Department of Hydrodynamic Systems

On the sensitivity of jets and shear layers or the solution of the Orr-Sommerfeld equation by the compound matrix method (CMM)

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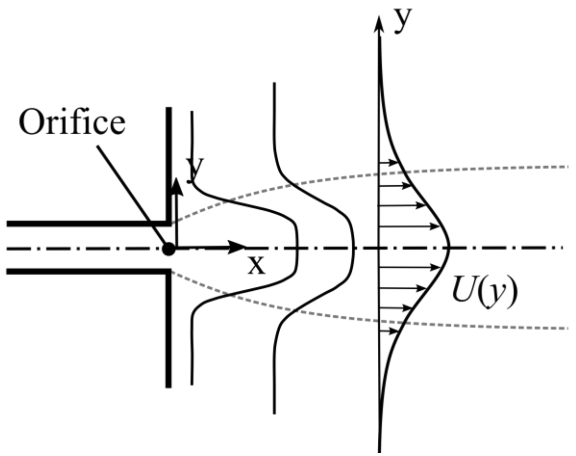
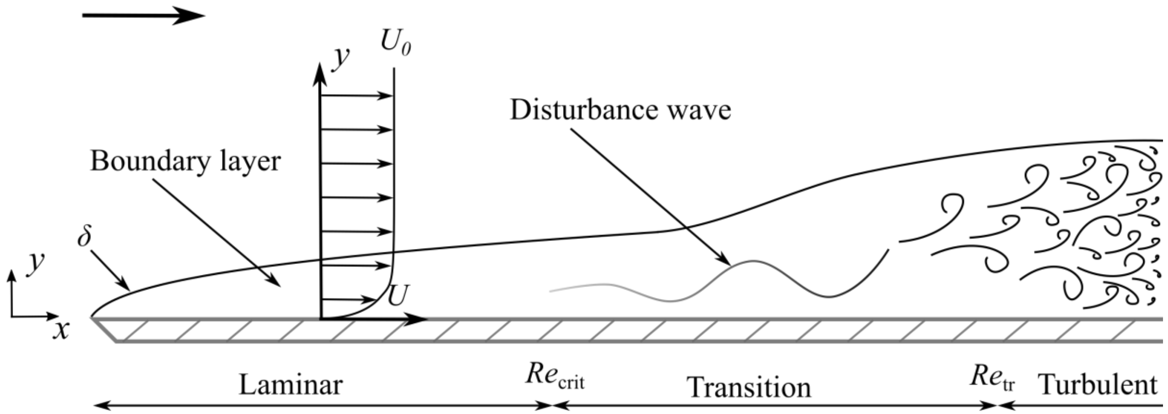
Motivation 1.

Determine the stability properties of self-similar flows



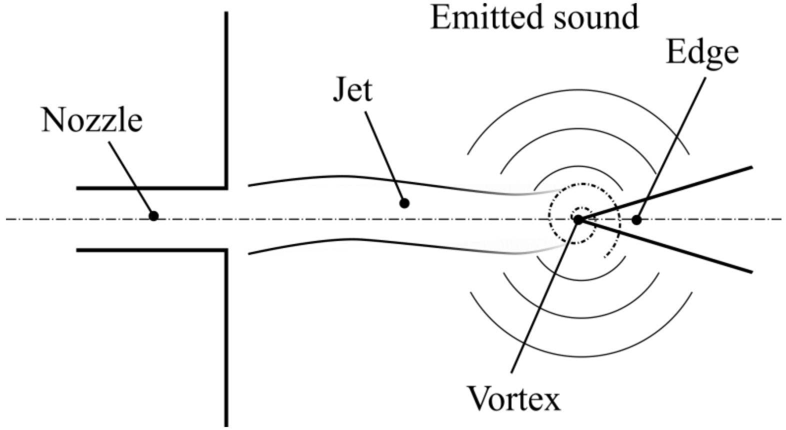


Jets **Boundary layers**

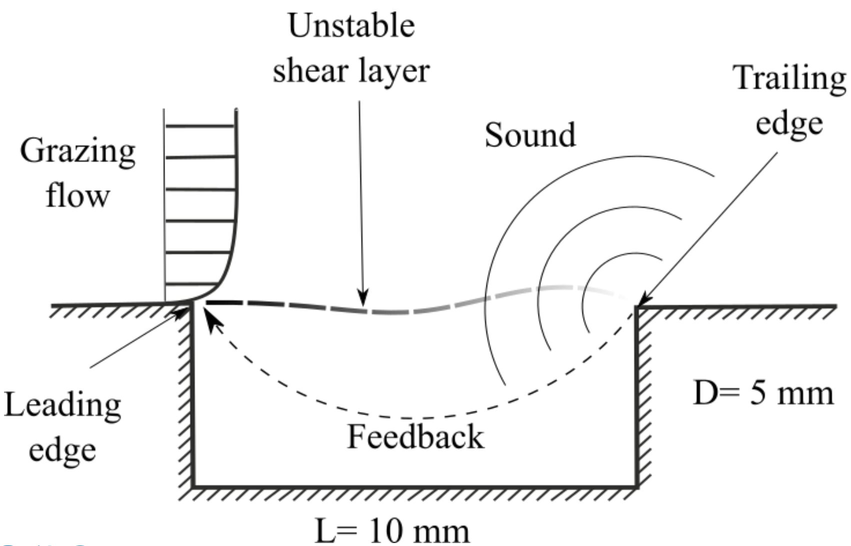
Motivation 2.

Modelling self-excited flows

Edge tone



Cavity tone

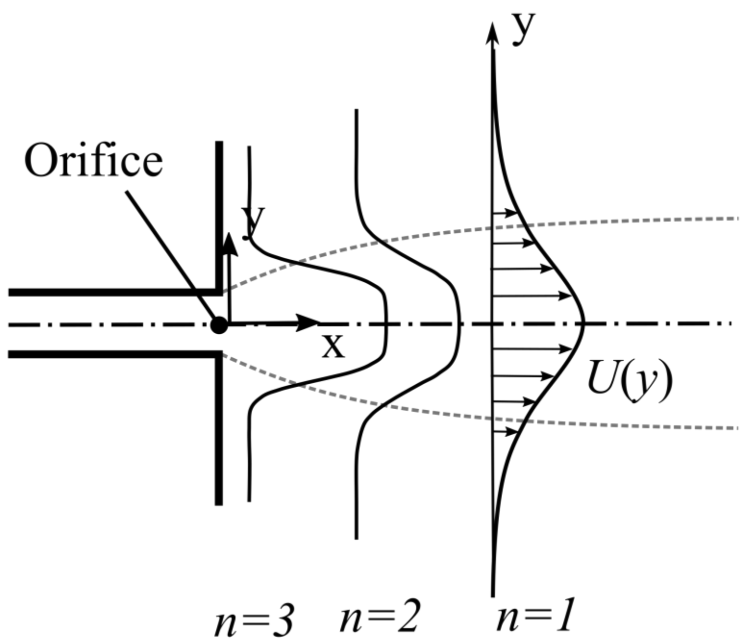


Applications

Organ pipes, flue instruments
Y branches in pipes

car door gaps
pantograph of trains
weapon bays on aircrafts

Analytical approximation of the jet velocity profile



Plane jet

$$U(y) = \text{sech}^2(y^n)$$

n=1: - the self-similar profile in the far field
- non-dimensional Bickley profile

The perturbed NS equations

The non-dimensional Navier-Stokes equations (conservation of mass and momentum):

$$\nabla \cdot \underline{u}_t = 0 \qquad \frac{\partial \underline{u}_t}{\partial t} + \underline{u}_t \cdot \nabla \underline{u_t} = -\nabla p_t + \frac{1}{Re} \Delta \underline{u}_t$$

Apply perturbation , $\underline{u}_p, p_p$

$$\underline{u}_t = \underline{U} + \underline{u}_p \qquad p_t = P + p_p$$

If $\underline{U}$, P are known, time-independent variables, describe the base flow, fulfil the governing equations. $\underline{u}_p = (u_p, v_p, w_p)$

If U(y) is independent of x (parallel flow, or self-similar flow), the perturbation can be assumed in a complex wave form:

