Buyer-Optimal Demand and Monopoly Pricing

Daniele Condorelli, Anne-Katrin Roesler, Balázs Szentes

May 24, 2016

Model

- there is a seller (of a single object) and a buyer
- buyer's value, v, $\sim$ CDF F on $[0, 1]$
- seller sets price p
- buyer observes $v\,(\sim F)$
- and accepts $p \Leftrightarrow v \geq p$
- seller's profit is p, buyer's payoff is $v - p$

Model I

- there is a seller (of a single object) and a buyer
- buyer chooses a CDF F supported on $[0, 1]$
- seller observes F and sets price p
- buyer observes $v\,(\sim F)$
- and accepts $p \Leftrightarrow v \geq p$
- seller's profit is p, buyer's payoff is $v - p$

Model II

- there is a seller (of a single object) and a buyer
- buyer's value, v, $\sim$ CDF F on $[0, 1]$
- buyer observes signal s about v
- seller sets price p
- and accepts $p \Leftrightarrow E(v|s) \geq p$
- seller's profit is p, buyer's payoff is $v - p$

Notations

$D(F,p) = 1 - F(p) + \Delta(F,p)$: *demand* at p

$\Delta(F,v)$: atom at v

$\Pi(F) = \max_p pD(F,p)$

$P(F) = \arg\max_p pD(F,p)$: profit maximizing prices

$U(F,p) = \int_p^1 v - p\,dF(v)$.

Reduce the problem to a one-dimensional one

1. For each p, find F_p which maximizes U s.t. $p \in P(F_p)$.

2. Find p^* s.t. F_{p^*} maximizes U in the class $\{F_p\}_p$.

Equal-revenue distributions

For each $\pi \in (0, 1]$, let F_π be defined as follows:

$$F_\pi(v) = \begin{cases} 0 & \text{if } v \in [0, \pi], \\ 1 - \frac{\pi}{v} & \text{if } v \in (\pi, 1), \\ 1 & \text{if } v = 1. \end{cases}$$

Note:

(1) $pD(F_\pi, p) = p[1 - F_\pi(p)] = p(\pi/p) = \pi$

(2) $\Pi(F_\pi) = \pi$ and $P(F_\pi) = [\pi, 1]$.

Lemma

Suppose $G \in \mathcal{F}$ and $p \in P(G)$. Then

(i) $F_p(v) \leq G(v)$ for all v.

(ii) $U(F_p, p) \geq U(G, p)$ and the inequality is strict if $F_p \neq G$

Proof of (i)

Since $p \in P(G)$,

$$vD(G, v) = v(1 - G(v) + \Delta(G, v)) \leq pD(G, p).$$

Proof of (i)

Since $p \in P(G)$,

$$vD(G, v) = v(1 - G(v) + \Delta(G, v)) \leq pD(G, p).$$

or equivalently,

$$1 - \frac{pD(G, p)}{v} + \Delta(G, v) \leq G(v).$$

Proof of (i)

Since $p \in P(G)$,

$$vD(G,v) = v(1 - G(v) + \Delta(G,v)) \leq pD(G,p).$$

or equivalently,

$$1 - \frac{pD(G,p)}{v} + \Delta(G,v) \leq G(v).$$

$\Delta(G,p) \in [0,1]$ and $D(G,p) \leq 1$, so

$$1 - \frac{p}{v} = F_\pi(v) \leq G(v).$$

Proof of (ii)

$$
\begin{aligned}
U\left(F_p, p\right) &= \int_p^1 v - p dF_p\left(v\right) \\
&\geq \int_p^1 v - p dG\left(v\right) = U\left(G, p\right)
\end{aligned}
$$

Theorem

In the unique equilibrium outcome,

- $F^* = F_{1/e}$
- $p^* = U(F^*, p^*) = \Pi(F^*) = 1/e.$

Proof

(1) Since $P\left(F_{1/e}\right) = [1/e, 1]$, the seller optimally sets $1/e$ after $F_{1/e}$.

Proof

(1) Since $P\left(F_{1/e}\right) = [1/e, 1]$, the seller optimally sets $1/e$ after $F_{1/e}$.

