Analysis of systems containing nonlinear and uncertain components by using Integral Quadratic Constraints

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Contents

Preliminary

LMIs

A Linear Matrix Inequality (LMI) is an expression of the form

$$F(x) = F_0 + x_1 F_1 + \ldots + x_m F_m = F_0 + T(x) > 0 \qquad (1)$$

where

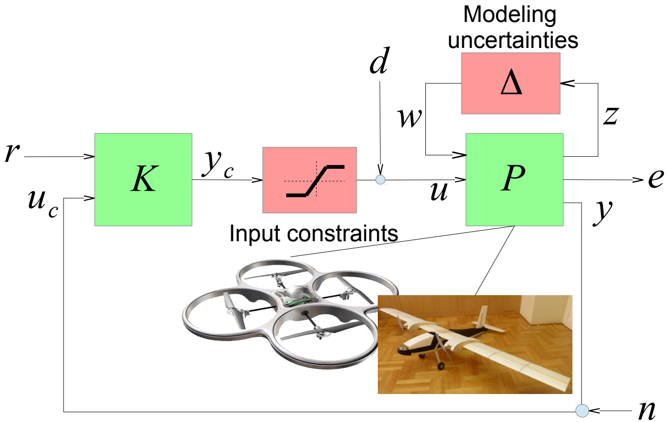
- $x = (x_1, \ldots, x_m)$ is a vector of real numbers called decision variables.
- $F_0, \ldots, F_m$ are real, symmetric matrices, i.e. $F_i = F_i^T \in \mathbb{R}^{n \times n}$
- the inequality '>0' means positive definite, i.e. $u^T F(x) u > 0, \forall u \neq 0$

The affine function $F(x)$ is often given with matrix argument in the form $F(X)$. The decision variable $X \in \mathbb{R}^{n \times n}$ is a matrix. This is a special case of (1) because by choosing a basis $E_1, \ldots, E_m$ s.t. $X = \sum_{i=1}^m x_j E_j$ then

$$F(X) = F\left(\sum_{i=1}^m x_j E_j\right) = F_0 + \sum_{i=1}^m x_j F(E_j) = F_0 + \sum_{i=1}^m x_j F_j = F(x)$$

Why do we like LMIs? An LMI defines a *convex* set, that is, the set $\{x \mid F(x) > 0\}$ is convex. The optimization of a convex (e.g. linear or affine) function $f(x) : \mathbb{R}^m \to \mathbb{R}$ with LMI constraints is thus a convex problem, which can be solved efficiently.

Solvers: LMIlab,SeDuMi,Yalmip



$$P \quad : \quad \dot{x} = F(x, u, d, w), \quad w = \Delta(z), \quad [z, e, y] = G(x, n) \quad \text{(System model)}$$
$$K \quad : \quad \dot{x}_c = K(x_c, u_c, r), \quad y_c = L(x_c, u_c, r) \quad \text{(Controller)}$$

x: states, **y**: measured outputs, **u**: control input, **d,n**: disturbance and noise, **r**: reference to be tracked, **e**: tracking error $(Cx - r)$,

System models (P=?)

Linear Time-Invariant (LTI):

- Time domain:
$$\begin{aligned} \dot{x} &= Ax(t) + Bv(t) \\ y &= Cx(t) + Dv(t), x(0) = x^* \end{aligned}$$
- Frequency domain ($j\omega$): $G(j\omega) = C(j\omega I - A)^{-1}B + D$.
If the Fourier transform $v(j\omega) = \mathfrak{F}(v(t))$ exists, then $y(j\omega) = G(j\omega)v(j\omega)$ when $x(0) = 0$
- S-domain (s): $G(s) = C(sI - A)^{-1}B + D$.
If the Laplace transform $v(s) = \mathfrak{L}(v(t))$ exists, then $y(s) = G(s)v(s)$ when $x(0) = 0$
- Convolution. Let $\delta(t)$ be the dirac delta defined as
$$\delta(t) = 0, \text{ if } t \neq 0 \text{ and } \int_{-\infty}^{\infty} \delta(t)dt = 1$$
The impulse response $g(t)$ of the LTI system is the output if $v(t) = \delta(t)$ and $x(0) = 0$.
$g(t)$ is the inverse Laplace transform of $G(s)$ For some $u(t)$ the output can be expressed by using $g(t)$ as follows:
$$y(t) = \int_0^t g(t - \tau)v(\tau)d\tau$$

System models (P=?)

Linear Parameter-Varying (LPV)

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))v(t)\\ y(t) &= C(\rho(t))x(t) + D(\rho(t))v(t),\\ &\quad \underline{\rho} \le \rho(t) \le \overline{\rho},\ \underline{\delta} \le \dot{\rho}(t) \le \overline{\delta}\end{aligned}$$

Nonlinear, input-affine: $\dot{x} = f(x(t)) + g(x(t))v(t)$

Stability: The origin is (exponentially) stable if $x(t) \to 0$ at exponential decay rate. This is equivalent to the existence of a positive definite Lyapunov function $V(x(t)) > 0$ satisfying $\dot{V}(x(t)) < 0$. For LTI systems

- $V(x) = x^T Xx$, $X > 0$ is enough and
- the stability is equivalent to that the eigenvalues of A (or the poles of $G(s)$) are strictly in the left half plane

Example Lyapunov stability. Let $\dot{x} = Ax$ be an autonomous, linear, time-invariant system. From Lyapunov theorem we know that it is stable is there exists a quadratic Lyapunov function $V(x) = x^T Xx$ with $X > 0$, s.t. $A^T X + XA < 0$. These conditions are equivalent to the following LMI:

$$\begin{pmatrix} X & 0 \\ 0 & -A^T X - XA \end{pmatrix} > 0$$

Signal spaces and operators

Signal spaces

Signal spaces: vector spaces of functions mapping the 'time axis' $\mathcal{T} \subset \mathbb{R}$ into a vector space $\mathcal{V} \subset \mathbb{R}^n$. Examples for $\mathcal{T}$ in case of continuous time systems are real numbers $\mathbb{R} = (-\infty, \infty)$ or $\mathbb{R}_+ = (0, \infty)$

