MISCELLANEOUS RESULTS ON SUPERSOLVABLE GROUPS

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Abstract. The paper contains two theorems generalizing the theorems of Huppert concerning the characterization of supersolvable and p-supersolvable groups, respectively. The first of these gives a new approach to prove Huppert's first named result. The second one has numerous applications in the paper. The notion of balanced pairs is introduced for non-conjugate maximal subgroups of a finite group. By means of them some new deep results are proved that ensure supersolvability of a finite group.

1. Introduction

We recall Huppert's characterizations for (p-)supersolvable groups.

(i) *Let p be some prime. A finite group is p-supersolvable iff it is p-solvable and the index of any maximal subgroup is either p or coprime to p.*

(ii) *A finite group is supersolvable iff all maximal subgroups of it have prime index.*

(See in [10, 9.2 – 9.5 Satz], pp. 717-718.) Among others it immediately follows that the class (formation) of finite supersolvable groups is saturated, i.e. the supersolvability of $G/\Phi(G)$ is equivalent to the supersolvability of G itself. Result (ii) turned out to be of fundamental importance and it inspired a long series of further achievements. Concentrating to various characterizations of finite supersolvable groups by means of the index of maximal subgroups or the existence of cyclic supplements to maximal subgroups we mention [7], [12] and [15] from the past; cf. also [16] (or [6, Thm. 2.2], p 483). Concerning more recent developments we refer to the articles [2], [4],[5], [8], [14] and [17] from the great number of contributions in this special area.

Notation and terminology. In the paper G will always denote a finite group, $\pi(G)$ stands for the set of primes dividing $|G|$, the order of G. We set $\pi := \{p_i | 1 \leq i \leq n\}$, such that the sequence $\{p_i\}_1^n$ is strictly decreasing. For each k, $1 \leq k \leq n$ let $\sigma_k = \{p_i | 1 \leq i \leq k\}$. We shall say that G has a *distinguished Sylow tower* if for all k, $1 \leq k \leq n-1$, there are normal Hall σ_k-subgroups G_k in G. E.g. supersolvable groups always have distinguished Sylow-towers. We choose the notation $p_1 = r$ and $p_n = s$. Let R and S always denote a Sylow r-subgroup and a Sylow s-subgroup of G, respectively.

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2. Preparatory results

Lemma 2.1. *Let H be a Hall σ-subgroup of the finite group G, and let L be a normal subgroup of G containing H. If H has a distinguished Sylow tower, then $G = LN_G(H)$.*

Proof. Using the condition imposed on H one may prove using induction on $|\sigma|$ that any two Hall σ-subgroups of L that have distinguished Sylow tower are conjugate in G. Making use of this, one may finish the proof in the usual way. $\square$

Lemma 2.2. *Let $p \in \pi$ be a prime and let σ be a subset of π containing p. Let M be a σ-supersolvable subgroup of G of index p. If $P \in Syl_p(G)$, then the group $N = N_G(P)$ is σ-supersolvable.*

Proof. We use induction on the order of G. Since $G = MN$, we get that $|N : N \cap M| = |G : M| = p$. If $N \neq G$ then the assertion follows by induction. So we may assume that P is normal in G. If $|Z(P) \cap M| = 1$ then $G/Z(P) \simeq M$ and we are done. Let $|P| = p^a$, we may assume that $a > 1$. Then $M \cap P$ and $M \cap Z(P)$ are nontrivial normal subgroups of M. Let L be a minimal normal subgroup of M contained in $M \cap Z(P)$. Then $|L| = p$, since M is σ-supersolvable and $p \in \sigma$. On the other hand L is normal in G. Set $\overline{G} = G/L$. Then $\overline{G}$ is σ-supersolvable by induction. Thus G is also σ-supersolvable. $\square$

As an application of the preceding lemma we prove

Theorem 2.3. *A finite group G is supersolvable iff the following condition α) holds:*

α) *For every prime $p \in \pi$ the group G has a maximal subgroup M_p such that*
(i) $|G : M_p| = p$,
(ii) *M_p is p-supersolvable.*

Proof. The necessity is obvious. To prove the sufficiency we proceed by induction on $|G|$. Choose $q \in \pi$ different from p. Then $|M_p : M_p \cap M_q| = q$ and $M_p \cap M_q$ is q-supersolvable. Thus M_p satisfies condition α). Hence by induction M_p is supersolvable. As s is the smallest prime dividing $|G|$, we obtain by $|G : M_s| = s$ that M_s is normal in G. We also have that $R \in Syl_r(G)$ is contained in M_s. Since M_s is supersolvable, R is characteristic in M_s, and hence R is normal in G. Let us choose in Lemma 2.2 σ to be equal to π, let $M := M_r$, and let $p := r$. Then we get that $G = N_G(R)$ is supersolvable. $\square$

The following result will be used in the proof of a theorem in the next section

Lemma 2.4. *Let $p \in \pi$ be a given prime and $P \in Syl_p(G)$. Assume that*
(i) $P \lhd G$,
(ii) $\Phi(P) = 1$,
(iii) *P contains a unique minimal normal subgroup of G.*

If G contains a maximal subgroup M of index p, then M is a p'-subgroup.