(2) Buyer's payoff:

$$\begin{aligned} U(F_\pi, \pi) &= \int_\pi^1 v - \pi dF_\pi(v) = \int_\pi^1 v f_\pi(v)\, dv + \Delta(F_\pi, 1) - \pi \\ &= \int_\pi^1 \frac{\pi}{v} dv = -\pi \log \pi, \end{aligned}$$

Proof

(1) Since $P\left(F_{1/e}\right) = [1/e, 1]$, the seller optimally sets $1/e$ after $F_{1/e}$.

(2) Buyer's payoff:

$$\begin{aligned} U\left(F_{\pi}, \pi\right) &= \int_{\pi}^{1} v - \pi dF_{\pi}(v) = \int_{\pi}^{1} v f_{\pi}(v)\, dv + \Delta\left(F_{\pi}, 1\right) - \pi \\ &= \int_{\pi}^{1} \frac{\pi}{v} dv = -\pi \log \pi, \end{aligned}$$

- $-\pi \log \pi$ is maximized at $1/e$
- $U\left(F_{1/e}, 1/e\right) = -1/e \log 1/e = 1/e$
- $\Pi\left(F_{1/e}\right) = 1/e.$

Welfare

- first-best total surpluss: 1
- equilibrium total surpluss: $2/e\,(\approx 0.73)$

$\Rightarrow$ welfare loss: $\approx 27\%$

Literature Review

Information rent in hold-up problems:

Lau (2008) and Hermalin and Katz (2009).

Equal-Revenue Distribution:

Bergemann and Schlag (2008): min-max regret criterion

Neeman (2003) minimizes profit/ expected value in auctions

Hart and Nisan (2012) approximate the seller's maximum revenue in a multiple-item auction.

An extention

- What if $Ev \le \mu$?

Model II

- there is a seller (of a single object) and a buyer
- buyer's value, v, $\sim$ CDF $F\,[0,1]$, $Ev = \mu$
- buyer observes signal s about v
- seller sets price p
- and accepts $p \Leftrightarrow E\,(v|s) \geq p$
- seller's profit is p, buyer's payoff is $v - p$

without loss: $E(v|s) = s \sim G\,[0,1]$

$\Leftrightarrow$ $v = s + \varepsilon_s$ for some ε_s s.t. $E(\varepsilon_s|s) = 0$

$\Leftrightarrow$ F is a mean-preserving spread of G

$$\mathcal{G}_F = \left\{ G \in \mathcal{G} : \int_0^x F(v)\,dv \geq \int_0^x G(s)\,ds \text{ for all } x \in [0,1]\,,\ \int_0^1 s dG(s) = \mu \right\}.$$

Buyer's Problem

$$\max_{G \in \mathcal{G}_F} \int_p^1 s - p dG(s)$$

$$\text{s.t. } p \in \arg\max_s sD(G, s).$$

Truncated Pareto

$$G_q^B(s) = \begin{cases} 0 & \text{if } s \in [0, q), \\ 1 - \frac{q}{s} & \text{if } s \in [q, B), \\ 1 & \text{if } s \in [B, 1]. \end{cases}$$

Lemma

For all $G \in \mathcal{G}_F$, $p \in P(G)$ $\exists! B \in [\pi(G), 1]$ such that

(i) G is a mean-preserving spread of G_π^B,

(ii) $G_\pi^B \in \mathcal{G}_F$,

(iii) $\int_\pi^1 s - \pi dG_\pi^B(s) \geq \int_p^1 s - pdG(s)$, strict if $D(G,p) < 1$.

Proof

Step 1: $\exists! B$ s.t. $\int s dG_{\pi}^{B}(s) = \mu$.

$p \in P(G)$, so $sD(G, s) \leq \pi$ for all $s \in [0, 1]$:

Proof

Step 1: $\exists! B$ s.t. $\int s dG_{\pi}^{B}(s) = \mu$.

$p \in P(G)$, so $sD(G, s) \leq \pi$ for all $s \in [0, 1]$:

$$1 - \frac{\pi}{s} + \Delta(G, s) \leq G(s) \Rightarrow G_{\pi}^{1}(s) \leq G(s).$$

Proof

Step 1: $\exists ! B$ s.t. $\int s d G_{\pi}^{B}(s)=\mu$.