Normed (signal) space $\mathcal{L}$: a linear vector space equipped with a norm $\|\cdot\|$

$$\begin{aligned} \|f\| = 0 \quad &\leftrightarrow \quad f \equiv 0 \\ \|\alpha f\| \quad &= \quad |\alpha| \|f\| \\ \|f + g\| \quad &\leq \quad \|f\| + \|g\| \end{aligned}$$

If the normed space is complete, i.e. its Cauchy sequences converge, the normed space is called *Banach space*. Examples for Banach spaces:

$$\mathcal{L}_p[0,\infty) : \ \|f\|_p = \left(\int_0^\infty |f_i|^p \ dt \right)^{1/p}$$

$$\mathcal{L}_p(-\infty,\infty) : \ \|f\|_p = \left(\int_{-\infty}^\infty |f_i|^p \ dt \right)^{1/p}$$

$$\mathcal{L}_\infty[0,\infty) : \ \|f\|_\infty = \text{ess} \sup_{t \in \mathbb{R}_+} |f(t)|$$

where $|f|$ is the standard Euclidean norm $|f| = (f^T f)^{1/2}$.

Signal spaces

Inner product space: linear vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ satisfying the following properties (where $f, g : \mathcal{T} \mapsto \mathbb{R}$ and $\alpha \in \mathbb{R}$)

$$\begin{aligned} \langle f, g \rangle &= \langle g, f \rangle \\ \langle \alpha f, g \rangle &= \alpha \langle f, g \rangle \\ \langle f_1 + f_2, g \rangle &= \langle f_1, g \rangle + \langle f_2, g \rangle \end{aligned}$$

The inner product induces a norm:

$$\|f\| = \sqrt{\langle f, f \rangle}$$

If an inner product space is complete it is called *Hilbert space*. Hilbert spaces will be denoted by $\mathcal{H}$ to be distinguished from the normed spaces. Examples for Hilbert spaces:

$$\begin{aligned} \mathcal{L}_2^m[0,\infty) \quad &: \quad \langle f, g \rangle = \int_0^\infty f(t)^T g(t) dt = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(j\omega)^* \hat{g}(j\omega) d\omega \\ \mathcal{L}_2^m(-\infty,\infty) \quad &: \quad \langle f, g \rangle = \int_{-\infty}^\infty f(t)^T g(t) dt = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(j\omega)^* \hat{g}(j\omega) d\omega \end{aligned}$$

where $\hat{f}(j\omega)$ is the Fourier transform of $f(t)$. The examples show that, if $p = 2$ then $\mathcal{L}_p[0,\infty)$ and $\mathcal{L}_p(-\infty,\infty)$ are *not only Banach but also Hilbert spaces.*

Operators

Operators: mapping from one normed (signal) space into another. Now 'one=another', i.e. $H : \mathcal{L} \to \mathcal{L}$

Properties of operators:

Composition: $H_1 H_2$ is also an operator defined by $(H_1 H_2)(f) = H_1(H_2(f))$
Sum: $\alpha H_1 + \beta H_2$ is also an operator defined by $(\alpha H_1 + \beta H_2)(f) = \alpha H_1(f) + \beta H_2(f)$
Linearity (we will assume it often): $H(\alpha f + \beta g) = \alpha H(f) + \beta H(g)$
Gain and boundedness:

$$\|H\| = \sup_{f \in \mathcal{L}, f \neq 0} \frac{\|Hf\|}{\|f\|} < \infty \tag{2}$$

Multiplicativity rule: $\|H_1 H_2\| \leq \|H_1\| \cdot \|H_2\|$

Example 1 ($\mathcal{RH}_\infty$): Let $\dot{x} = Ax + Bu, y = Cx + Du$, $x(0) = 0$ be a finite dimensional, linear, time-invariant (LTI) dynamical system *with poles strictly in the left half plane*. The LTI system defines an operator in terms of convolution

$$(Gu)(t) = (g * u)(t) = \int_0^t g(t-\tau)u(\tau)d\tau \qquad G(s) = C(sI - A)^{-1}B + D$$

where $g(t) = \mathfrak{L}^{-1}\{G\}$ is the weighting function (impulse response). Since all poles are stable, this operator is bounded on every on $\mathcal{L}_p[0, \infty)$ space. (All poles on the left half plane is a necessary and sufficient condition for an operator to be bounded on $\mathcal{L}_p[0, \infty)$ spaces.)

If we consider G as an operator on $\mathcal{L}_2[0,\infty)$ then its bound can be determined as follows:

$$\|G\| = sup_{f\in\mathcal{L}_2[0,\infty),f\neq 0}\frac{\|Hf\|}{\|f\|} = \sup_{\omega\in[0,\infty)} \sigma_{max}(G(j\omega)) \quad \text{(the well-known } \mathcal{H}_\infty \text{ } \textit{norm}\text{)}$$

If we consider G as an operator on $\mathcal{L}_\infty[0,\infty)$ then its bound will be calculated as

$$\|G\| = sup_{f\in\mathcal{L}_\infty[0,\infty),f\neq 0}\frac{\|Hf\|}{\|f\|} = \int_0^\infty |g(t)|\, dt \quad \text{(i.e., the } \mathcal{L}_1 \text{ norm of the impulse response)}$$

Example 2 ($\mathcal{RL}_\infty$): If the LTI system has poles both in the right and the left half plane, but there is no pole on the imaginary axis, then G is a bounded operator on $\mathcal{L}_p[-\infty,\infty)$. The operator is defined in terms of convolution:

$$(Gu)(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \tag{3}$$

where $g(t) = \mathfrak{L}^{-1}\{G(s)\}$ If $G \in \mathcal{RL}_\infty$ is considered on $\mathcal{L}_2[-\infty,\infty)$ then

$$\|G\| = \sup_{\omega\in[0,\infty]} \sigma_{max}(G(j\omega))$$

Remark. The system norms can be computed in MATLAB by `norm` function.