$$v_p = v(y)\ e^{i(\alpha x - \omega t)}$$

α: **wavenumber,** ω: **angular frequency**

Temporal and spatial instability

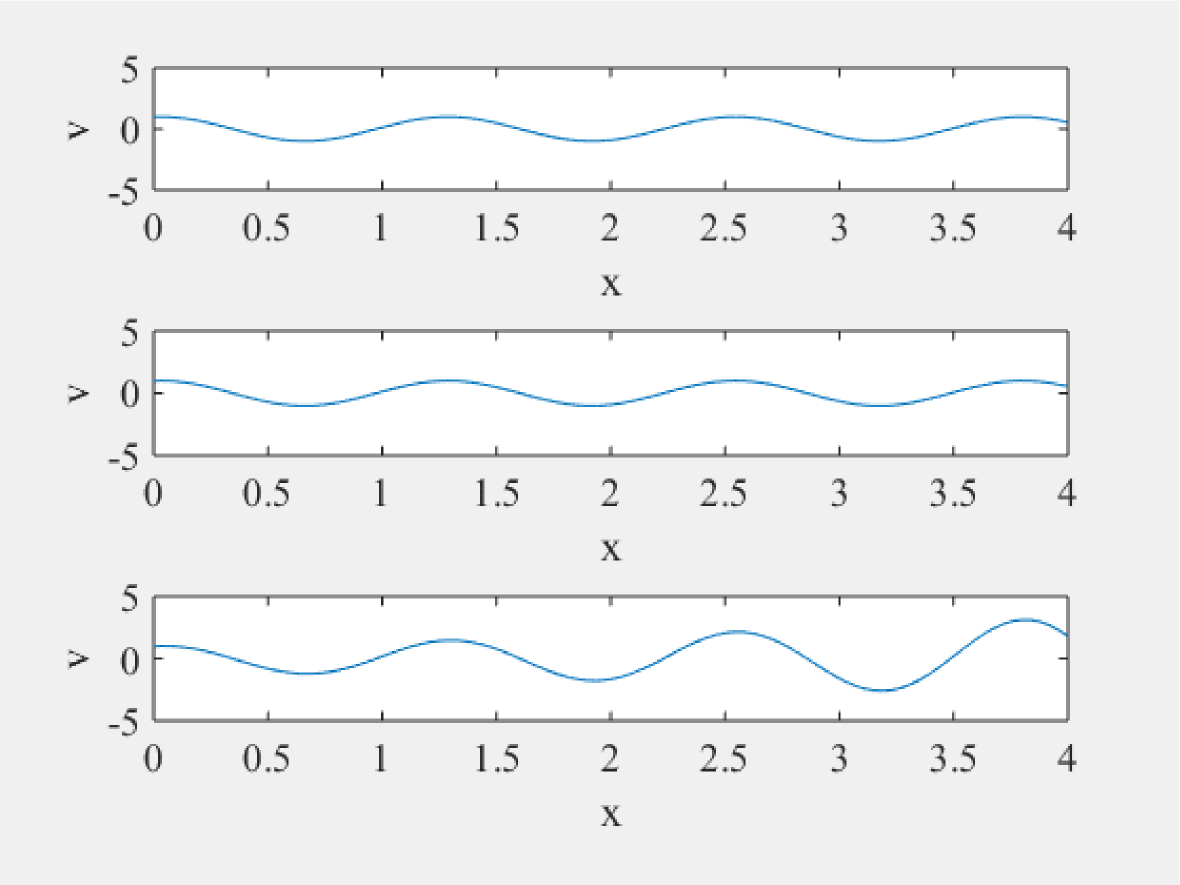
μ : **growth rate**

$v_p = v(y)\ e^{i(\alpha x - \omega t)}$

$\omega = 5$
$\alpha = 5$

$\omega = 5 + 0.3i$
$\alpha = 5$
$\mu_t = Im(\omega)$

$\omega = 5$
$\alpha = 5 - 0.3i$
$\mu_x = -Im(\alpha)$



Here, spatial stability investigation: $\mu = \mu_x = -Im(\alpha)$

The Orr-Sommerfeld equation

Substituting the perturbation into the NS equation, linearising and simplifying, the equation system can be expressed in terms of *v*, leading to a fourth order linear ordinary differential equation:

$$v^{iv} - 2\alpha^2 v'' + \alpha^4 v = i\,Re\,[(\alpha U - \omega)[v'' - \alpha^2 v] - \alpha U'' v]$$

$$v_p = v(y)\;\; e^{i(\alpha x - \omega t)}$$

$U(y), U''(y)$ **describe the flow (known)**
Re **is a parameter (known)**
$v(y)$ **is the eigenfunction (unknown)**
α, ω **are the eigenvalue pairs (one is a parameter, the other one is unknown)**

The OS equation in the far field

The Orr-Sommerfeld equation:

$$v^{iv} - 2\alpha^2 v'' + \alpha^4 v = \mathrm{i}\, Re\, [(\alpha U - \omega)[v'' - \alpha^2 v] - \alpha U'' v]$$

In the far field ($y \to \infty$) it becomes a constant coefficient linear differential equation because $U(y)$ is a constant:

$$U_\infty = \lim_{y\to\infty} U(y)\,, \qquad \lim_{y\to\infty} U''(y) = 0$$

$$\tilde{v}^{iv} - 2\alpha^2 \tilde{v}'' + \alpha^4 \tilde{v} = i\, Re\, [(\alpha U_\infty - \omega)[\tilde{v}'' - \alpha^2 \tilde{v}]]$$

Analytical solution in the far-field:

$$\tilde{v}(\mathrm{y}) = \sum_{i=1}^{4} \tilde{k}_i\, \tilde{v}^{(i)}(y) = \sum_{i=1}^{4} \tilde{k}_i\, e^{\lambda_i y}$$

$$\underline{\lambda} = [-\alpha, -Q, \alpha, Q]^{\mathrm{T}}$$

$$Q := \sqrt{\alpha^2 - \mathrm{i}Re(\alpha U_\infty - \omega)}$$

The OS equation in the far field 2.

Analytical solution in the far field:

$$\tilde{v}(\mathrm{y}) = \sum_{i=1}^{4} \tilde{k}_i \, \tilde{v}^{(i)}(y) = \sum_{i=1}^{4} \tilde{k}_i \, e^{\lambda_i y}$$

$$\underline{\lambda} = [-\alpha, -Q, \alpha, Q]^{\mathrm{T}}$$

$$Q := \sqrt{\alpha^2 - \mathrm{i}Re(\alpha U_\infty - \omega)}$$

If $Re \gg 1$ **then** $|Q| \gg |\alpha|$ **the differential equation becomes stiff.**

Special method: the Compound Matrix Method

Transformation of variables

In the far field the perturbation vanishes $v\,(y\to\infty) = v'(y\to\infty) = 0$

$$\tilde{v}(\mathrm{y}) = \sum_{i=1}^{4} \tilde{k}_i\, e^{\lambda_i y} \qquad \underline{\lambda} = [-\alpha, -Q, \alpha, Q]^{\mathrm{T}} \longrightarrow \tilde{k}_3 = \tilde{k}_4 = 0$$

The „true" solution reduces to

$$v(y) = \sum_{i=1}^{2} k_i\, v^{(i)}(y)$$

In the far field: $v^{(i)}(y\to\infty) = \tilde{v}^{(i)}(y)$

Let us transcribe the 4th order diff. eq. into a first order diff. eq. system:

$$\underline{\phi} = [v, v', v'', v''']$$

Transformation of variables 2.

$$\underline{\phi} = [v, v', v'', v'''] = [\phi_1, \phi_2, \phi_3, \phi_4]$$

Use the so-called compound matrix variables:

$$\eta_1 = \det\begin{vmatrix}\phi_1^{(1)} & \phi_1^{(2)} \\ \phi_2^{(1)} & \phi_2^{(2)}\end{vmatrix} = \phi_1^{(1)}\phi_2^{(2)} - \phi_1^{(2)}\,\phi_2^{(1)} = \; v^{(1)}\,v'^{(2)} - \; v^{(2)}\,v'^{(1)} \ldots$$

$$\eta_2 = \det\begin{vmatrix}\phi_1^{(1)} & \phi_1^{(2)} \\ \phi_3^{(1)} & \phi_3^{(2)}\end{vmatrix},\ \eta_3 = \det\begin{vmatrix}\phi_1^{(1)} & \phi_1^{(2)} \\ \phi_4^{(1)} & \phi_4^{(2)}\end{vmatrix},\ \eta_4 = \det\begin{vmatrix}\phi_2^{(1)} & \phi_2^{(2)} \\ \phi_3^{(1)} & \phi_3^{(2)}\end{vmatrix}, \ldots, \eta_6$$

In the far field:

$$v^{(1)}(y) \sim e^{-\alpha y} \qquad \longleftrightarrow \qquad \eta_1 = \; v^{(1)}\,v'^{(2)} - \; v^{(2)}\,v'^{(1)} \; \sim \; -(Q+\alpha)\,e^{-(\alpha+Q)y}$$

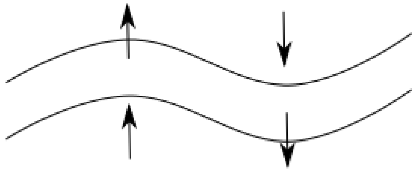
$$v^{(2)}(y) \sim e^{-Qy} \qquad \qquad \eta_2 \sim -\,(Q^2-\alpha^2)e^{-(\alpha+Q)y}, \ldots, \eta_6 \sim e^{-(\alpha+Q)y}$$

The problem of stiffness is solved! Furthermore, by introducing the $\underline{\tilde{\eta}} = \frac{\underline{\eta}}{-(Q+\alpha)e^{-(\alpha+Q)y}}$ variable, non-homogenous bc.s can be prescribed at „infinity"

Boundary conditions in the near field

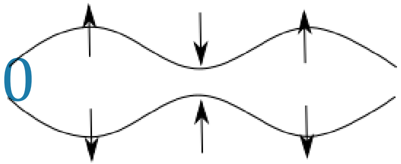
The far field boundary conditions ($v\,(y \to \infty) = v'(y \to \infty) = 0$) are fulfilled after transformation. Two other boundary conditions are necessary. Near field boundary conditions

Symmetric profile $v'(0) = v'''(0) = 0 \to \tilde{\eta}_5(0) = 0$



Jet

Antisymmetric profile $v(0) = v''(0) = 0 \to \tilde{\eta}_2(0) = 0$



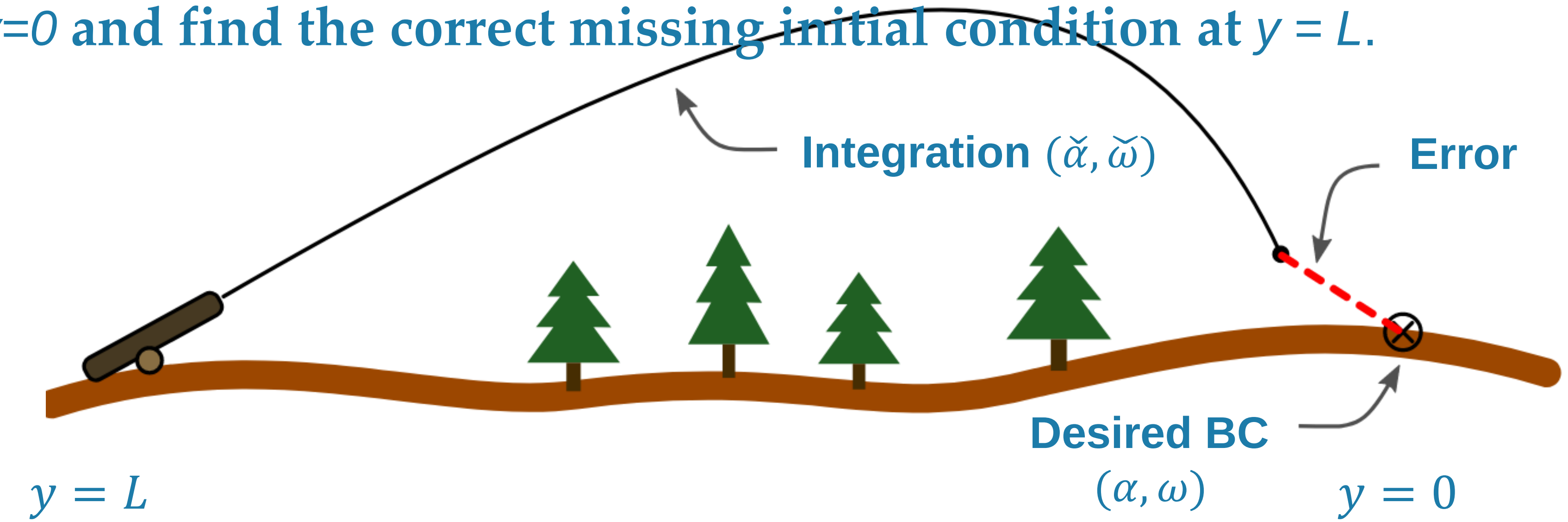
Wall $v(0) = v'(0) = 0 \to \tilde{\eta}_1(0) = 0$