$p \in P(G)$, so $s D(G, s) \leq \pi$ for all $s \in[0,1]$:

$$1-\frac{\pi}{s}+\Delta(G, s) \leq G(s) \Rightarrow G_{\pi}^{1}(s) \leq G(s).$$

Hence

$$\int_{0}^{1} s d G_{\pi}^{1}(s) \geq \int_{0}^{1} s d G(s)=\mu \geq p D(G, p)=\pi=\int_{0}^{1} s d G_{\pi}^{\pi}(s).$$

$\int_{0}^{1} s d G_{\pi}^{B}(s)$ is strictly increasing $\Rightarrow B \in[\pi, 1]$ s.t. $\int_{0}^{1} s d G_{\pi}^{B}(s)=\mu$.

Part (i) G is a MPS of G_π^B

Part (i) G is a MPS of G_{π}^{B}

If $s \leq B$, then, since $G_{\pi}^{B}(x) = G_{\pi}^{1}(x)$ on $[0, B)$,

$$\int_0^s G(x)\,dx \geq \int_0^s G_{\pi}^{B}(x)\,dx.$$

Part (i) G is a MPS of G_{π}^{B}

If $s \leq B$, then, since $G_{\pi}^{B}(x) = G_{\pi}^{1}(x)$ on $[0, B)$,

$$\int_0^s G(x)\,dx \geq \int_0^s G_{\pi}^{B}(x)\,dx.$$

If $s \geq B$ then

$$\begin{aligned}
\int_0^s G(x)\,dx &= [1-\mu] - \int_s^1 G(x)\,dx \\
&\geq [1-\mu] - (1-s) \\
&= [1-\mu] - \int_s^1 G_{\pi}^{B}(x)\,dx = \int_0^s G_{\pi}^{B}(x)\,dx.
\end{aligned}$$

Part (ii): $G_\pi^B \in \mathcal{G}_F$

F is a MPS of G

G is a MPS of G_π^B

So, F is a MPS of G_π^B

Part (iii):

WHTS: $\int_{\pi}^{1} s - \pi dG_{\pi}^{B}(s) \geq \int_{p}^{1} s - p dG(s)$, strict if $D(G, p) < 1$

Part (iii):

WHTS: $\int_{\pi}^{1} s - \pi dG_{\pi}^{B}(s) \geq \int_{p}^{1} s - pdG(s)$, strict if $D(G,p) < 1$

$$\int_{\pi}^{1} s - \pi dG_{\pi}^{B}(s) = \mu - \pi \geq \int_{p}^{1} s - pdG(s)$$

Lemma

For all $G \in \mathcal{G}_F$, $p \in P(G)$ $\exists! B \in [\pi(G), 1]$ such that

(ii) $G_\pi^B \in \mathcal{G}_F$,

(iii) $\int_\pi^1 s - \pi dG_\pi^B(s) \geq \int_p^1 s - pdG(s)$, strict if $D(G, p) < 1$.

Lemma

For all $G \in \mathcal{G}_F$, $p \in P(G)$ $\exists! B \in [\pi(G), 1]$ such that

(ii) $G_\pi^B \in \mathcal{G}_F$,

(iii) $\int_\pi^1 s - \pi dG_\pi^B(s) \geq \int_p^1 s - pdG(s)$, strict if $D(G, p) < 1$.

- We can restrict attention to $\left\{G_\pi^B\right\}$
- Buyer's payoff if G_π^B: $\mu - \pi$
- $p^* = \min\left\{\pi : \exists\ B \in [\pi, 1] \text{ s.t. } G_\pi^B \in \mathcal{G}_F\right\}$

Theorem

$\left(G_{p^*}^{B^*}, p^*\right)$ is buyer-optimal. If (G, p) is buyer-optimal, then

(i) $p = p^*$,

(ii) $D(G, p) = 1$ and

(iii) G is a MPS of $G_{p^*}^{B^*}$.