Adjoint operators:
Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. The Hilbert adjoint H^* of H is the operator $H^* : \mathcal{H} \to \mathcal{H}$ s.t.

$$\langle Hf, g\rangle = \langle f, H^* g\rangle \quad \forall f, g \in \mathcal{H}$$

An operator is *self-adjoint* if $H^* = H$. Examples:

- Let $H \in \mathcal{RH}_\infty$ be an operator with state space realization $H(s) = C(sI - A)^{-1}B + D$. Then $H^*(s) = H(-s)^T = -B^T(sI + A^T)^{-1}C^T + D^T$. So, if $H(s)$ stable then its adjoint will be unstable.
- More generally, if $H \in \mathcal{RL}_\infty$ then $H^*(s) = H(-s)^T$.

Properties of Hilbert adjoint. *The Hilbert adjoint H^* exists uniquely and it is a linear operator with $\|H^*\| = \|H\|$. Furthermore, for bounded operators $H, H_1, H_2 : \mathcal{H} \to \mathcal{H}$ the following equations hold:*

$a)$ $(\alpha H)^* = \alpha H^*$ $\quad$ $b)$ $(H_1 + H_2)^* = H_1^* + H_2^*$ $\quad$ $c)$ $(H^*)^* = H$
$d)$ $(H_1 H_2)^* = H_2^* H_1^*$ $\quad$ $e)$ $\|H^* H\| = \|HH^*\| = \|H\|^2$ $\quad$ $f)$ $(H^*)^{-1} = (H^{-1})^*$

Self-adjoint operators. A bounded linear operator $H : \mathcal{H} \rightarrow \mathcal{H}$ is self-adjoint if $H^* = H$.

Remark. If $H \in \mathcal{RH}_\infty$ and H is self-adjoint, then it is constant.

Quadratic forms.

The quadratic form $\sigma(f) = \langle Hf, f \rangle$ defined by a self-adjoint operator H is positive semidefinite (positive definite) (denoted by $H > (\geq) 0$ if $\langle Hf, f \rangle \geq (>) 0$ for all $f \in \mathcal{H}$.

Let $\Phi = \Phi^* : \mathcal{H} \rightarrow \mathcal{H}$. Then $\sigma(f) = \langle \Phi f, f \rangle$ is a quadratic form not only on $\mathcal{H}$ but also on its subspace $\tilde{\mathcal{H}} \subset \mathcal{H}$. $\Phi \geq 0$ obviously implies that $\sigma \geq 0$ on $\tilde{\mathcal{H}}$, but the reverse implication is not at all clear. In the particular case when $\Phi = \Phi^* \in \mathcal{RL}_\infty$ and $\mathcal{H} = \mathcal{L}_2(-\infty, \infty)$, $\tilde{\mathcal{H}} = \mathcal{L}_2(0, \infty)$ then the reverse implication holds, too:

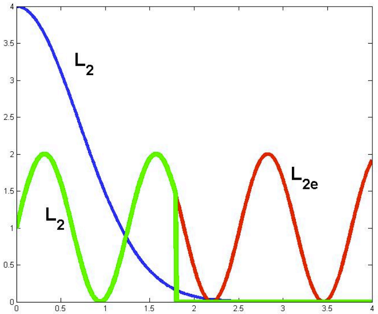
$$\sigma(f) \geq 0 \text{ for all } f \in \mathcal{L}_2[0, \infty) \quad \Leftrightarrow \quad \Phi(j\omega) \geq 0 \tag{4}$$

This makes it possible to define IQCs with noncausal Φ while the signals remain in $\mathcal{L}_2[0, \infty)$.

Methods for analyzing the stability of feedback interconnections

Extended spaces

The signal spaces examined so far contains only signals which have finite integral. Many important signals e.g. $(\sin(t), e^t)$ do not satisfy this requirement. The signal spaces are *extended* now to contain signals, which are integrable over a finite time interval.



Extended normed spaces:

$$\mathcal{L}_e = \{f : \mathcal{T} \to \mathcal{V} : \ \|f_T\| < \infty, \ \forall T \geq 0\}$$

where $f_T = P_T f$ with the truncation operator

$$P_T f = \begin{cases} f(t), & t \leq T \\ 0, & t > T \end{cases}$$

Examples: $\sin(t), e^t \in \mathcal{L}_{pe}[0, \infty)$, $2^k \in l_{pe}(Z_+)$

Operator on an extended space: Let $H : \mathcal{L}_e \to \mathcal{L}_e$ be an operator defined on an extended space. The notion of 'gain' and 'boundedness' can be extended in the following way:

$$\|H\| = \sup_{f \in \mathcal{L}_e, \|P_T f\|_{\mathcal{L}} \neq 0, T \geq 0} \frac{\|P_T(Hf)\|_{\mathcal{L}}}{\|P_T f\|_{\mathcal{L}}} \tag{5}$$

Causality

Causality: the value at a certain time does not depend on future values of the argument

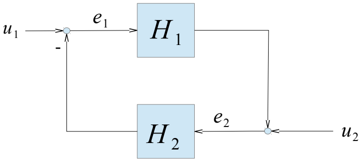
$$P_T H P_T = P_T H, \quad \forall T \in \mathcal{T}$$

Anticausality: the future matters only, i.e. $(I - P_T)H = (I - P_T)H(I - P_T)$

An important consequence of causality: If an operator is causal then it is bounded on $\mathcal{L}_e$ if and only if it is bounded on $\mathcal{L}$ and the gain defined over the two signal spaces ($\mathcal{L}_e$ and $\mathcal{L}$) are equal:

$$\|H\| = \sup_{f \in \mathcal{L}_e, \|P_T f\|_{\mathcal{L}} \neq 0, T \geq 0} \frac{\|P_T(Hf)\|_{\mathcal{L}}}{\|P_T f\|_{\mathcal{L}}} = \sup_{f \in \mathcal{L}, f \neq 0} \frac{\|Hf\|_{\mathcal{L}}}{\|f\|_{\mathcal{L}}}$$

This lemma enables us to use the $\mathcal{L}_2$-space, instead of the extended $\mathcal{L}_{2e}$ during the stability analysis.



H_1, H_2 are operators on normed space $\mathcal{L}_e$.

Well-posedness: the interconnection makes sense, that is the system does not have finite escape time, u uniquely determines e and the mapping $u \mapsto e$ is causal.

