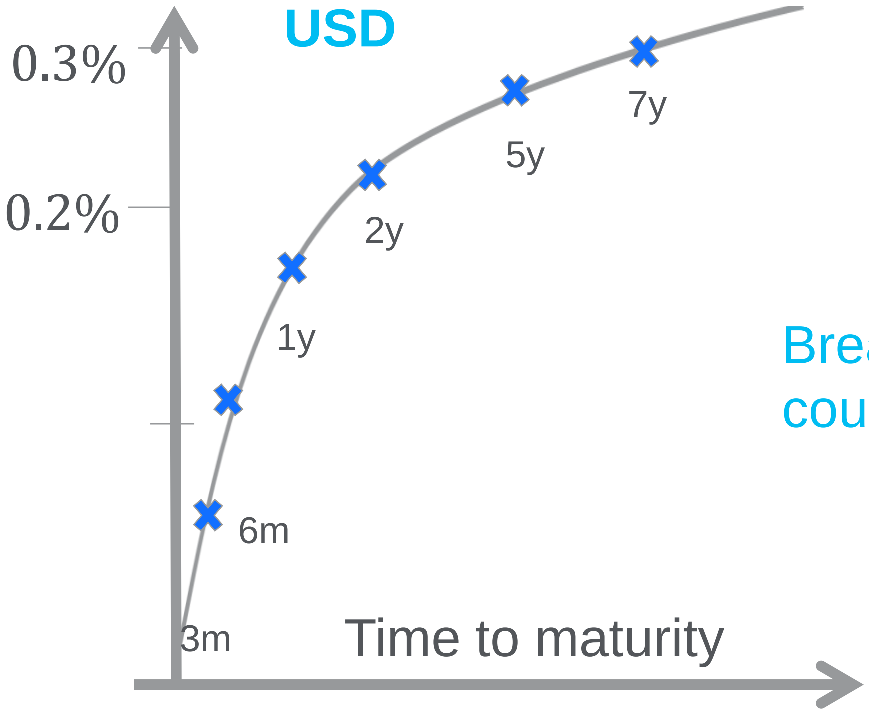
1 EUR = 1 USD

I have 1 M EUR worth of bonds, will buy 1M USD CDS protection

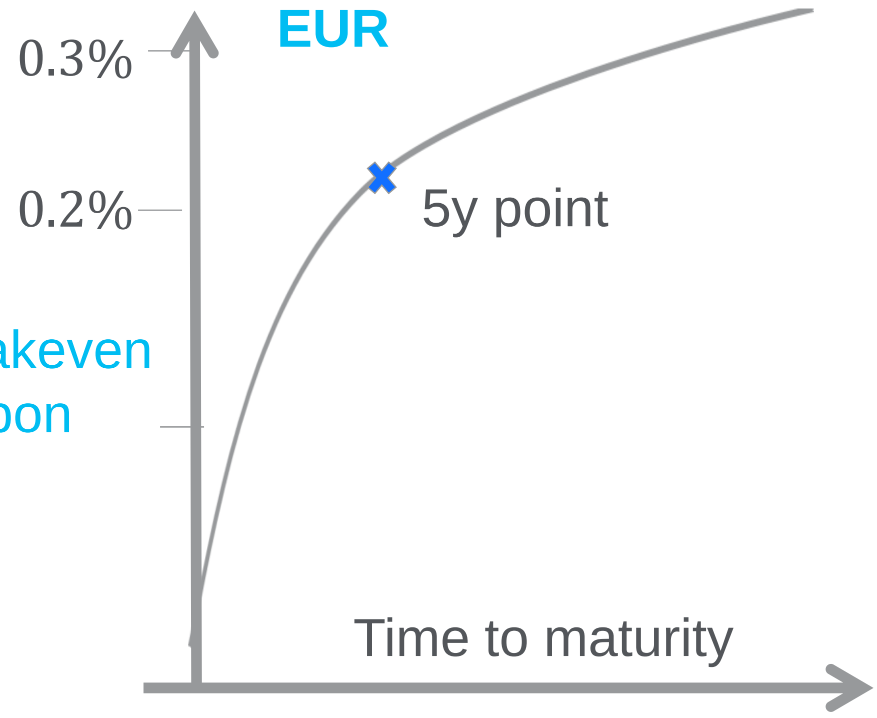
EUR/USD has changed: now 1 USD = 0.88 EUR, but I have 1M EUR of bonds.