Shooting method

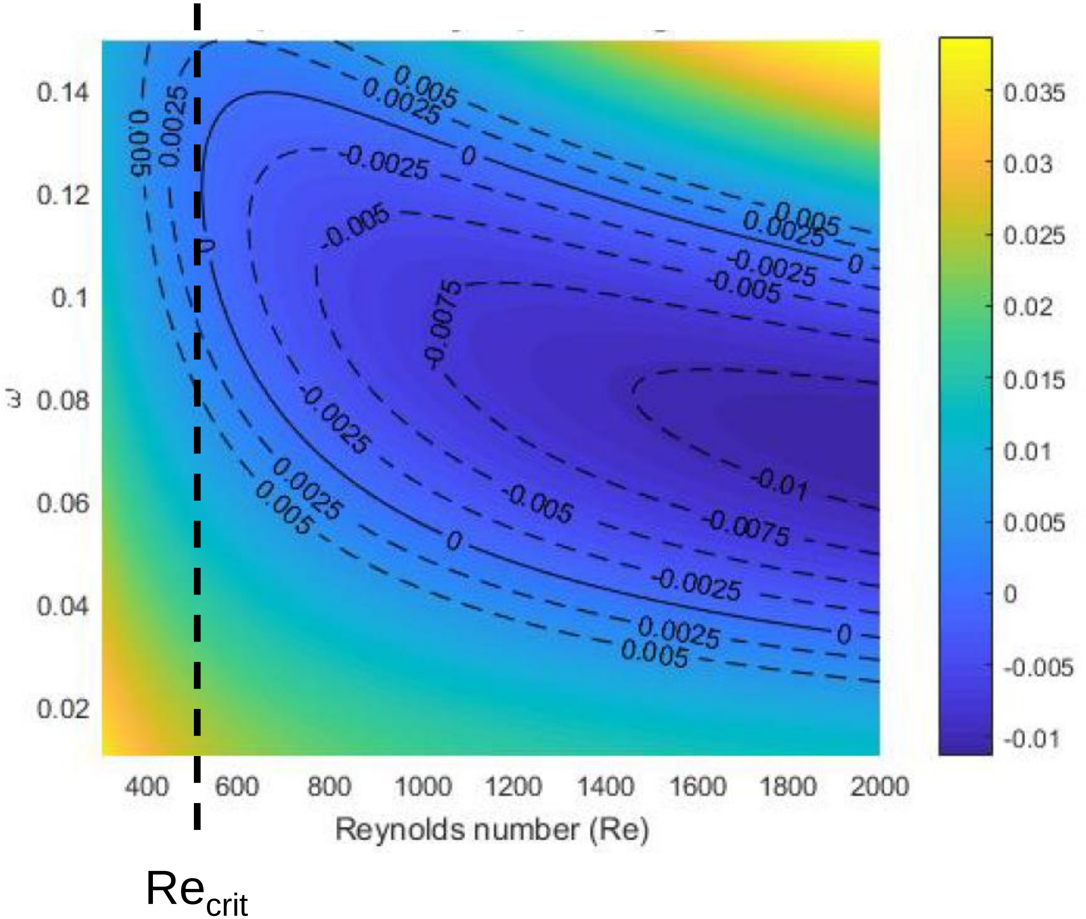
The boundary value problem should be solved as an initial value problem. The solution can be initialised at a point ($y_0 = \mathrm{L}$), where the velocity is approximately constant $U\,(y \to \infty) \approx U(L)$

The integration should start from there towards $y = 0$.

The α, ω parameters should be tuned to fulfil the boundary conditions at y=0 and find the correct missing initial condition at $y = L$.



Thumb curve

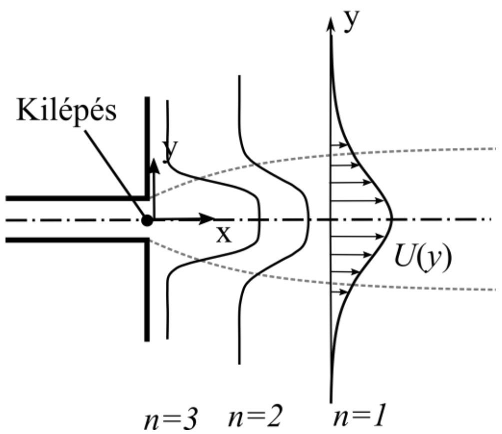


$\mu = Im(\alpha)$

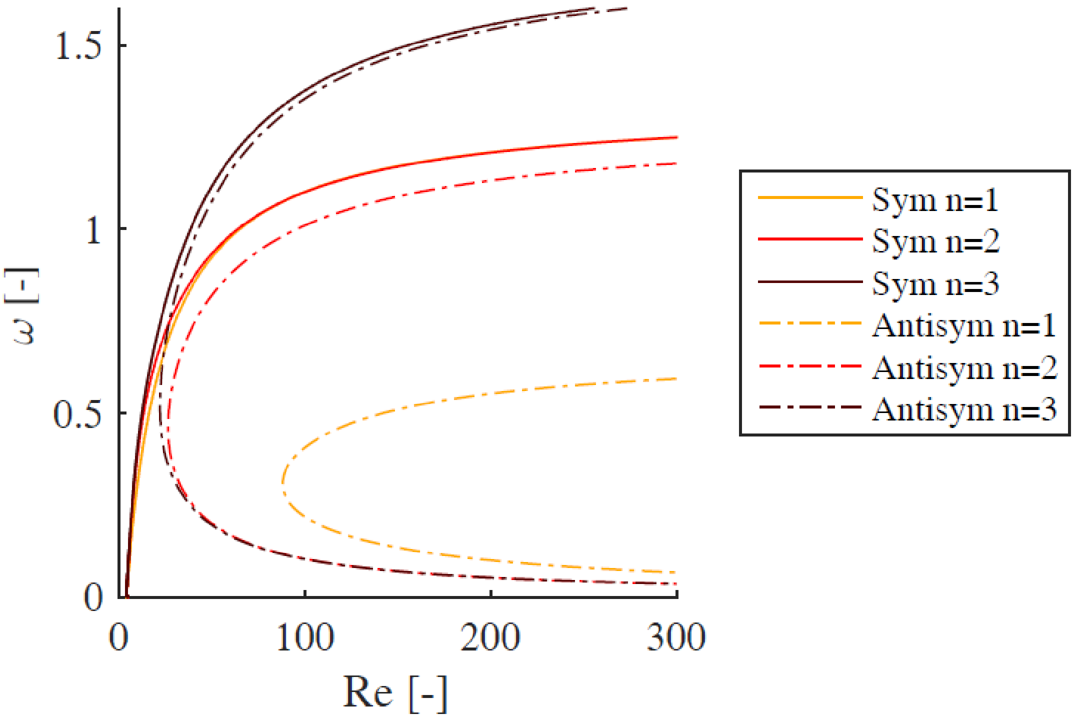
The curves show constant α_i contours. The solid curve separates the stable and the unstable regions. The critical Reynolds number can be read from the curve.

Jet: thumb curves

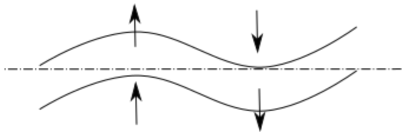
$U(y) = \mathrm{sech}^2(y^n)$



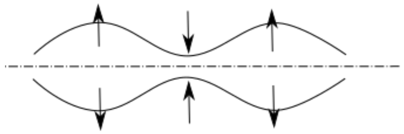
Bickley profile: $n = 1$



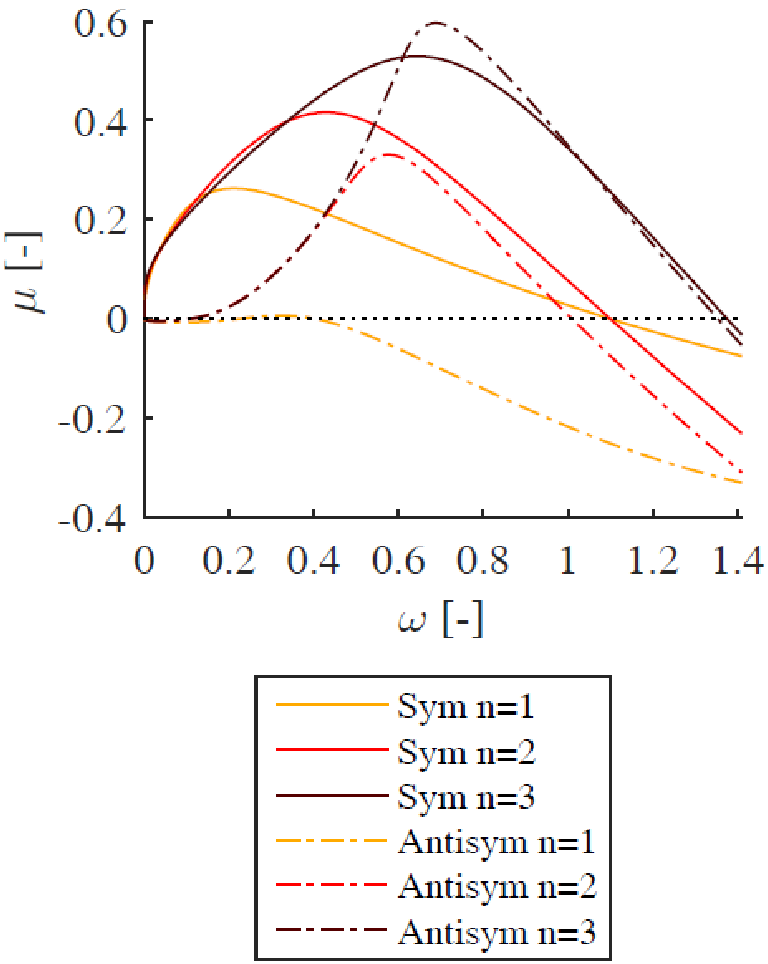
Symmetric:

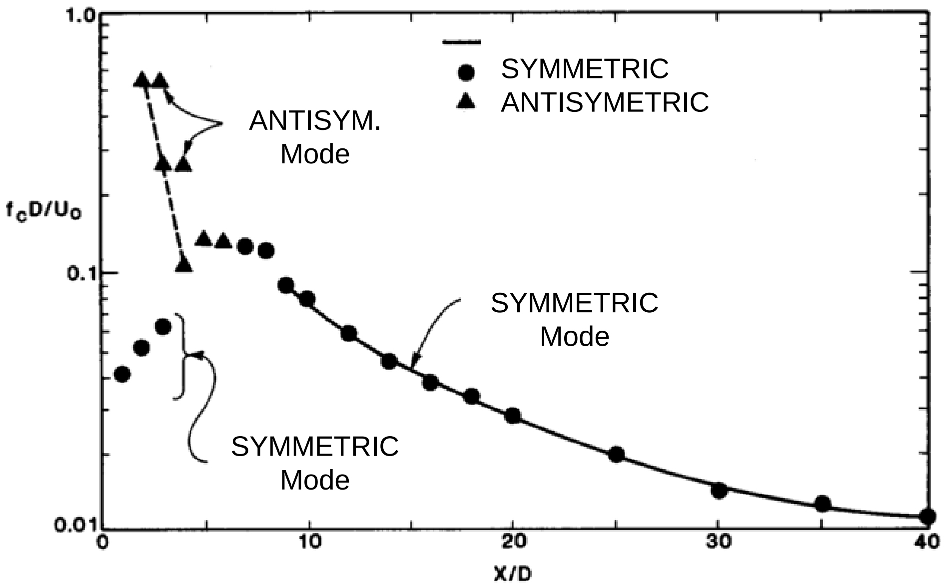


Antisymmetric:



Results: sim.–antisym.





Thomas, F. O., Goldschmidt, V. W., 1985. The possibility of a resonance mechanism in the developing two-dimensional jet. *Phys. Fluids* 28

Numerical velocity profiles I.

The analytical profiles do not describe

- The planar jet with parabolic velocity profile at the orifice
- The effect of the nozzle wall thickness
- The transition of the velocity profiles. The parameter n must be integer.

↓

Calculate the base flow with CFD software. (Computational Fluid Dynamics = numerical flow simulation)

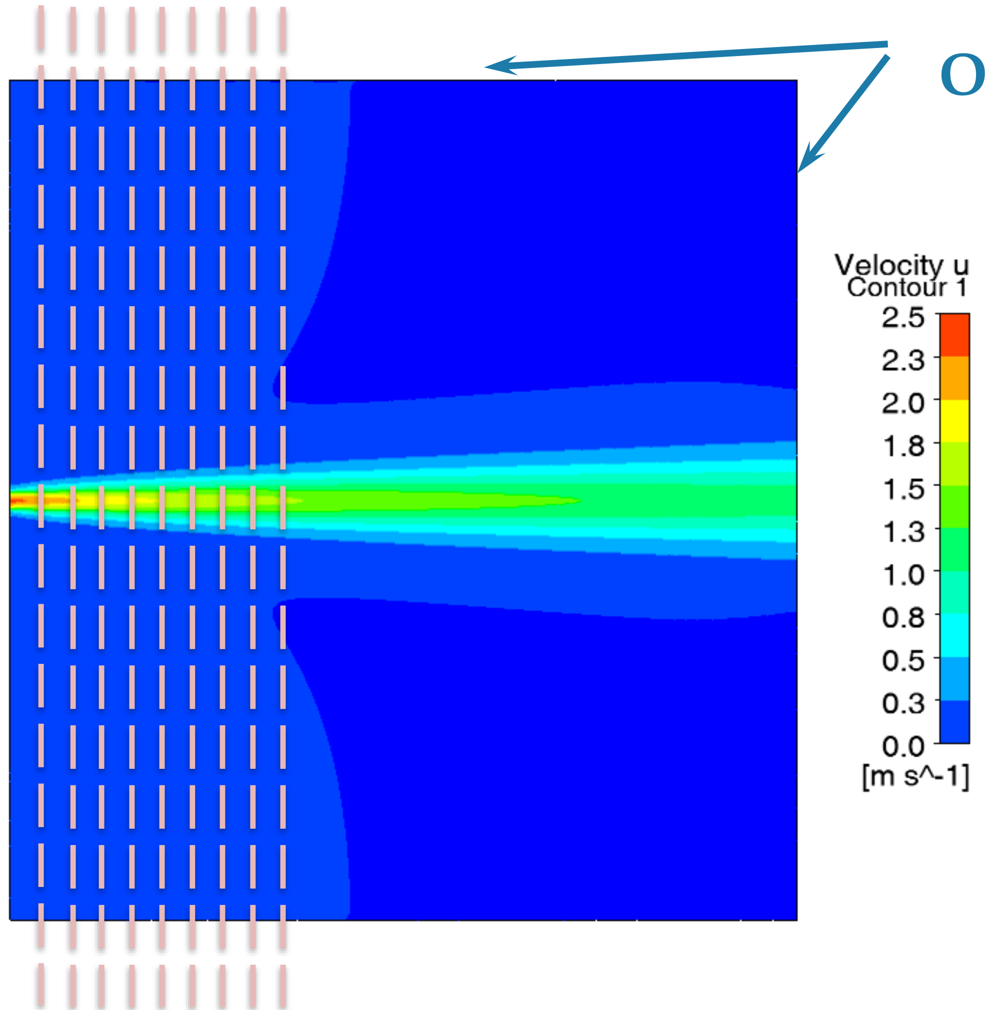
Numerical velocity profiles II.

Wall/opening

Prescribed velocity profile

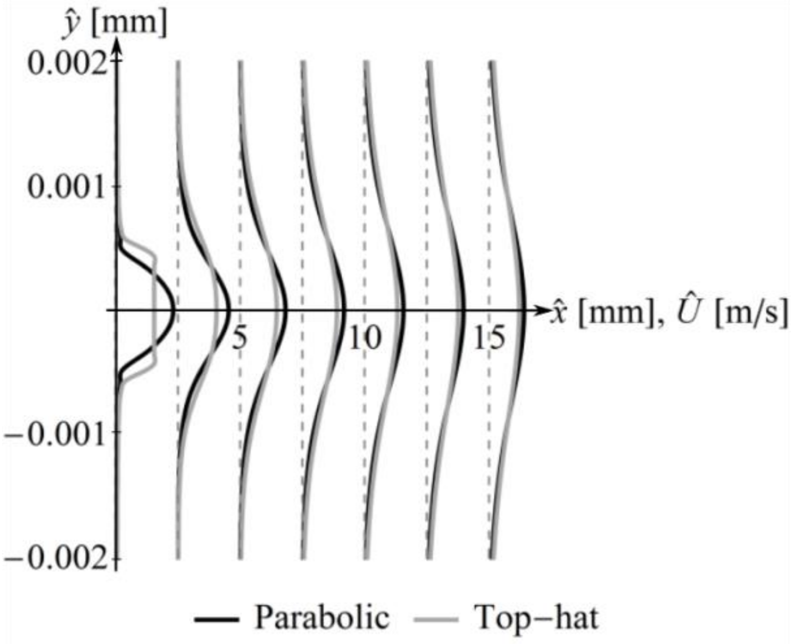
Long, parallel nozzle: parabolic

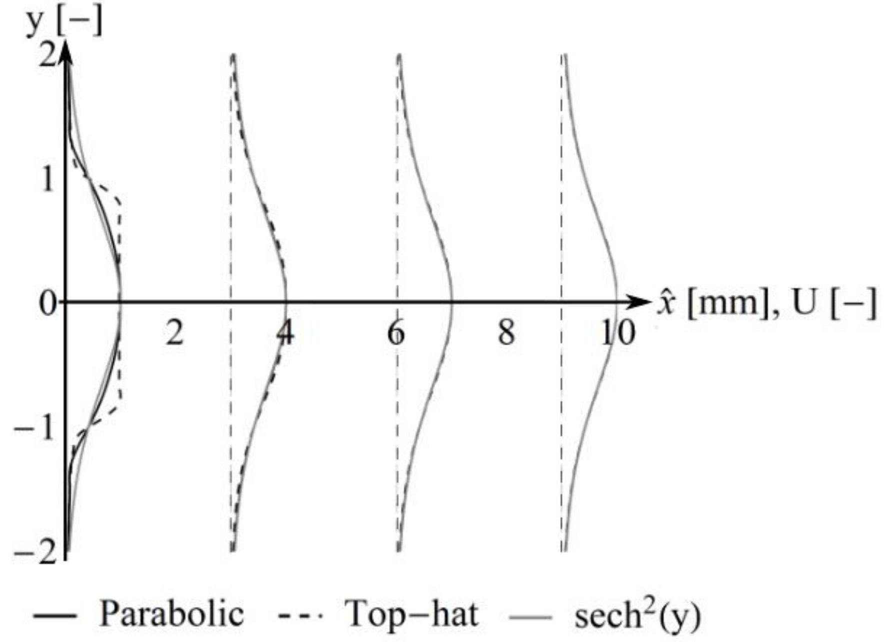
Short, convergent nozzle: „top hat" profile



$U(y) \longrightarrow$ Matlab

Numerical velocity profiles III.