Example1: Let $H_1(s) = 1/(s+1)$, $H_2(x) = -x - x^2$, $u_1(t) = \theta(t)$ (unit step function), $u_2 = 0$. Then the closed loop realizes the differential equation $\dot{x} = x^2 + 1$. The solution is $x(t) = \tan(t)\theta(t)$, which goes to infinity if $t \to \pi/2$. This means the closed loop has finite escape time so it is ill-posed.

Example2: Let $H_1 = 1$, $H_2(x) = e^{-sT} - 1$, $u_2 = 0$. Then the closed loop realizes the mapping $y(t) = u_1(t+T)$, so the closed loop is not causal.

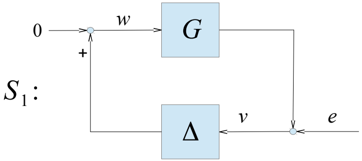
Definition. The interconnection is well-posed if for any $u_1, u_2 \in \mathcal{L}_e$ there exists a solution $e_1, e_2 \in \mathcal{L}_e$ and they depend causally on u_1 and u_2.

Stability: A well-posed system is stable if there exists c_1, c_2, c_3, c_4 s.t.

$$\begin{aligned} \|e_{1T}\| &\leq c_1\|u_{1T}\| + c_2\|u_{2T}\| \\ \|e_{2T}\| &\leq c_3\|u_{1T}\| + c_4\|u_{2T}\| \end{aligned}$$

where $e_{iT} = P_T e_i$.

Integral Quadratic Constraints



G: LTI transfer function defining a bounded, causal operator on the extended Hilbert-space $\mathcal{H}_e$
Δ: bounded, causal operator on the same $\mathcal{H}_e$ space.

We particularly interested in the special case when $\mathcal{H}_e = \mathcal{L}_{2e}[0,\infty)$

Integral Quadratic Constraint (IQC). Let Π be a bounded, self-adjoint operator. Then we say, Δ *satisfies the IQC* defined by Π ($\Delta \in \mathrm{IQC}(\Pi)$) if

$$\sigma_\Pi(v,\Delta(v)) = \left\langle \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} \right\rangle \geq 0 \quad \forall v \in \mathcal{L}_2[0,\infty)$$

We call Π the *multiplier* that defines the IQC. (Note that the IQC is defined on $\mathcal{L}_2$ even though the operators are defined on the extended $\mathcal{L}_{2e}$ space.)

Remark. If $\Delta \in \mathrm{IQC}(\Pi_1)$ and $\Delta \in \mathrm{IQC}(\Pi_2)$ then $\Delta \in \mathrm{IQC}(\tau_1\Pi_1 + \tau_2\Pi_2)$, $\tau_i \geq 0$.

Remark. Since we defined the IQC over $\mathcal{L}_2[0,\infty)$ space, Π can be taken as a transfer function $\Pi(j\omega) = \Pi(j\omega)^*$. The condition above then reduces to

$$\sigma_\Pi(v,\Delta(v)) = \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{\Delta(v)}(j\omega) \end{bmatrix} \geq 0, \qquad \forall\ v \in \mathcal{L}_2[0,\infty)$$

The IQC Theorem

IQC Theorem:
Assume there exists a bounded, self-adjoint operator Π so that all of the following statements hold

i) the interconnection $(G, \tau\Delta)$ well-posed for all $\tau \in [0,1]$

ii) For all $\tau \in [0,1]$ $\tau\Delta$ satisfies the IQC defined by Π, i.e.

$$\sigma_\Pi(v, \Delta(v)) = \left\langle \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} \right\rangle \geq 0 \quad \forall v \in \mathcal{L}_2[0,\infty]$$

iii) there exists $\varepsilon > 0$ s.t.

$$\sigma_\Pi(Gv, v) \leq -\varepsilon \|v\|^2 \qquad (*)$$

then the interconnection is stable.

Remark. Since we are on the $\mathcal{L}_2[0,\infty)$ space, (4) applies and thus $(*)$ is equivalent to

$$\begin{bmatrix} G \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I \end{bmatrix} \leq -\varepsilon I, \quad \text{that is} \quad \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I \quad \forall \omega \in \mathbb{R}$$

By IQC Theorem the stability analysis reduces to finding a suitable multiplier.

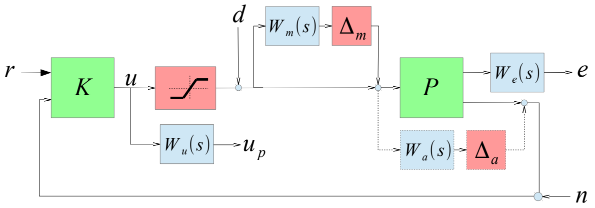
Remark. Note that, item i) is not too strict condition, since in general

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{pmatrix}, \quad \Pi_{11} \geq 0, \Pi_{22} \leq 0$$

then the fact Δ satisfies the IQC implies that the IQC is satisfied by all $\tau\Delta$, $\tau \in [0,1]$.

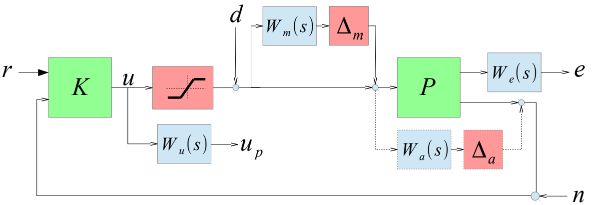
System analysis by using IQCs

The closed loop system



1. **Nominal model** P, defining an LTI causal and bounded operator.
2. **Controller.** We assume that the controller has already been designed so it is known now.
3. **External inputs.** Inputs coming from the environment. They can be disturbances, sensor- or actuator noises. The reference signal, which has to be tracked by the output of the plant, is also an external signal. The external signals are assumed to come from some extended Hilbert space $\mathcal{H}_e$. Typically $\mathcal{H}_e = \mathcal{L}_{2e}[0, \infty)$.
   IQCs can also be used to formulate the properties of the external signals:
   $$\sigma_\Psi(d) = \int_{-\infty}^{\infty} \hat{d}(j\omega)^* \Psi(j\omega) \hat{d}(j\omega) d\omega \geq 0$$
   holds for input $d \in \mathcal{L}_2[0, \infty]$ with some $\Psi \in \mathcal{RL}_\infty$ then this property can be taken into account during the analysis procedure.
4. **Performance outputs.** Inner signals or outputs, the behavior of which are important for us. They can also be weighted - $W_e(s)$, $W_u(s)$