What I need is a quanto CDS!

- Quanto CDS pays out in a currency other than the most liquid one: USD. It is typically very illiquid and only one tenor (5y) marked by the traders.
- I need protection for 3 years, what do I do?
- Use the liquid USA CDS marks + the single EUR mark and infer the rest using a model.



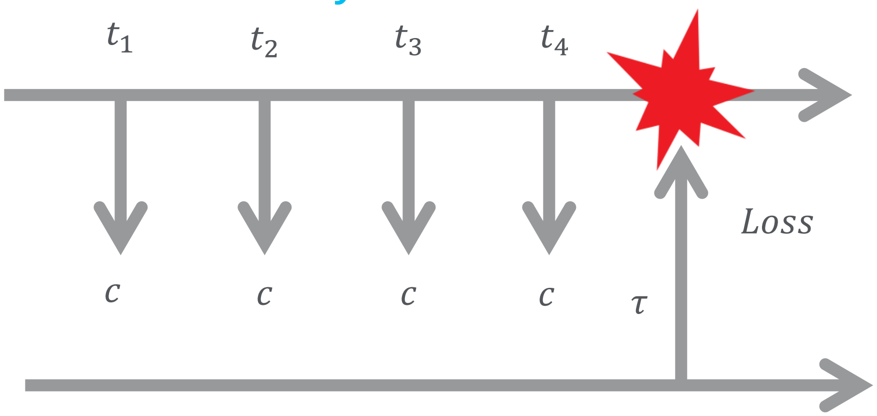
Breakeven coupon



USD vs. EUR CDS

USD CDS:

Protection buyer cash-flows



Protection seller cash-flows

Fair value of cash-flows

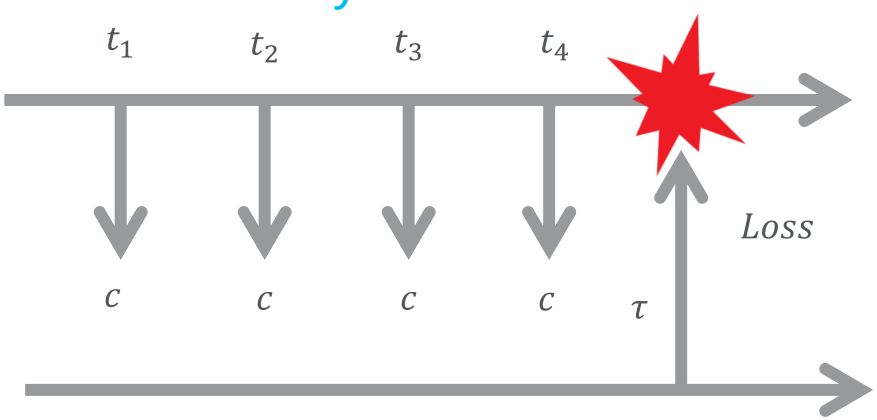
$$= 1(t_1 < \tau) * c + 1(t_2 < \tau) * c + \cdots$$

$$= \left(1 - 1(\tau < T)\right) * Loss$$

$$FX = \frac{EUR}{USD}$$ (USD I get for 1 EUR)

EUR CDS:

Protection buyer cash-flows



Protection seller cash-flows

$$= FX_{t_1} * 1(t_1 < \tau) * c + FX_{t_2} * 1(t_2 < \tau) * c + \cdots$$

$$= FX_\tau\left(1 - 1(\tau < T)\right) * Loss$$

Quantos CDS evaluation

Let's see how we can approach quantos.