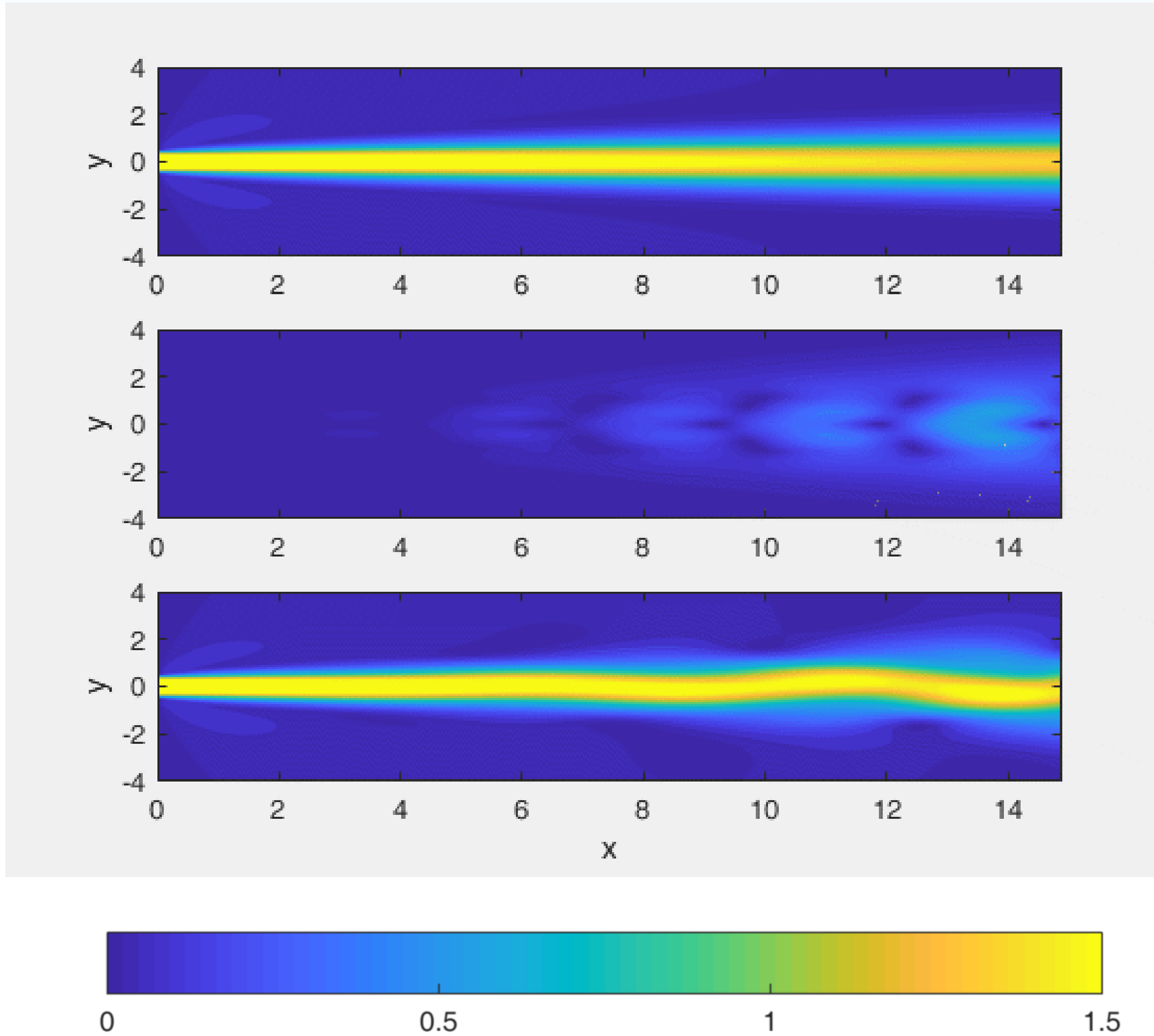
$$\mathrm{Re}(x) = \frac{L(x)\ U_{\mathrm{Max}}(x)}{\nu_{\mathrm{air}}}$$