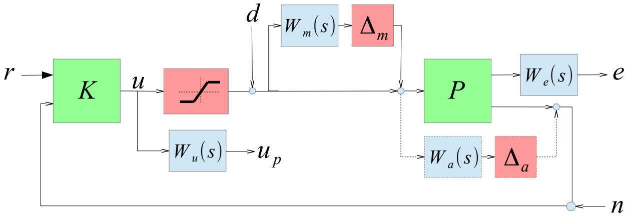
The closed loop system



5. **Nonlinear elements and time delays.** The nonlinear components can either be static (e.g. saturations, deadzones) or dynamic (e.g. friction).
6. **Uncertainty.** Models the difference between the mathematical model and the real system. Uncertainty can be present due to approximation or identification errors, change of parameters and nonlinearities due to wear or change of operating conditions. Typical uncertainty models:
   - *LTI Dynamic uncertainty.* It represents unmodeled dynamics or model error from identification. It is defined by an unknown, stable transfer function with bounded $\mathcal{H}_\infty$ norm. Typically, $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$ and $W(s)$ weighting function is applied to describe the frequency distribution of the uncertainty. It can be inserted into the system either by additive or multiplicative structure.
   - *Parametric uncertainty.* It is used to model uncertain gain, uncertainty in the location of poles or zeros or unknown changes in physical parameters.
   - *Polytopic uncertainty.* Special class of parametric uncertainty. The possible parameter values come from a convex polytope:

$$p(t) \in \Delta = \{p \mid p = \sum \lambda_i p_i,\ \lambda_1 \geq 0,\ \sum \lambda_i = 1,\ i = 1, \ldots, N\}$$

The closed loop system



The aim of the analysis is to check whether the system satisfy the *robust performance* criterium. Robust performance comprises the following two conditions:

- **robust stability.** The system has to be stable for all possible values of uncertainties, nonlinearities, delays, etc.
- **performance.** The performance outputs should satisfy the prescribed specification. The performance is generally formalized by a quadratic relation, e.g.

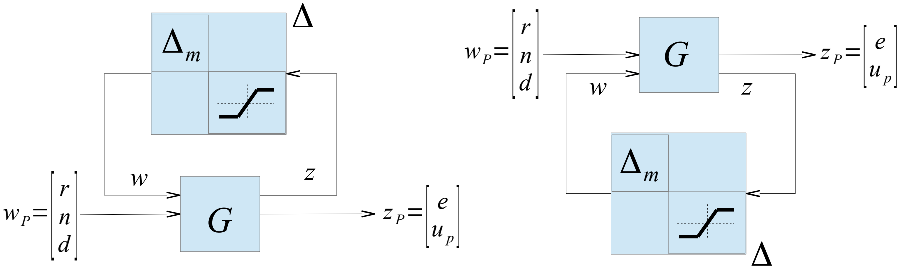
$$\sigma_P(z_P, w_P) = \int_0^\infty \begin{bmatrix} z_P(t) \\ w_P(t) \end{bmatrix}^T P \begin{bmatrix} z_P(t) \\ w_P(t) \end{bmatrix} dt \leq 0, \quad z_P, w_P \in \mathcal{L}_2[0, \infty)$$

  Important performance measure: the induced $\mathcal{L}_2$-norm (i.e the $\mathcal{H}_\infty$ operator norm in LTI case):

$$\frac{\|z_P\|_{\mathcal{L}_2}}{\|w_P\|_{\mathcal{L}_2}} \leq \gamma \Rightarrow \sigma_P(z_P, w_P) = \int_0^\infty \begin{bmatrix} z_P(t) \\ w_P(t) \end{bmatrix}^T \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & -\gamma I \end{bmatrix} \begin{bmatrix} z_P(t) \\ w_P(t) \end{bmatrix} dt \leq 0$$

LFT representation

The first step of analysis is pulling out the unknown and nonlinear blocks:



Upper LFT: $z_P = \mathcal{F}_u(G,\Delta)w_P = [G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}G_{12}]w_P$ (left)

Lower LFT: $z_P = \mathcal{F}_l(G,\Delta)w_P = [G_{11} + G_{12}\Delta(I - G_{22}\Delta)^{-1}G_{21}]w_P$ (right)

System analysis by using IQCs

STEP 1.

1. Construct a suitable, linearly parameterized set of IQCs for each uncertain, nonlinear block.

$$\Delta_k \in \mathrm{IQC}(\Pi_k(\lambda_{\Pi_k})), \quad \textbf{for all } \lambda_{\Pi_k} \in \Lambda_{\Pi_k}$$

   Build one diagonal IQC, that is satisfied by the augmented block $\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_N)$:

$$\Delta \in \mathrm{IQC}(\Pi(\lambda_\Pi)), \quad \Pi(\lambda_\Pi) = \mathrm{diag}(\Pi_1(\lambda_{\Pi_1}), \ldots, \Pi_N(\lambda_{\Pi_N})), \quad \lambda_\Pi = (\lambda_1, \ldots, \lambda_N)$$

   The IQC defines quadratic inequality condition between w and z : $\sigma_{\Pi(\lambda_\Pi)}(z, w) \geq 0$

2. You may choose (linearly parameterized) IQC conditions for the external inputs: $\sigma_{\Psi(\lambda_\Psi)}(w_P) \geq 0$

3. Prescribe the performance requirements by using IQC: $\sigma_{P(\gamma)}(z_P, w_P) \leq 0$

S-procedure. Let $\sigma_k : \mathcal{H} \to \mathbb{R}$ be quadratic forms defined as

$$\sigma_k(f) = \langle \Phi_k f, f \rangle, \quad k = 0, 1, \ldots, N$$

where Φ_k are linear, bounded, self-adjoint operators on $\mathcal{H}$. We consider the following two problems:

$$S_1 \quad : \quad \sigma_0(f) \leq 0 \text{ for all } f \in \mathcal{H} \text{ s.t. } \sigma_k(f) \geq 0,\ k = 1, \ldots, N.$$

$$S_2 \quad : \quad \text{there exists } \tau_k \geq 0,\ k = 1, \ldots, N, \text{ such that}$$

$$\sigma_0(f) + \sum_{k=1}^{N} \tau_k \sigma_k(f) \leq 0, \quad \forall f \in \mathcal{H}$$

It is obvious that S_2 implies S_1. The opposite direction holds only in special cases.