- Take the expectations of both cash-flows, you will find:
- You can price a quanto CDS exactly as you price a USD one, but survival probability
  - $SP(t) = E(1_{\tau>t})$
  - Will need to be replaced by:
  - $SP_Q(t) = \frac{E(X_t 1_{\tau>t})}{E(X_t)}$, where X_t is the EUR/USD exchange rate.

Model building steps:

- What are the market observables that I am looking to fit?
- Determine the price for the plain vanilla products:
  - **FX forward**
  - **Vanilla CDS (Credit default swap)**
  - Less important, but still there: options on CDS (swaptions).
- Look at how the quanto trade is built from these components? Is there a cross-impact between these? E.g., simply buying a CDS and an FX forward in a portfolio will not have any cross-currency impact, unless they are connected somehow (e.g., by multiplication, some non-linear operation).

My first try: Assume FX and Credit are uncorrelated.

$$SP_Q(t) = \frac{E(X_t 1_{\tau>t})}{E(X_t)} = E(1_{\tau>t}) = SP(t)$$

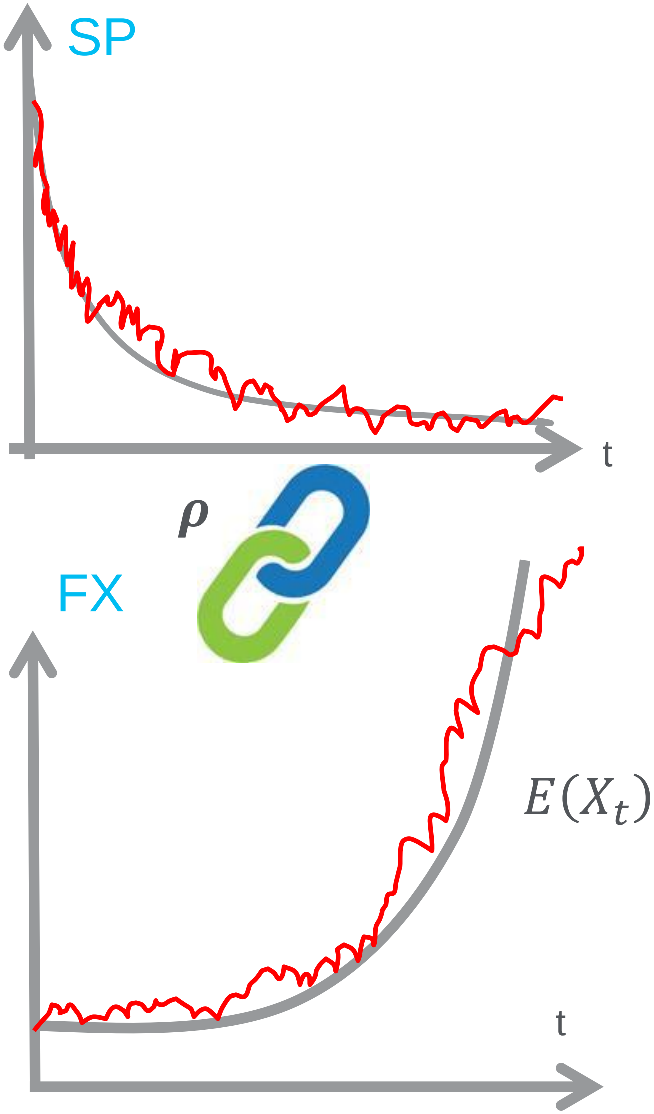
- I am getting the same survival probability, so I am out of luck here! I have to find something more complicated.

Why is the uncorrelated assumption not realistic? Why is it not working?

- If the Greek economy is not doing that well ⇔ EUR is weaker versus USD ⇔ I must build a model that accounts for correlation between credit quantity and strength of a currency.