Proof. Assume that $|P| = p^a$, $a \geq 2$. Then $P \cap M$ is a nontrivial normal subgroup of G. Thus $P \cap M$ must contain the unique minimal normal subgroup of G. Let H be a Zassenhaus-complement to P in G. By the Theorem of Maschke, we get an H-invariant complement N to $P \cap M$ in P. Also N is normal in G, contradicting (iii). $\square$

3. A generalization of Huppert's fundamental theorem

Let k be an integer. Let σ be a nonempty subset of π and let H_σ be the set of all Hall σ-subgroups of G. Define the set H_k by $\mathsf{H}_k = \{\mathsf{H}_\sigma \mid \sigma \subseteq \pi, |\sigma| = k\}$. A maximal subgroup M will be called *k-sentinel* if for some $H \in \mathsf{H}_\sigma \in \mathsf{H}_k$, $N_G(H) \leq M$. In this section we will prove the following generalization of Huppert's fundamental theorem :

Theorem 3.1. *Let G be a finite group and let k be an integer satisfying $1 \leq k \leq n$. Assume that*

(i) *G has a Hall σ_k-subgroup with a distinguished Sylow tower;*
(ii) *for all $\sigma \subseteq \pi$ with $|\sigma| = k$, $\mathsf{H}_\sigma \neq \emptyset$;*
(iii) *every k-sentinel subgroup M of G has prime index in G.*

Then G is supersolvable.

Remark 3.2. *Observe that in the case $k = 1$ (i) and (ii) are automatically satisfied by Sylow's theorem. So in this case (iii) is the only requirement, which is in fact a weaker condition than Huppert's original one.*

Proof. (of Theorem 3.1): The proof is by induction on $|G|$ and k. Let $K \in \mathsf{H}_{\sigma_k}$ having a distinguished Sylow tower (guaranteed by (i)). We prove that K is normal in G. Otherwise there would be a maximal subgroup M containing $N_G(K)$. By (iii) M is of prime index p in G. Here $p = p_\ell$, where $k+1 \leq \ell \leq n$. Let $C = Core_G(M)$, then $K \leq C$ as G/C is a permutation group of degree p_ℓ. By Lemma 2.1 $G = CN_G(K)$. This leads to the contradiction $G = M$. Thus K has to be normal in G. Hence $R \in Syl_r(G)$ is also normal in G. Let $\overline{G} = G/R$. Then $\overline{G}$ is supersolvable by induction. Note that in the case $k = 1$ $\overline{G}$ satisfies the conditions with $k = 1$, and in the cases $k \geq 2$ it satisfies the conditions with $k-1$ in the place of k. In particular, G is a solvable group. Since the conditions of the theorem are inherited to factor groups, we have that G has a unique minimal normal subgroup N. Since R is normal in G, $N \leq R$. Let L be a Hall r'-subgroup of G. As $\overline{G} \simeq L$, L is supersolvable. Let $\tau = \sigma_{k+1} \backslash \{p_1\}$ and let F be a Hall τ-subgroup of G contained in L. Then F cannot be normal in G, otherwise it would contain the minimal normal subgroup, which is an r-group. Thus $L \leq N_G(F) < G$. Let us choose a maximal subgroup M in G containing $N_G(F)$. By assumption M is of prime index r in $N_G(R)$. By Lemma 2.4 if R were elementary abelian, then M would be an r'-group, and hence $|R| = r$, and thus G would be supersolvable. Thus we may assume that $\Phi(R) \neq 1$. Then $N \leq \Phi(R)$; our aim is to show $|N| = r$. Since $R \cap M$ is normal in M and $R \cap M$ is maximal in R, $R \cap M$ is normal in G. If $\Phi(R \cap M)$ is not 1 then by induction $G/\Phi(R \cap M)$ and thus $M/\Phi(R \cap M)$ and $M/\Phi(M)$ are also supersolvable. Then every maximal subgroup of M is of prime index, thus by induction M is supersolvable. Using Lemma 2.2 for $p = r$ we have that G is also supersolvable. Thus $\Phi(M \cap R) = 1$. As $\tilde{G} = G/\Phi(R)$ is supersolvable, its maximal subgroups have prime index. Let $\tilde{R}$ be the image of R in $\tilde{G}$. Then $\tilde{R}$ is the direct sum of L-invariant one-dimensional subspaces $\tilde{R}_i$ $i = 1, ..., t$. Let R_i be the inverse image of $\tilde{R}_i$ in G and $Q_i = \langle R_j | j \neq i\rangle$. Then Q_i is of index r in R. Then Q_iL is a maximal subgroup of G of index r. As $R \cap Q_iL = Q_i$ we may assume by the above that $\Phi(Q_i) = 1$, i.e. all Q_i are elementary abelian. Since $\Phi(R) \neq 1$ we have that $t \leq 2$. Thus $|R : \Phi(R)| = r^2$. Let a and b be two elements with $a \in R_1 \backslash \Phi(R)$ and