STEP 2.

The interconnection satisfies the robust performance requirements if there exist $\lambda_\Pi, \lambda_\Psi$ s.t.

$$\sigma_{P(\gamma)}(G_{21}w + G_{22}w_P, w_P) + \sigma_{\Pi(\lambda_\Pi)}(G_{11}w + G_{12}w_P, w) + \sigma_{\Psi(\lambda_\Psi)}(w_P) < 0$$

which, over the $\mathcal{L}_2[0, \infty)$ space, is equivalent to the frequency domain inequality:

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty]$$

where $\Pi(j\omega)$ collects all multipliers $\Pi(\lambda_\Pi), \Psi(\lambda_\Psi), P(\gamma)$ and thus depends on $\lambda_\Pi, \lambda_\Psi, \gamma$.

Transforming frequency domain inequalitis to LMI

We want to check the feasibility of

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\omega \in [0, \infty]$$

Let

$$\Pi = \begin{bmatrix} (j\omega I - A_\pi)^{-1} B_\pi \\ I \end{bmatrix}^* M_\pi \begin{bmatrix} (j\omega I - A_\pi)^{-1} B_\pi \\ I \end{bmatrix}$$

where $B_\pi = [B_{\pi,v} \; B_{\pi,w}]$ and A_π is stable. For simplicity, the dependence of $\Pi(j\omega)$ on λ is omitted. Then we have

$$\begin{bmatrix} (j\omega I - A_\pi)^{-1} B_\pi \\ I \end{bmatrix} \cdot \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* M_\pi \begin{bmatrix} (j\omega I - A_\pi)^{-1} B_\pi \\ I \end{bmatrix} \cdot \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0$$

To perform the multiplications we use the following lemma:

$$\text{If } G_i(s) = C_i(sI - A_i)^{-1} B_i + D_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] \text{ then}$$

$$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]$$

Transforming frequency domain inequalitis to LMI

Then we find that the original inequality can be formulated as

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0 \qquad (\diamond)$$

where

$$A = \begin{bmatrix} A_\pi & B_{\pi,v}C_G \\ 0 & A_G \end{bmatrix}, \quad B = \begin{bmatrix} B_{\pi,v}D_G + B_{\pi,w} \\ B_G \end{bmatrix}$$

and

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = -\left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_G & D_G \\ \hline 0 & 0 & I \end{array}\right]^T M_\pi \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_G & D_G \\ \hline 0 & 0 & I \end{array}\right]$$

From (4) it follows that $(\diamond)$ is equivalent to the existence of $\epsilon > 0$ s.t.

$$\epsilon \|w\|^2 \leq \int_{-\infty}^{\infty} \begin{bmatrix} (j\omega I - A)^{-1}B\hat{w}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B\hat{w}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} \qquad (6)$$

$$= \int_0^\infty (x^T Q x + 2x^T S w + w^T R w) dt \qquad (7)$$

for all pairs $(x, w) \in \mathcal{L}_2[0, \infty)$, where $\dot{x} = Ax + Bw$, $x(0) = 0$, $w \in \mathcal{L}_2[0, \infty)$. This is a linear-quadratic optimal control problem.

Kalman-Yakubovic-Popov Lemma

The following statements are equivalent:

- there exist of $\epsilon > 0$ s.t.

$$\int_0^\infty (x^T Qx + 2x^T Sw + w^T Rw)dt \geq \int_0^\infty |x|^2 + |w|^2 dt$$

  for all pairs $(x, w) \in \mathcal{L}_2[0, \infty)$, where $\dot{x} = Ax + Bw$, $x(0) = 0$, $w \in \mathcal{L}_2[0, \infty)$.

- we have

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0 \quad \forall \omega \in [0, \infty)$$

- there exists $P = P^T$ s.t.

$$\begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0 \quad \text{LMI condition!}$$

Linear dependence of $\Pi(j\omega)$ on the parameters $\lambda = (\lambda_\Pi, \lambda_\Psi, \gamma)$ is generally shifted into $M_\pi(\lambda)$, which results in parameter dependent $Q(\lambda)$, $S(\lambda)$ and $R(\lambda)$.

List of IQCs

LTI uncertainty

1. D-scaling:

Let $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_m)$ LTI operator with norm $\|\Delta_i\|_\infty \leq 1$. In frequency domain we can choose an IQCwith multiplier

$$\Pi(j\omega) = \begin{bmatrix} -D(j\omega)^* D(j\omega) & \\ & D(j\omega)^* D(j\omega) \end{bmatrix} \text{ s.t. } \Delta D = D\Delta$$

where $D(j\omega)$ is a free variable. Since

$$\begin{pmatrix} I \\ \Delta \end{pmatrix}^* \begin{bmatrix} D(j\omega)^* D(j\omega) & \\ & -D(j\omega)^* D(j\omega) \end{bmatrix} \begin{pmatrix} I \\ \Delta \end{pmatrix} = -\Delta^* D^* D\Delta + D^* D = D^*[I - \Delta^*\Delta]D > 0$$

The condition of stability in this special case can be given as follows

$$\begin{pmatrix} G \\ I \end{pmatrix}^* \begin{bmatrix} D(j\omega)^* D(j\omega) & \\ & -D(j\omega)^* D(j\omega) \end{bmatrix} \begin{pmatrix} G \\ I \end{pmatrix} = -D^* D + G^* D^* DG < 0 \;\Leftrightarrow\; \|DGD^{-1}\| < 1$$

This is the *D-iteration* part of the D-K iteration used in robust control design. The design of scaling $D(j\omega)$ is a construction of a suitable multiplier.