What impact will a Greek default have on the Euro?

Continuous model for credit and FX:



Make survival probability stochastic:

$$SP(t) = e^{-\int_0^t h(s)ds}$$

$$dh = \kappa(h_*(t) - h(t))dt + \sigma_h dW_h$$

(h_* is chosen to fit the survival curve in expectation)

EUR/USD forward stochastic:

- $dX_t = (r_{USD} - r_{EUR})X_t dt + X_t \sigma_{FX} dW_{FX}$

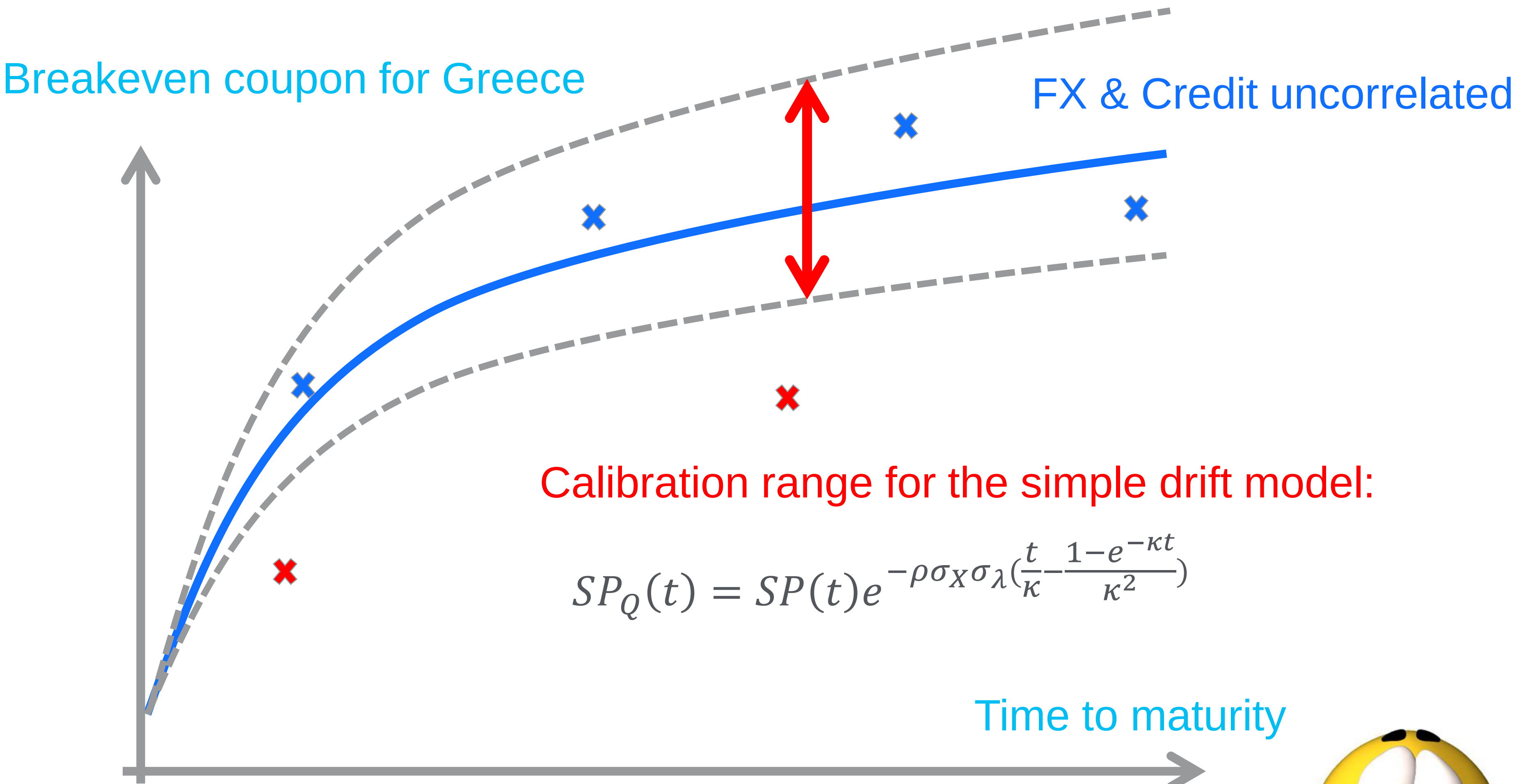
Correlate these two models:

- $dW_{FX} dW_h = \rho dt$

This can be solved exactly!

$$SP_Q(t) = SP(t)e^{-\rho\sigma_X\sigma_h\left(\frac{t}{\kappa} - \frac{1-e^{-\kappa t}}{\kappa^2}\right)}$$

Model spreads vs. Market observations for the diffusion model

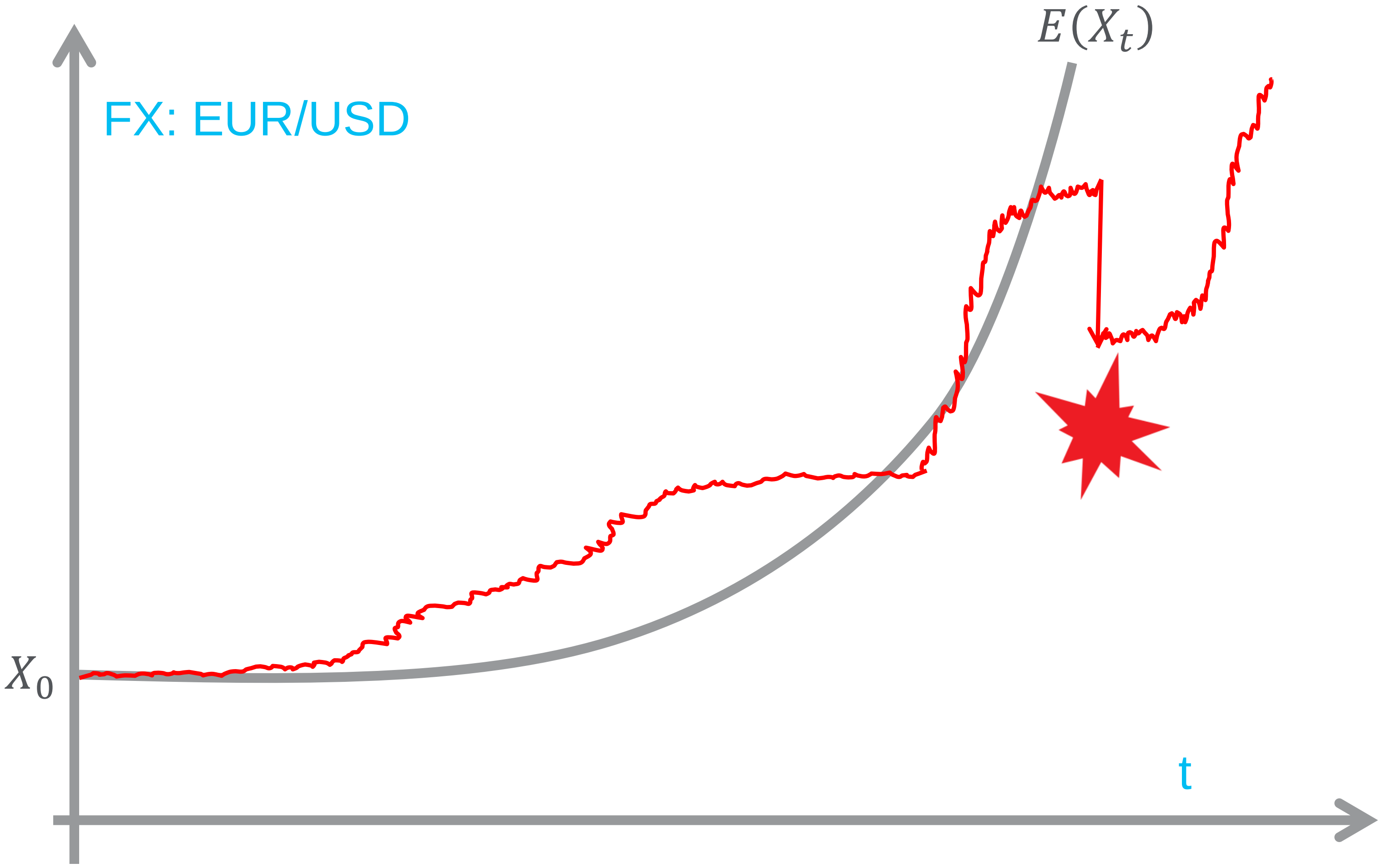


$$SP_Q(t) = SP(t)e^{-\rho\sigma_X\sigma_\lambda\left(\frac{t}{\kappa}-\frac{1-e^{-\kappa t}}{\kappa^2}\right)}$$

