

Contractible and Definable Contracts

Balázs Szentes
(joint with Michael Peters)

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Motivation

1. Self- and Cross-Referential Contracts

- a. Meet the Competition

- b. Trade Agreements

- c. Tax Exemptions

2. Common Agency Problems

Example: The Prisoner's Dilemma

- two agents, actions: (C, D)
- action and contracts are contractible

$c_0 =$ "I play C no matter what"

$c_1 =$ "I play C if the other player offers c_0 "

$=$ "I play C if the other player offers 'I play C if no matter what'"

What do I Want to Model?

(symmetric, two-player games)

A : action space

C : **space of contracts**, $c : C \rightarrow 2^A$ (or A)

Game:

Stage 1. Players offer contracts: (c_1, c_2)

Stage 2. Players take actions: (a_1, a_2) , such that $a_i \in c_i(c_j)$

Contract Space

$$C \subset \{c \mid c: C \rightarrow 2^A\}$$

Universality

If $c: C \rightarrow 2^A$, then $c \in C$

Our Approach

1. **Restriction:**

Contracts must be *described* in a formal language

(the language contains arithmetics)

contract: describes a mapping $\{\text{text}\} \rightarrow 2^A$

2. Describable contracts can be **coded:**

bijection between contracts and \mathbb{N}

contract: describes a mapping $\mathbb{N} \rightarrow 2^A$

3. Contracts are *describable* mappings from codes to actions

The Gödel Coding

1. \exists a *recursive* function $\beta : \mathbb{N}^2 \rightarrow \mathbb{N}$ s.t. $\forall \{a_i\}_{i=1}^n \exists$ an $a \in \mathbb{N}$ s.t.:

(i) $\beta(0, a) = n$, and

(ii) $\beta(i, a) = a_i$ for $i \in \{1, \dots, n\}$.

Definition: $[\{a_i\}_{i=1}^n] = \min a$.

2.

space	.	,	!	?	a	b	c	d	...
1	2	3	4	5	6	7	8	9	...

 ab cc! \rightarrow (6, 7, 1, 8, 8, 4)

$[ab\ cc!] = [(6, 7, 1, 8, 8, 4)]$.

Definition. $f : \mathbb{N}^k \rightarrow 2^{\mathbb{N}}$ is *definable* if there exists a first-order predicate ϕ in $k + 1$ free variables such that $b \in f(a_1, \dots, a_k)$ if and only if $\phi(a_1, \dots, a_k, b)$ is true.

Example. Consider the following function:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is an even number,} \\ 1 & \text{if } a \text{ is an odd number.} \end{cases}$$

A corresponding predicate ϕ :

$$\phi(x, y) \equiv \{\{y = 1\} \vee \{y = 0\}\} \wedge \{\exists z : 2z = y + x\}.$$

The Gödel Code of f is $[\phi]$.

Example: The Prisoner's Dilemma

	C_2	D_2
C_1	1, 1	-1, 2
D_1	2, -1	0, 0

c_i : contract of player i

$$c_i^{n_j}([c_j]) = \begin{cases} C_i & \text{if } [c_j] = g_i(n_j), \\ D_i & \text{otherwise.} \end{cases}$$

Cross referential contracts \Leftrightarrow fixed point: (n_1^*, n_2^*)

$$\begin{bmatrix} n_j^* \\ c_i^{n_j^*} \end{bmatrix} = g_i(n_j^*).$$

Notation 1: $\langle n \rangle$: $[\langle n \rangle] = n$.

Notation 2: $\varphi^{(n_1, n_2)}$: plug (n_1, n_2) into (x_1, x_2) in φ

example: " $x_1 \leq 5^{x_2}$ " $(9, 2)$ is " $9 \leq 5^2$ "

Result: $f(n_1, n_2) = [\langle n_1 \rangle^{(n_1, n_2)}]$ is a *recursive* function.

Fixed Point

Consider a contract in two free variable:

$$c_i^{x_1, x_2} \left([c_j] \right) = \begin{cases} C_i & \text{if } [c_j] = [\langle x_j \rangle^{(x_1, x_2)}], \\ D_i & \text{otherwise.} \end{cases}$$

The Gödel Code of it is $[c_i^{x_1, x_2}] = \gamma_i$.

What is $c_i^{\gamma_1, \gamma_2}$? Evaluate $c_i^{x_1, x_2}$ at $(x_1, x_2) = (\gamma_1, \gamma_2)$:

$$c_i^{\gamma_1, \gamma_2} \left([c_j] \right) = \begin{cases} C_i & \text{if } [c_j] = [\langle \gamma_j \rangle^{(\gamma_1, \gamma_2)}], \\ D_i & \text{otherwise.} \end{cases}$$

Key observation:

$$[\langle \gamma_j \rangle^{(\gamma_1, \gamma_2)}] = \left(c_j^{x_1, x_2} \right)^{(\gamma_1, \gamma_2)} = c_j^{\gamma_1, \gamma_2}$$

Hence:

$$c_i^{\gamma_1, \gamma_2} \left([c_j] \right) = \begin{cases} C_i & \text{if } [c_j] = [c_j^{\gamma_1, \gamma_2}] , \\ D_i & \text{otherwise.} \end{cases}$$