$\pi = pD(G, p)$

By the Lemma

$$\begin{aligned}
\int_p^1 s - p dG(s) &\leq \int_\pi^1 s - \pi dG_\pi^B(s) = \mu - \pi \\
&\leq \mu - p^* = \int_{p^*}^1 s - p^* dG_{p^*}^{B^*}(s).
\end{aligned}$$

$\pi = pD\left(G, p\right)$

By the Lemma

$$\begin{aligned} \int_p^1 s - p dG\left(s\right) \leq \int_\pi^1 s - \pi dG_\pi^B\left(s\right) = \mu - \pi \\ \leq \ \ \mu - p^* = \int_{p^*}^1 s - p^* dG_{p^*}^{B^*}\left(s\right). \end{aligned}$$

$\Rightarrow$ the inequalities equalities.

$\pi = pD(G, p)$

By the Lemma

$$\begin{aligned} \int_p^1 s - p dG(s) &\leq \int_\pi^1 s - \pi dG_\pi^B(s) = \mu - \pi \\ \leq \quad \mu - p^* &= \int_{p^*}^1 s - p^* dG_{p^*}^{B^*}(s). \end{aligned}$$

$\Rightarrow$ the inequalities equalities.

By the Lemma, first inequality is strict unless $D(G, p) = 1 \Rightarrow$ (ii).

$\pi = pD(G, p)$

By the Lemma

$$\int_p^1 s - p dG(s) \le \int_\pi^1 s - \pi dG_\pi^B(s) = \mu - \pi$$
$$\le \ \mu - p^* = \int_{p^*}^1 s - p^* dG_{p^*}^{B^*}(s).$$

$\Rightarrow$ the inequalities equalities.

By the Lemma, first inequality is strict unless $D(G, p) = 1 \Rightarrow$ (ii).

The second inequality is strict unless $\pi = p^* \Rightarrow$ (i).

$\pi = pD\left(G, p\right)$

By the Lemma

$$\begin{aligned} \int_p^1 s - p dG\left(s\right) &\leq \int_\pi^1 s - \pi dG_\pi^B\left(s\right) = \mu - \pi \\ &\leq \mu - p^* = \int_{p^*}^1 s - p^* dG_{p^*}^{B^*}\left(s\right). \end{aligned}$$

$\Rightarrow$ the inequalities equalities.

By the Lemma, first inequality is strict unless $D\left(G, p\right) = 1 \Rightarrow$ (ii).

The second inequality is strict unless $\pi = p^* \Rightarrow$ (i).

Since $\pi = p^*$, part (i) of Lemma $\Rightarrow$ (iii).

Observation.

Suppose that G is a signal distribution. Then $\Pi(G) \geq p^*$.

Observation.

Suppose that G is a signal distribution. Then $\Pi(G) \geq p^*$.

Proof.

By the lemma, $\exists B \in [\Pi(G), 1]$ s.t.

(i) $G^B_{\Pi(G)}$ is a signal distribution and

(ii) $\Pi(G) \in P\left(G^B_{\Pi(G)}\right)$.

Hence, $\Pi(G) \geq p^*$.

Example: $F(v) = v$

perfect learning: $p = 0.5$, $D(F, p) = 0.5$, $U(F, p) = 0.125$, DWL$= 0.125$

Example: $F(v) = v$

perfect learning: $p = 0.5$, $D(F, p) = 0.5$, $U(F, p) = 0.125$, DWL$= 0.125$

optimal learning: $p^* \approx 0.2$

Example: $F(v) = v$

perfect learning: $p = 0.5$, $D(F, p) = 0.5$, $U(F, p) = 0.125$, DWL$= 0.125$

optimal learning: $p^* \approx 0.2$, $B^* \approx 0.87$

Example: $F(v) = v$

perfect learning: $p = 0.5$, $D(F, p) = 0.5$, $U(F, p) = 0.125$, DWL$= 0.125$

optimal learning: $p^* \approx 0.2$, $B^* \approx 0.87$, $U\left(0.2, G_{0.2}^{0.87}\right) \approx 0.3$

Example: $F(v) = v$

perfect learning: $p = 0.5$, $D(F, p) = 0.5$, $U(F, p) = 0.125$, DWL$= 0.125$

optimal learning: $p^* \approx 0.2$, $B^* \approx 0.87$, $U\left(0.2, G_{0.2}^{0.87}\right) \approx 0.3$, DWL$= 0$

Related Questions

1. Optimal Distributions in the Moussa-Rosen model

3. Costly Learning

2. Intermediation