2. Time-varying, causal, linear, structured uncertainty:

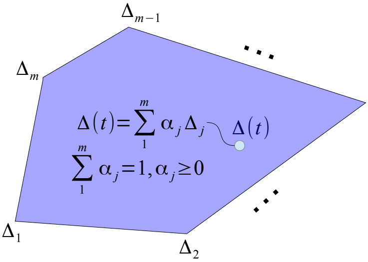
$(\Delta z)(t) = \Delta(t)z(t)$, $\Delta(t) = \mathrm{diag}(\delta_1(t)I, \ldots, \delta_m(t)I)$ $|\delta_i(t)| \leq 1$ for $t \geq 0$. Then

$$\Pi = \left\{ \begin{pmatrix} Q & S^T \\ S & R \end{pmatrix} \mid R = -Q,\ Q = \mathrm{diag}(Q_1, \ldots, Q_m) > 0, \right.$$
$$\left. S = \mathrm{diag}(S_1, \ldots, S_m),\ S_j + S_j^T = 0 \right\}$$

3. Time-varying, causal, linear, polytopic uncertainty:

$(\Delta z)(t) = \Delta(t)z(t)$, $\Delta(t) \in \mathrm{co}\{\Delta_1, \ldots, \Delta_N\}$, $\Delta_i \in \mathbb{R}^{n \times n}$. Then

$$\Pi = \left\{ \begin{pmatrix} R & S^T \\ S & Q \end{pmatrix} \mid Q < 0,\ \begin{pmatrix} I \\ \Delta_j \end{pmatrix}^T \Pi \begin{pmatrix} I \\ \Delta_j \end{pmatrix} \right.$$

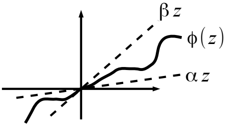


Sector bounded nonlinearities

4. Memoryless nonlinearity in a sector

Let $w(t) = (\Delta z)(t) = \phi(z(t), t)) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function contained in a sector $[\alpha, \beta]$, i.e.

$$\alpha z^2 \leq \phi(z, t)z \leq \beta z^2, \quad \forall z \in \mathbb{R}, t \geq 0$$



Then $\beta z - \phi(z, t)$ and $\phi(z, t) - \alpha z$ have the same sign, that is $(\beta z - \phi(z, t))(\phi(z, t) - \alpha z) \geq 0$. This implies the following constant multiplier:

$$\Pi(j\omega) = \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -1 \end{bmatrix} \qquad (8)$$

5. The "Popov" IQC

If $w(t) = (\Delta z)(t) = \phi(z(t)) : \mathbb{R} \to \mathbb{R}$ be a continuous function, $z(0) = 0$ and both $w(\cdot)$ and $\dot{z}(\cdot)$ are square summable, then $\int_0^\infty \dot{z}(t)w(t) = 0$ holds. This implies in frequency domain the following IQC:

$$\Pi(j\omega) = \pm \begin{bmatrix} 0 & j\omega\lambda \\ -j\omega\lambda & 0 \end{bmatrix}, \quad \lambda \in \mathbb{R} \qquad (9)$$

Note that, this multiplier is not proper, so in general it is combined with other multipliers or instead of Δ, the modified $\tilde{\Delta} = \Delta \circ \frac{1}{s+1}$ is considered. The modified (proper) multiplier is

$$\Pi(j\omega) = \pm \begin{bmatrix} 0 & \frac{j\omega}{1+j\omega}\lambda \\ -\frac{j\omega}{1+j\omega}\lambda & 0 \end{bmatrix}, \quad \lambda \in \mathbb{R}$$

Remark. The sum of (8) and (9) gives the Popov criterion for memoryless, sector bounded, continuous nonlinearities.

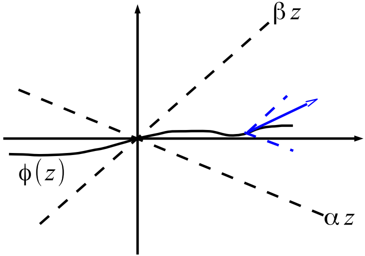
Let $w(t) = (\Delta z)(t) = \phi(z(t)) : \mathbb{R} \to \mathbb{R}$ be a static nonlinearity with the following properties:

i) $\phi(0) = 0$

ii) With some $\alpha \leq \beta$

$$\alpha \leq \frac{\phi(z_1) - \phi(z_2)}{z_1 - z_2} \leq \beta$$

iii) there exists $k > 0$ s.t. $|\phi(z)| \leq k|z|$



6. Zames-Falb multiplier [Zames,Falb 1968]:
The following multiplier was derived by Zames and Falb in [Zames,Falb 1968]:

$$\Pi(j\omega) = T^T \begin{bmatrix} 0 & 1 + H(j\omega)^* \\ 1 + H(j\omega) & 0 \end{bmatrix} T$$

where H is a strictly proper rational transfer function with impulse response h and the following constraints are satisfied:

- $h(t) \leq 0$ for all $t \in \mathbb{R}$. If ϕ is an odd function then this constraint is not needed.
- $\mathcal{L}_1$-norm constraint: $\|h\|_1 = \int_{-\infty}^{\infty} |h(t)| dt \leq 1$
- $$T = \begin{bmatrix} \frac{\beta}{\beta - \alpha} & -\frac{1}{\beta - \alpha} \\ -\alpha & 1 \end{bmatrix}$$

Slope restricted nonlinearities

Sketch of proof: We prove only the special case, when ϕ is odd and $\beta = 1$, $\alpha = 0$. The normed saturation nonlinearity satisfies these properties.