One result

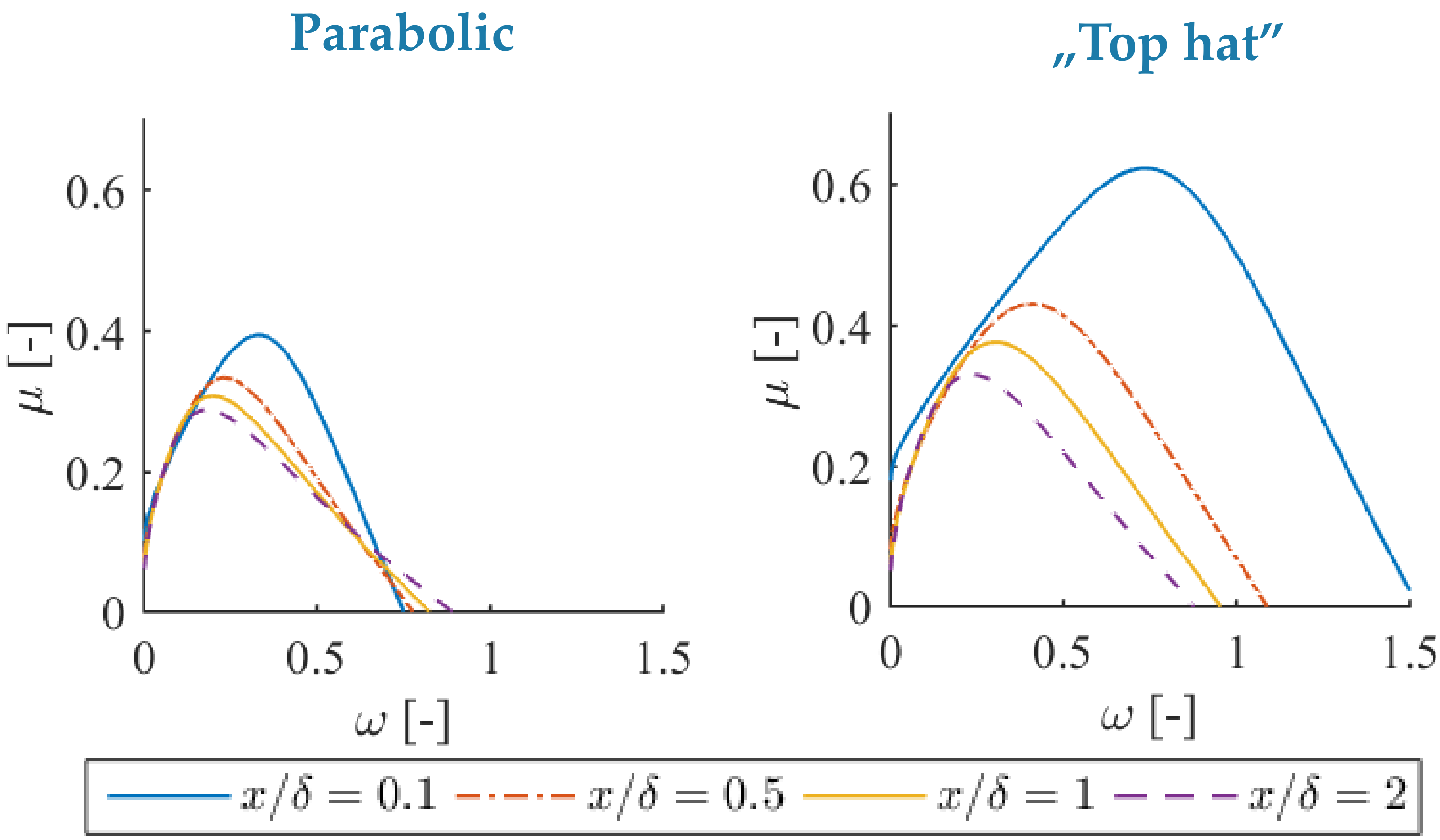
U

$|\underline{u}_p|$

$|\underline{u}_t| = |\underline{U} + \underline{u}_p|$



Growth rates for various velocity profiles

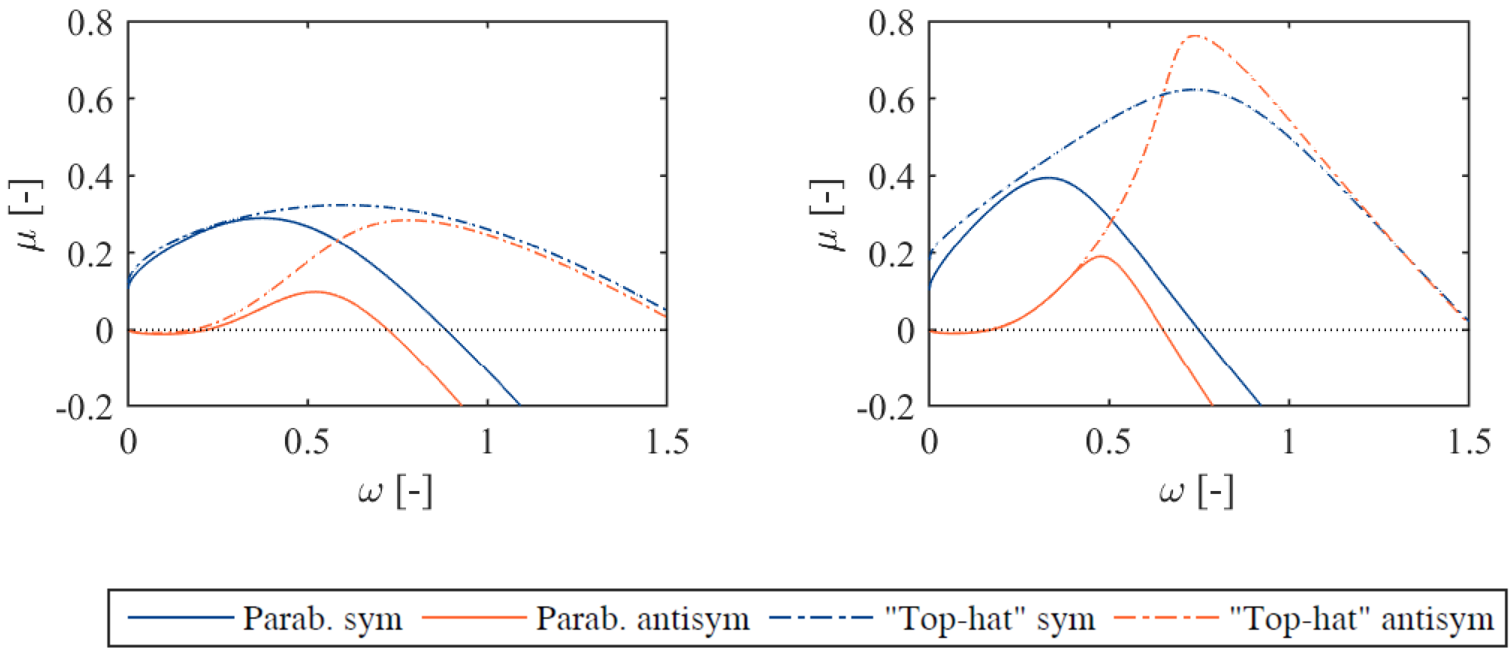


Wall BC, and symmetric perturbation

Symmetric-antisymmetric

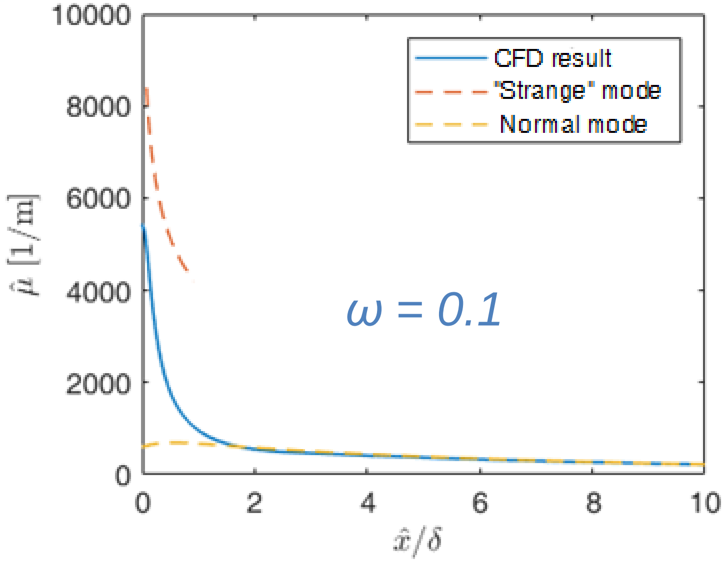
Boundary condition around the orifice

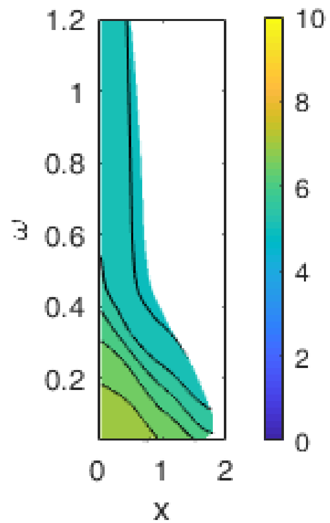
Opening | Wall



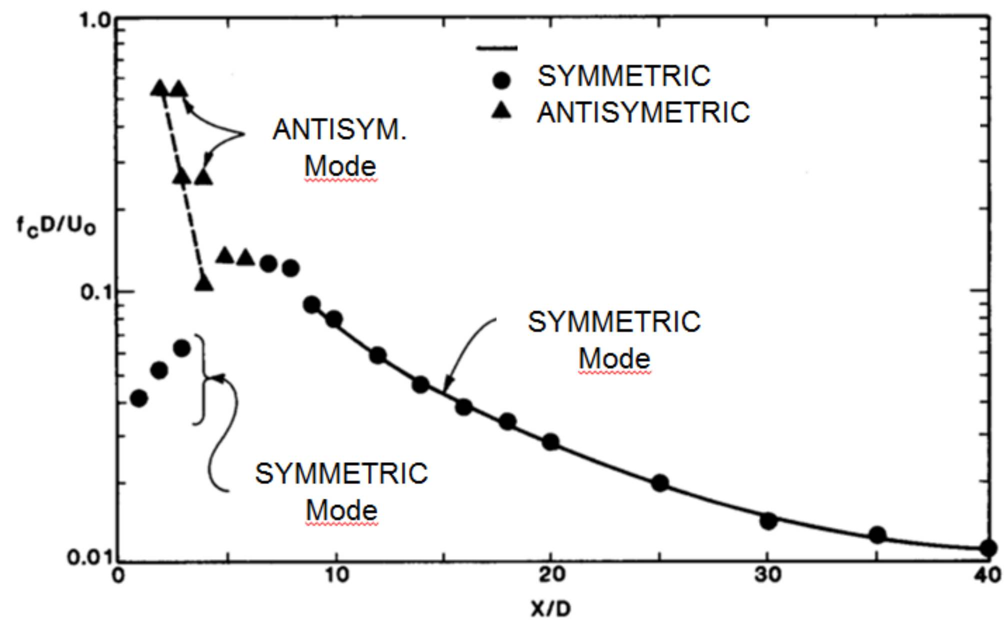
$x/\delta = 0.1$

”Strange” mode





μ



Why is the jet more sensitive at the exit?

There are three reasons for the increased sensitivity of laminar, plane jets close to the orifice.

1., the steeper velocity profile amplifies the disturbances at a higher growth rate.

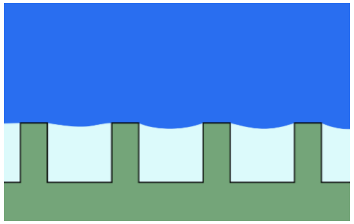
2., the local length scale is smaller close to the orifice, meaning that the dimensional growth rate of disturbances is larger than that in downstream region.

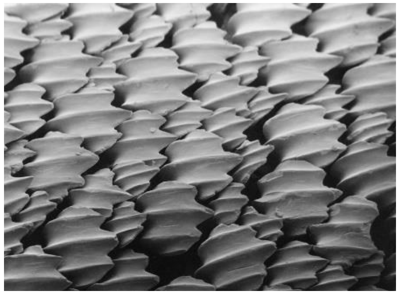
3., the disturbances excited upstream have more space to grow exponentially.

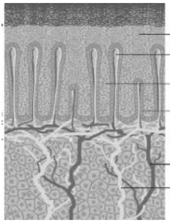
Application for boundary layers: motivation

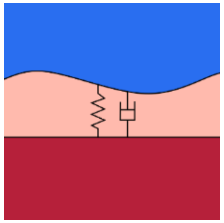
Aim: drag reduction

Examples in nature:

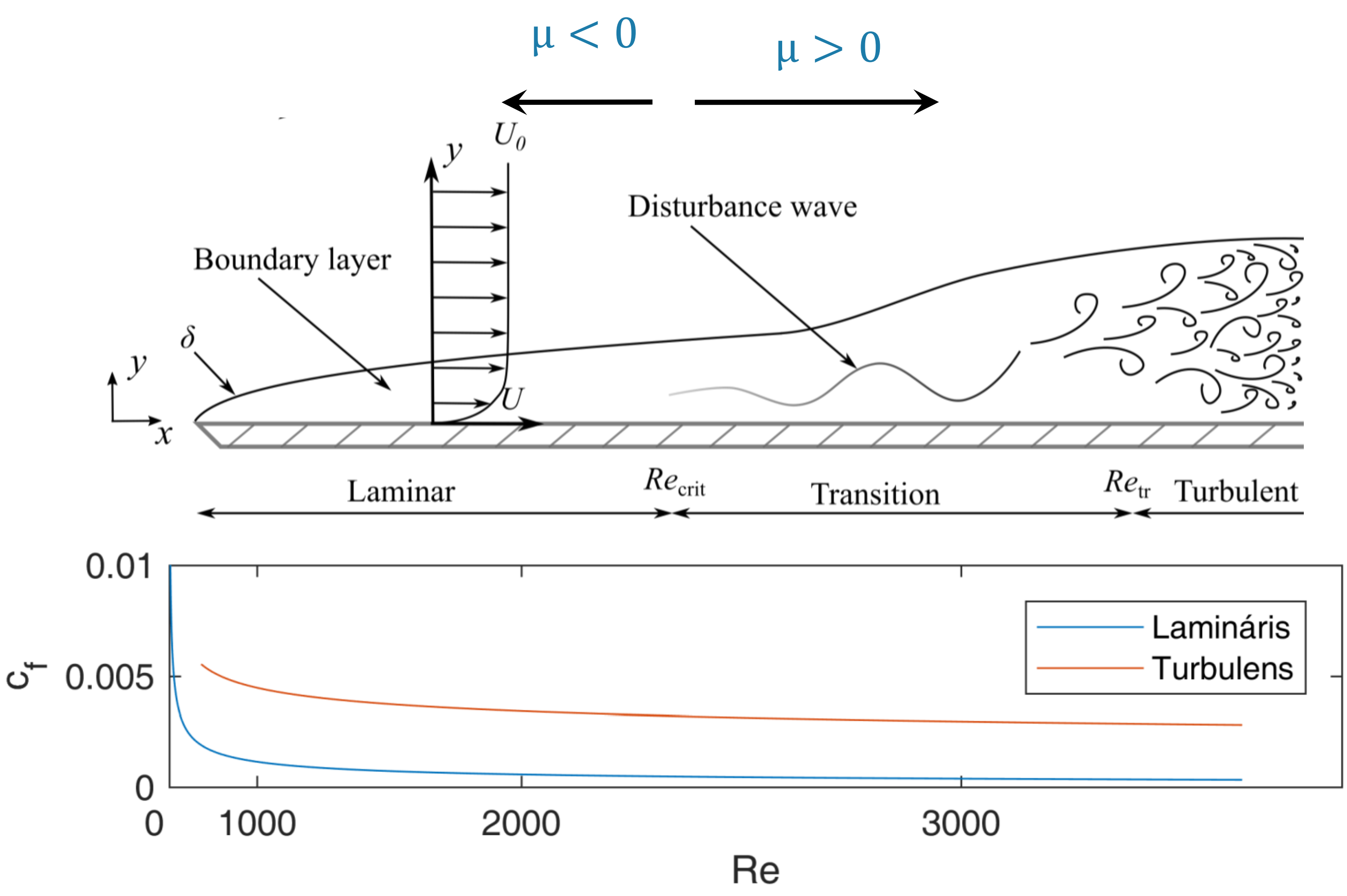








Laminar-turbulent transition I.