$b \in R_2 \backslash \Phi(R)$, respectively. Then $a^r, b^r \in \Phi(R) \le Z(R)$ implies by $R = \langle a, b\rangle$ that $R' = \langle [a, b]\rangle$ has order r, therefore $R' = N$. □

Remark 3.3. *The method of the above proof gives a conceptual proof for Huppert's original result. The main steps are the following: By induction on $|G|$ we have that R is normal and G has a unique minimal normal subgroup N, since the conditions are inherited to epimorphic images. Now $\Phi(R) \neq 1$. For a maximal subgroup M of index r we may suppose that $\Phi(R \cap M) = 1$. In the end we have that R' is a normal subgroup of order r, thus since G/R' is supersolvable, and we get that G is also supersolvable.*

We shall show that in the case $k = n - 1$ (similarly to the case $k = 1$) the condition (i) in Theorem 3.1 can be dropped. To do so we need

Theorem 3.4. *Let G be a finite p-solvable group, $H \in Hall_{p'}(G)$. If for every maximal subgroup A of G satisfying $N_G(H) \le A$, $|G : A| = p$ also holds, then G is p-supersolvable.*

Proof. The proof is by contradiction. Let G be a minimal counterexample. Then $O_{p'}(G) = 1$. Since the condition is inherited to epimorphic images, G has a unique minimal normal subgroup N which is an elementary abelian p-group. As $|N| \neq p$, H cannot be a maximal subgroup containing $N_G(H)$.
Case 1: $N \not\le \Phi(G)$. Then there is a maximal subgroup M of G with $N \not\le M$. Thus $G = MN$ and $M \cap N = 1$. Since $G/N \simeq M$ M is p-supersolvable by induction. Then M' is p-nilpotent, thus $O_{p'}(M') \in Hall_{p'}(M')$.
Case 1/a: Let $O_{p'}(M') \neq 1$. Then let $H \in Hall_{p'}(G)$ contained in M. Then $O_{p'}(M') = M \cap NO_{p'}(M') = H \cap NO_{p'}(M')$. Since $NO_{p'}(M')$ is a normal subgroup of G and $O_{p'}(G) = 1$, thus $N_G(H) \le N_G(O_{p'}(M')) = M$. By assumption $|G : M| = p$, so $|N| = p$, and we have that G is p-supersolvable.
Case 1/b: Let $O_{p'}(M') = 1$. Let $P \in Syl_p(G)$. Observe that $P \cap M \neq 1$, as $P \cap M = 1$ would imply that $M = H$. Since $P = N(M \cap P)$, $M \cap P \in Syl_p(M)$ and $M' = O_p(M') \le M \cap P$. So $M \cap P$ is normal in G and $N \not\le M \cap P$, which is a contradiction. Hence Case 1/b cannot hold.
Case 2: In this case $N \le \Phi(G)$. Let $\overline{G} = G/N$. Then $\overline{G}$ is p-supersolvable by induction, and so is $G/\Phi(G)$. Hence by Huppert's theorem G is also p-supersolvable. □

From this we obtain

Theorem 3.5. *Let G be a finite group. Suppose that for every prime $p \in \pi$, G has a Hall p'-subgroup $G_{p'}$. Assume further that every maximal subgroup M of G satisfying $N_G(G_{p'}) \le M$ for some $p \in \pi$ has prime index in G. Then G is a supersolvable group.*

Proof. The existence of Hall p'-subgroups yields solvability by P. Hall's criterion, [9]. Thus, since the conditions of Theorem 3.1 are satisfied for every $p \in \pi$, G is p-supersolvable for every $p \in \pi$. Hence G is supersolvable. □

We conclude the section with the following special result

Theorem 3.6. *Let G be a finite group. Assume that the Sylow p-subgroups of G are all abelian, with distinct invariants. Then G is supersolvable.*