Since $|v(t)| \geq |\varphi(v(t))|$ and $|\varphi(v(t))| \geq |h * \varphi(v(t))|$ thus

$$[v(t) - \varphi(v(t))] \cdot [\varphi(v(t)) + (h * \varphi(v(t)))] \geq 0$$

Consequently

$$0 \leq \int_0^\infty 2[v - \varphi(v)] \cdot [\varphi(v) + h * \varphi(v)]dt =$$

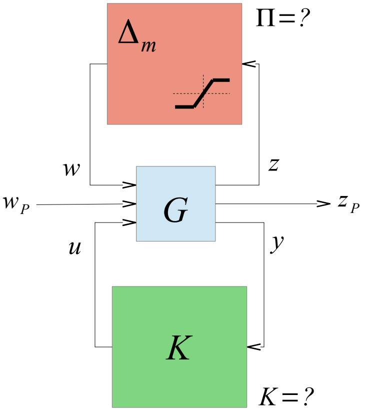
$$\int_{-\infty}^{\infty} 2Re[\hat{v}(j\omega) - \widehat{\varphi(v)}(j\omega)]^* [\widehat{\varphi(v)}(j\omega) + H(j\omega)\widehat{\varphi(v)}(j\omega)]d\omega$$

$$= \begin{pmatrix} \hat{v}(j\omega) \\ \widehat{\varphi(v)}(j\omega) \end{pmatrix}^* \begin{bmatrix} 0 & 1 + H(j\omega) \\ 1 + H(j\omega)^* & -2(1 + \mathrm{Re}H(j\omega)) \end{bmatrix} \begin{pmatrix} \hat{v}(j\omega) \\ \widehat{\varphi(v)}(j\omega) \end{pmatrix}$$

Remark 1. The filter H can be non-causal (poles on the right half plane!). The construction of H is not easy, only approximate solutions exist. E.g. [Chen and Wen, 1995]

Beyond the analysis

Controller synthesis



$$\begin{aligned} \text{Analysis} &\equiv \text{find a multiplier} \\ \text{Synthesis} &\equiv \text{find a multiplier} \\ & \text{and a controller} \end{aligned}$$

so that the closed-loop satisfies the robust performance.

The synthesis cannot be transformed to a convex optimization problem.

Iterative design is needed: the multiplier and the controller are tuned alternately until the performance requirements are met.

- convergence cannot be guaranteed in general
- numerical problems
- problem-specific solvers are needed

Useful extension of the LTI dynamics:

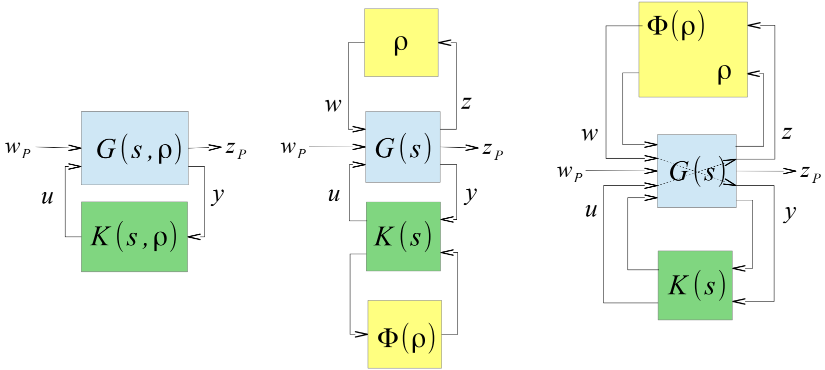
$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t), \qquad y = C(\rho(t))x(t) + D(\rho(t))u(t)$$

where $\rho(t) \in \mathbb{R}^p$ is the measured, time-varying scheduling parameter, which has in general, well-known magnitude and rate bounds: $\underline{\rho}_i \leq \rho_i(t) \leq \overline{\rho}_i$, $\underline{\delta}_i \leq \dot{\rho}_i(t) \leq \overline{\delta}_i$.

Properties:

- good modeling capabilities - by letting $\rho(t) = f(x(t))$ the nonlinear behavior can be embedded into the LPV structure
- powerful analysis and design tools of LTI system theory remain applicable

Controller synthesis:



Beyond the IQCs
(Hard constraints in control)

Set-theoretic methods

Hard constraints:

$$u(t) \in U,\ x(t) \in X \quad X, U \text{ are convex sets, polytopes}$$

These constraints cannot be handled by IQCs! Different approach is needed!

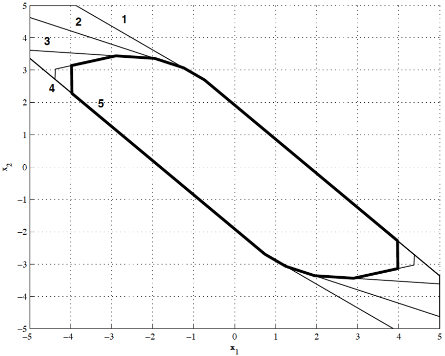
The problem seems easier in discrete-time:

$$\dot{x} = A_c x(t) + B_c u(t) + E_c d(t) \quad \Leftrightarrow_{ZOH} \quad x(k+1) = Ax(k) + Bu(k) + Ed(k)$$

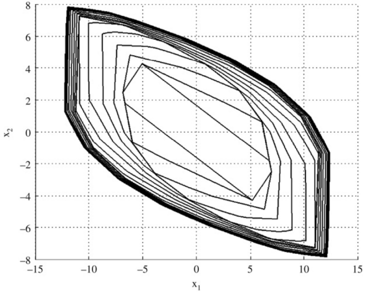
$$A = e^{AT_s}, [B\ E] = \int_0^{T_s} e^{A(t-\tau)} [B_c\ E_c] d\tau$$

1. If A_c is not stable and u is subject to hard constraints the system can be stabilized only on a closed set of states. How can this set be determined?
2. If A_c is stable and $d(k) \in D$ and D is convex (polytope) then the states converge to a closed set around the origin. This is the minimal disturbance invariant set. How does it look like?
3. The maximal disturbance invariant set contained in X is the maximal subset of X, which cannot be leaved by the trajectories of the system in the presence of constrained disturbance $d(k)$ either. How can this set be computed efficiently?

Examples



Maximal disturbance invariant set contained in X



Minimal disturbance-invariant set.

Problems: numerical difficulties, exponentially growing complexity, increasing number of vertices
Problem to be solved: find at each step the best inner and outer approximation by using only fixed number of vertices.