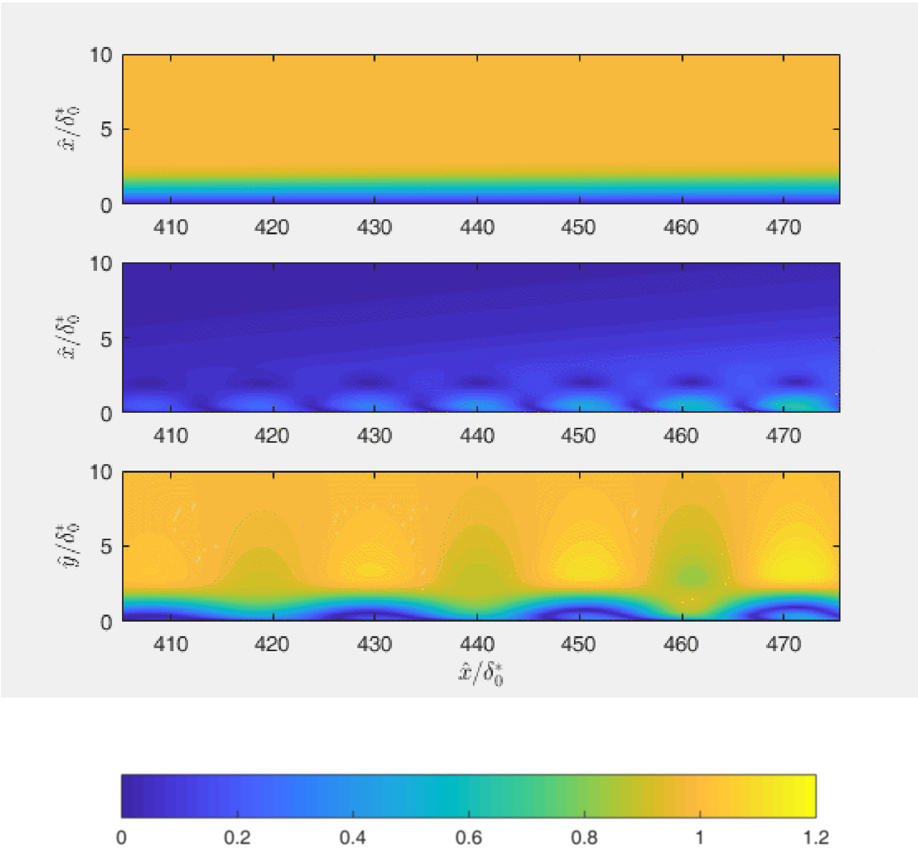


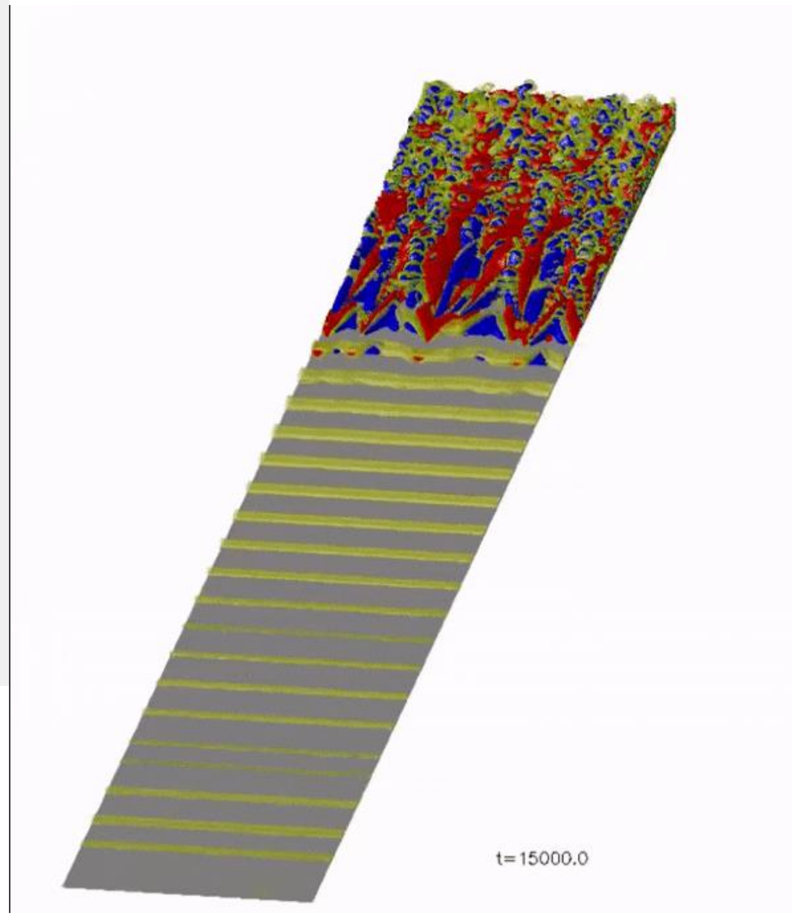
Laminar-turbulent transition II.

U

$|\underline{u}_p|$

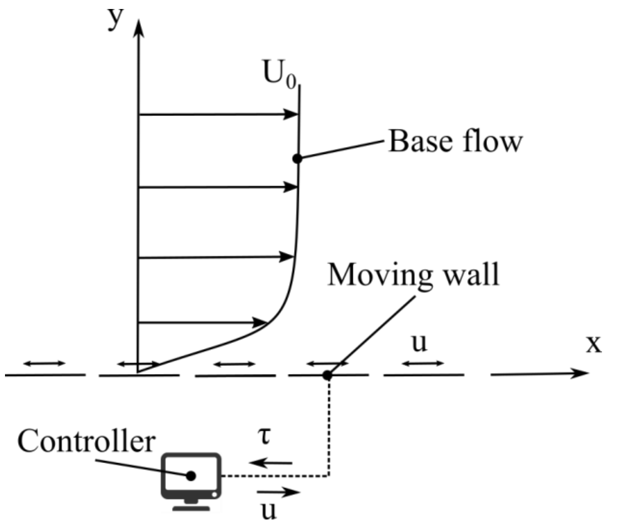
$|\underline{u}_t|$





Forrás: YouTube

Simple controller I.



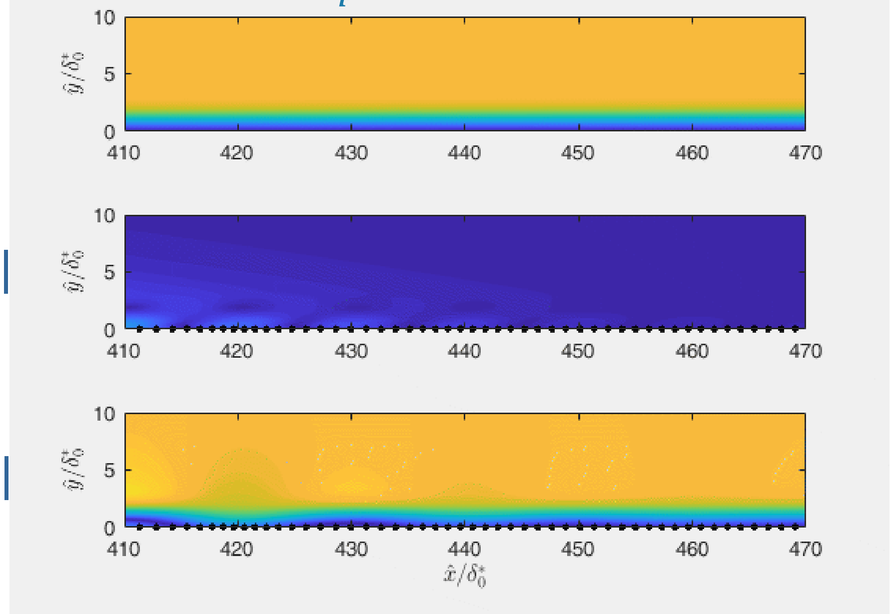
$$u(t) = \tilde{c}_p \tau(t - \Delta t)$$