The Gödel Code of it is: $[c_i^{\gamma_1, \gamma_2}] \rightarrow$ cross-referential

$$g_i(n_1, n_2) = \left[\langle n_j \rangle^{(n_1, n_2)} \right], \text{ and } n_j^* = [c_j^{\gamma_1, \gamma_2}].$$

Complete Information Environment

- n players.
- player i has a finite action space A_i
- her payoff is $u_i(a_1, \dots, a_n)$
- C_i : set of definable functions from $\mathbb{N}^n \rightarrow 2^{\mathbb{N}}$

The Game

Stage 1: Players simultaneously offer contracts

Stage 2: Players simultaneously take actions

but: Player i can take action a_j^i at stage 2 iff $j \in c_i([c_1], \dots, [c_n])$

equilibrium: pure SPNE

Note: if Player i 's contract is a function, c_i , then Player i 's action: $a_{c_i}^i([c_1], \dots, [c_n])$
and her payoff: $u_i(a_{c_1}([c_1], \dots, [c_n]), \dots, a_{c_n}([c_1], \dots, [c_n]))$

Folk Theorem:

Let $u_i^* = \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i})$.

Theorem. The allocation a is an equilibrium outcome if and only if $u_i(a) \geq u_i^*$ for all i .

Proof: a_{ij} : player i ' minmaxing action against player j

$$c_{x_1, \dots, x_n}^i \left(\left\{ [c^j]_{j \neq i} \right\} \right) = \begin{cases} a_i & \text{if } |\{k : [\langle x_k \rangle^{(x_1, \dots, x_n)}] \neq [c_k]\}| \neq 1, \\ a_{ij} & \text{if } \{k : [\langle x_k \rangle^{(x_1, \dots, x_n)}] \neq [c_k]\} = \{j\}. \end{cases}$$

$$\gamma_i = [c_{x_1, \dots, x_n}^i]$$

$$c_{\gamma_1, \dots, \gamma_n}^i \left(\left\{ [c^j]_{j \neq i} \right\} \right) = \begin{cases} a_i & \text{if } |\{k : [\langle \gamma_k \rangle^{(\gamma_1, \dots, \gamma_n)}] \neq [c_k]\}| \neq 1, \\ a_{ij} & \text{if } \{k : [\langle \gamma_k \rangle^{(\gamma_1, \dots, \gamma_n)}] \neq [c_k]\} = \{j\}. \end{cases}$$

$$\langle \gamma_k \rangle^{(\gamma_1, \dots, \gamma_n)} = c_{\gamma_1, \dots, \gamma_n}^k$$

Remark:

The previous result holds if

1. $C_i = \{\text{finite texts in the formal language}\}$
2. The payoffs are the same as before if c_i is a definable function

(The Gödel coding can be included in the contract.)

Interpretation:

contractibility of contracts substitutes a mechanism designer

What if there are types?

can we get the same as with the **Revelation Principal**

Bayesian Model

- n players
- player i 's finite action space: A^i .
- player i 's private information: $t_i \in T^i$.
- payoff: $u_i(a_i, a_{-i}, t)$

the game is the same as before

Consider the following mechanisms:

1. Players decide whether or not to participate.

Participants send *public* messages from the countable space M .

Non-participant, i , sends $d : M^{n-1} \rightarrow 2^{A_i}$.

2. The MD can restrict the action spaces of the participants. The restrictions cannot depend on d , but can depend on the identity of the non-participant.
3. Players take actions simultaneously from the restricted spaces.

Set of Equilibria:

Without Loss:

1. each player participates.
2. messages are subsets of the type space.

Three constraints:

1. best response in the last stage
2. incentive to reports the true subset
3. participation

equilibrium: pure BNE

mechanism: deterministic

Theorem:

An allocations can be implemented with a mechanism iff it can be implemented with contractible contracts.

Example

$$N = 2$$

$$T^1 = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}, \Pr(t_i) = 1/4$$

$$A_1 = \{1, 2\}, A_2 = \{a_1, a_2, b_1, b_2, g_{\alpha_1}, g_{\alpha_2}, g_{\beta_1}, g_{\beta_2}\}$$

$$u_j(\alpha_i, k, x) = u_j(\beta_i, k, x) = -1, \text{ if } k \neq i, x \neq g_t, (i, j = 1, 2),$$

$$u_j(\alpha_i, i, a_i) = u_j(\beta_i, i, b_i) = 10, (i, j = 1, 2),$$

$$u_j(\alpha_i, i, a_l) = u_j(\beta_i, i, b_l) = 9 \text{ if } l \neq i, (i, j = 1, 2),$$

$$u_1(t, i, g_t) = 0, u_2(t, i, g_t) = 15, i = 1, 2, t \in T^1,$$

$$u_j(t, i, g_{t'}) = 0 \text{ if } t' \neq t, (i, j = 1, 2).$$

Allocation: $f(\alpha_i) = (c_i, a_1)$ and $f(\beta_i) = (c_i, a_1)$ ($i = 1, 2$).

The expected payoff is 9.5.

partition: $\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}$

restriction on player 2 a_1 if $\{\alpha_1, \alpha_2\}$ is reported

g_{α_1} if player 1 did not participate.

Conclusion

- Contract space is universal (game independent)
- Cross-referential contracts are in the space
- Decentralized games implement a large set of outcomes