✖ :market observables we cannot fit

We need to build stronger connection between credit and FX.

- Try adding jumps to FX, so that if a default happens the currency loses a percentage its value instantaneously.



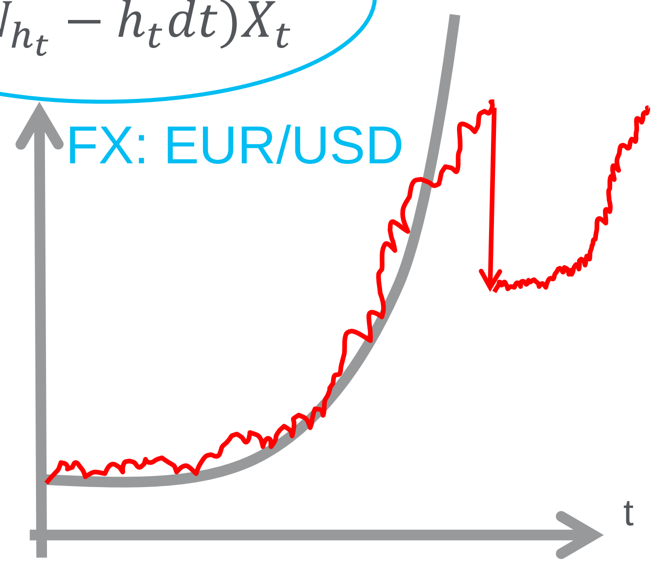
Building a Jump-diffusion model

- Add a jump to my FX process so that the expectation of forwards will remain the same - Poisson-process with a compensator:

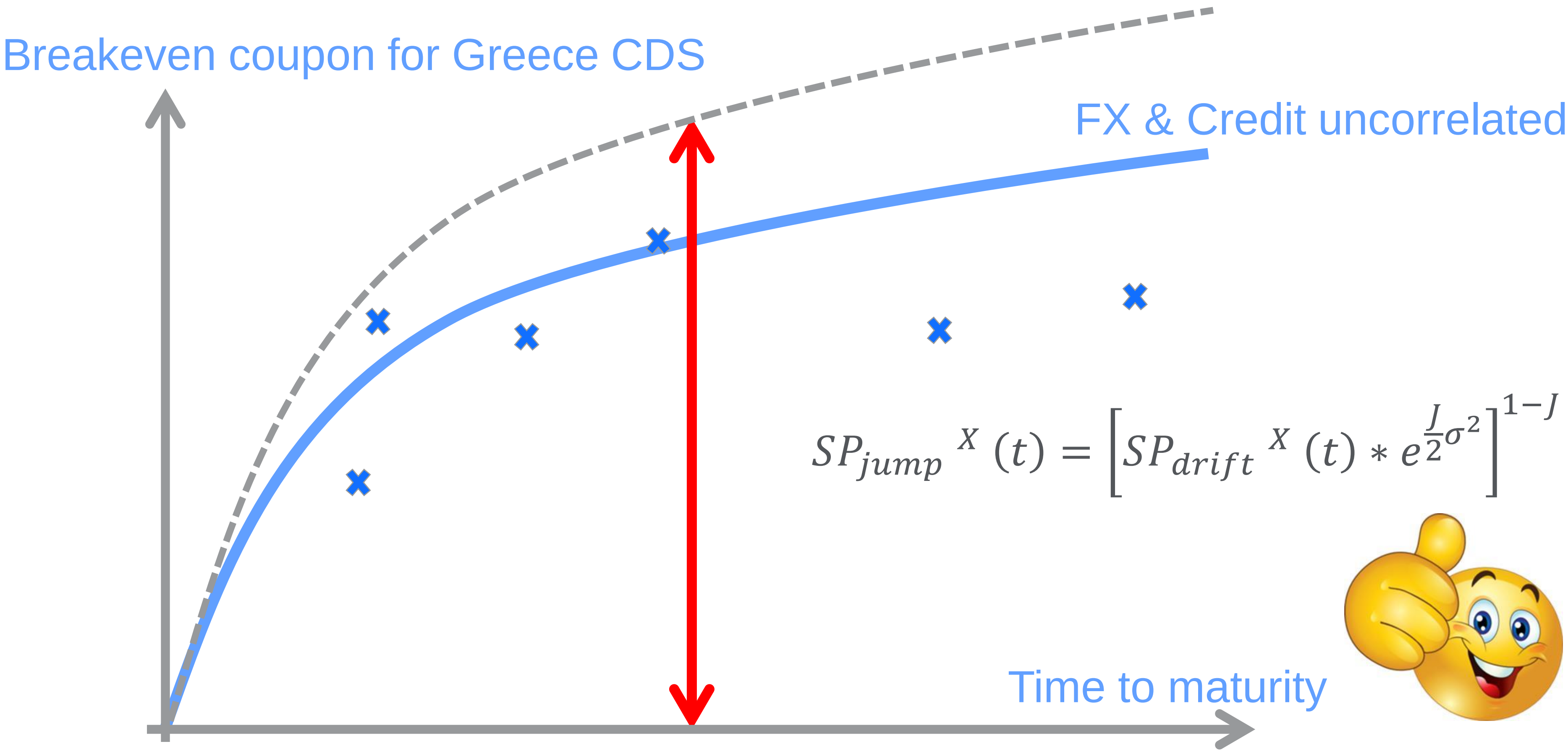
$$dN_{h_t} - h_t dt$$

- My old FX model:
  - $dX_t = (r_{USD} - r_{EUR})X_t dt + \sigma_{FX} dW_{FX}$
- Will change to:
- $dX_t = (r_{USD} - r_{EUR})X_t dt + X_t \sigma_{FX} dW_{FX} + J(dN_{h_t} - h_t dt)X_t$
- N_{h_t} Poisson-process
- This is still solvable:

$$SP_{jump}{}^{X}(t) = \left[SP_{drift}{}^{X}(t) * e^{\frac{J}{2}\sigma^2}\right]^{1-J}$$



Jump-diffusion model vs. market observables



$$SP_{jump}{}^{X}(t) = \left[SP_{drift}{}^{X}(t) * e^{\frac{J}{2}\sigma^2}\right]^{1-J}$$

✖ :market observables

Calibration range for the jump-diffusion model:

Summary and references

There is an extensive literature on credit derivatives, the selection below is only a small fraction of what is available.

Summary:

- Introducing credit default swaps.
- Motivation for quanto trades.
- How to price quantos with diffusion/jump-diffusion models.

References:

- Valuation of Credit Contingent Options with Applications to Quanto CDS, *Anlong Li*
- Credit Derivatives Pricing Models: Models, Pricing and Implementation, *Philipp J. Schonbucher*

- **Thank you!**