$c_p = 0.5, \Delta t = 0$

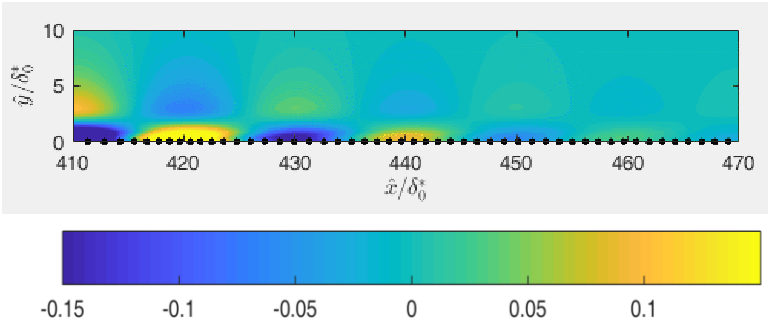
U

$|\underline{u}_p|$

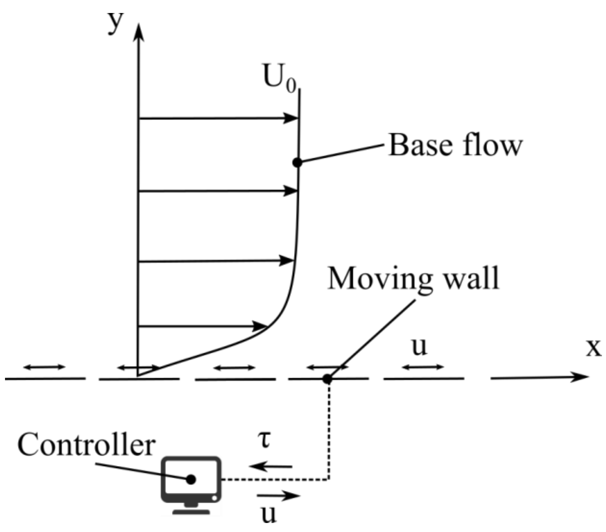
$|\underline{u}_t|$



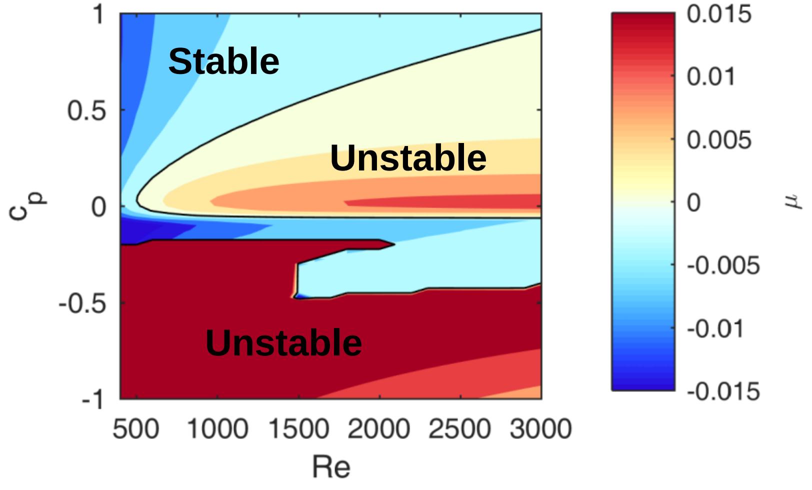
u_p



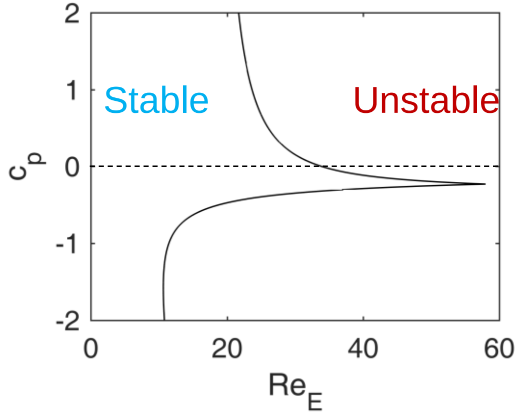
Simple controller II.



Orr-Sommerfeld equation

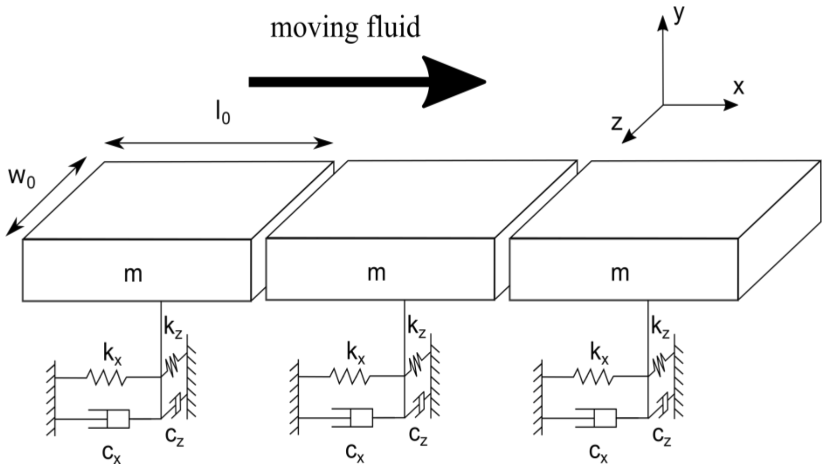


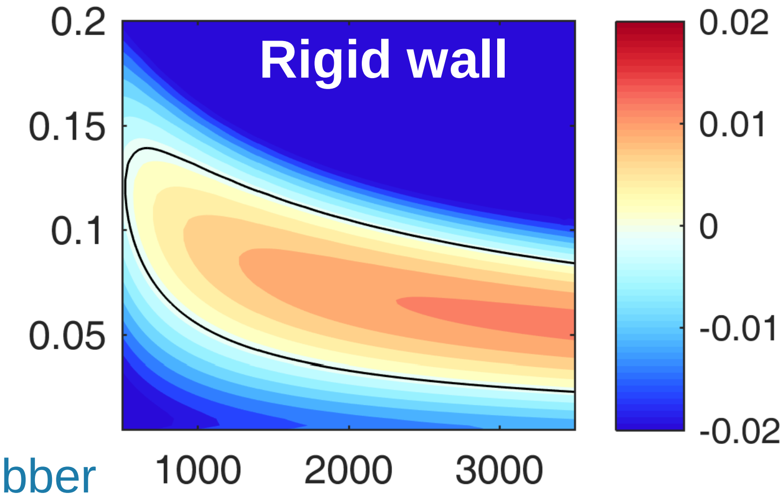
Reynolds-Orr equation

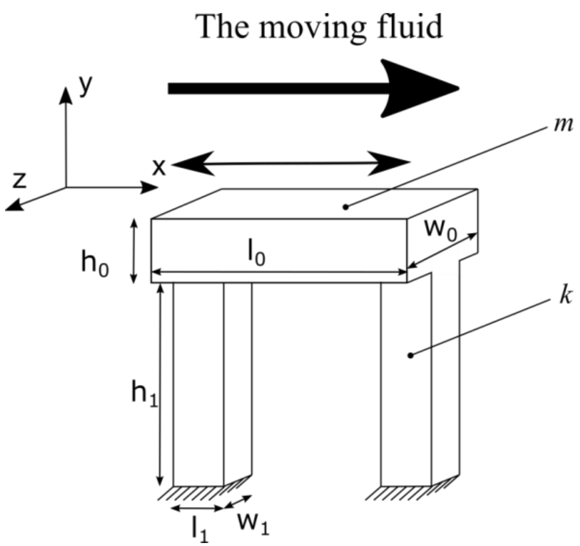


$$u(t) = \tilde{c}_p \tau(t - \Delta t)$$

Passive coating







Material: silicone rubber

$l_1 = 25.5\ \mu m$

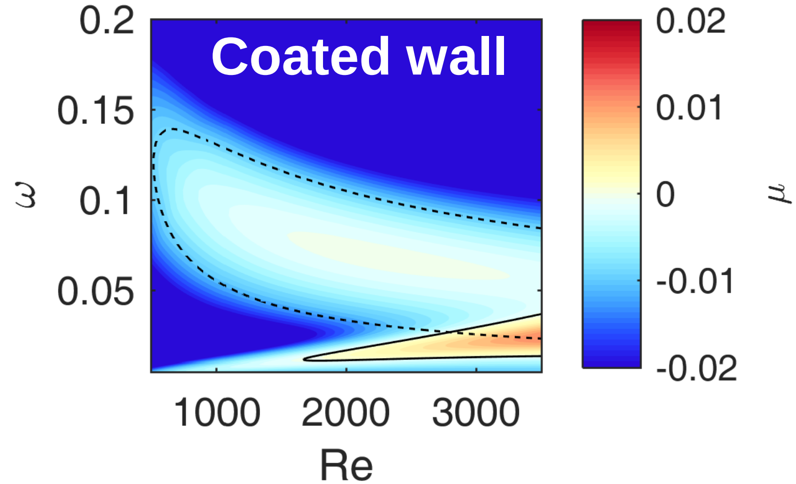
$l_2 = 8653\ \mu m$

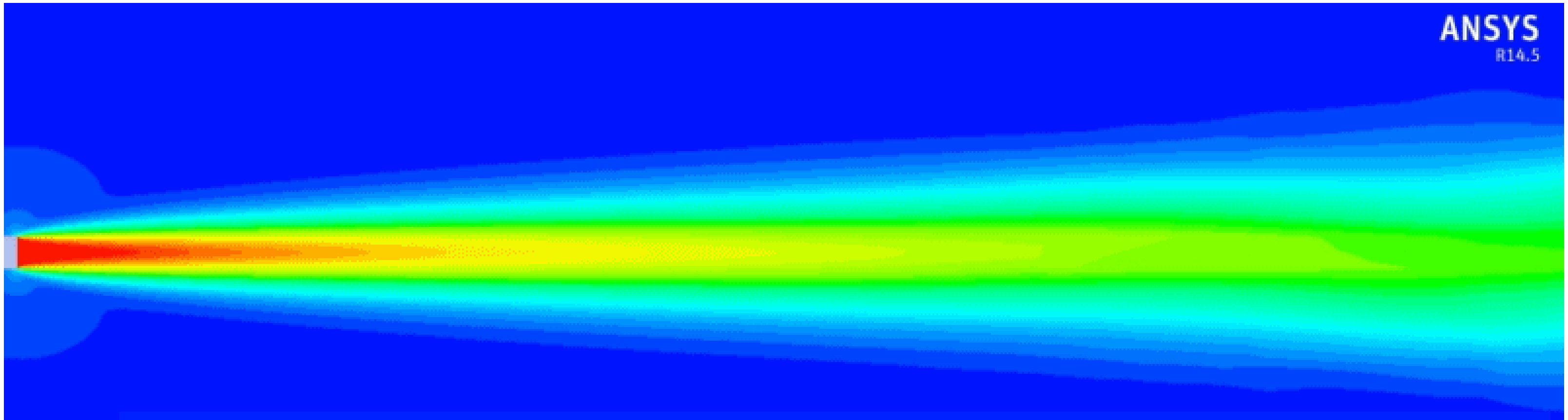
$h_1 = 168\ \mu m$

$h_2 = 14.5\ \mu m$

$w_1 = 10\ \mu m$

$w_2 = 33\ \mu m$





Thank you for your attention!