Proof. We will construct a chief series between $\Phi(R)$ and G such that all chief factors have prime order. Since $\Phi(R) \le \Phi(G)$ will be satisfied, by 3.1 the supersolvability of G will follow. We observe first that since S has distinct invariants, G has a normal s-complement. See 2.7 Satz, p. 419. in [10]. Repeating this argument, we have that G has a distinguished Sylow tower. In particular R is normal in G. Thus $\Phi(R) \le \Phi(G)$. Let now R have invariants $(r^{n_1}, ..., r^{n_m})$ $n_1 > n_2 > ... > n_m$. Let further a_i $i = 1, ..., m$ be basis elements with $|a_i| = n_i$ $i = 1, ..., m$. Let us define the characteristic subgroups R_i $i = 1, .., m$ in R as follows: $R_0 = \Phi(R)$, $R_i = \Phi(R)\Omega_{n_{m+1-i}}(R)$ $i = 0, ..., m$. It is easy to see that the subgroups R_i are normal in G satisfying $|R_{i+1} : R_i| = r$ $i = 0, .., m-1$. Since by induction we may suppose that G/R is supersolvable, one may simply complete the subgroups R_i to a chief series of the desired type. □

We mention the following consequence:

Theorem 3.7. *Let G be a finite group. Assume that the Sylow subgroups of G are all abelian and they have distinct invariants. If for any pair (p, q) of distinct primes from π $p \not\equiv 1 \pmod q$ holds, then G is an abelian group.*

Proof. The result is a consequence of 3.6 and a result of Rédei [13]. □

4. Balance in finite groups

Our first result is

Theorem 4.1. *Let G be a finite group and let H and K be maximal subgroups of G. Assume that the following conditions are satisfied:*

(i) *H and K are non-conjugate supersolvable groups,*
(ii) *$|G : H|$ and $|G : K|$ are prime powers.*

Then G is a solvable group.

Proof. We prove by induction on $|G|$. The assumptions are obviously hereditary to factor groups. Therefore it is enough to find a proper solvable normal subgroup. Let $|G : H| = p^a$ and $|G : K| = q^b$ and $p \ge q$. Note, that $p = r$ can be assumed. Namely for $r > p$ we have that R is normal in H and K hence also in G and so we are done. We may also assume that $p \ne q$. If $p = q = r$ then either $R \cap H \ne 1$ or $R \cap K \ne 1$. Otherwise H and K would be Hall r'-subgroups having distinguished Sylow towers. Just like in the proof of Lemma 2.1 we have that H and K are conjugate, contradicting assumption (i). Say $R \cap H \ne 1$. Then choosing R suitably $R \cap H \in Syl_r(H)$ and this implies $R \cap H \lhd G$ and so we are done. Thus we may assume that $p = r > q$. By Burnside's $p^a q^b$-theorem, [1] we may directly assume that $|\pi| \ge 3$.
We will prove the rest of the theorem in several steps:

Step 1: We may assume that $H \in Hall_{r'}(G)$ and q is the maximal prime in $\pi \backslash \{p\}$. We argue by contradiction. If $p = r$ and $R \cap H \ne 1$ then we have seen above that $R \cap H \lhd H$ can be assumed, and this implies $R \cap H \lhd G$. Thus we may assume that $H \in Hall_{r'}(G)$. Let t be the maximal prime in $\pi \backslash \{p\}$. Suppose $t \ne q$. Since $(|G : K|, |G : H|) = 1$, $G = HK$. Thus $|G : H| = |K : K \cap H|$. From this we have that $|G : H \cap K| = |G : K||K : K \cap H| = |G : K||G : H| = p^a q^b$.Since $t \ne q$ we may choose $T \in Syl_t(G)$ contained in $H \cap K$.The supersolvability of H implies that T is normal in H. Then $T^G = T^{HK} = T^K \le K$. Hence T^G is a solvable normal subgroup in G. Thus we may assume that $t = q$.

Step 2: Let $\sigma = \{r, q\}$. We can assume that the Hall σ'-subgroups of G are abelian. Let $Q \in Syl_q(G)$ be contained in H, then Q is normal in H. We want to show that $H' \leq Q$. Assume $H' \not\leq Q$. Then there is a prime $u \in \pi$, with $p = r \neq u \neq q$ dividing $|H'|$. Since H is supersolvable, H' is nilpotent. Let $U \in Syl_u(G)$ be contained in $H \cap K$. Let U_0 be the unique Sylow u-subgroup of H'. Then $U_0 \neq 1$ is a proper normal subgroup of H. We have that $U_0 \leq U$. Then $N = U_0^G = U_0^{HK} = U_0^K \leq U^K \leq K$ is a solvable normal subgroup of G and we are done. Thus we may assume that $H' \leq Q$. Let L be a Zassenhaus complement to Q in H. Then $L \simeq H/Q$ is abelian and $L \in Hall_{\sigma'}(G)$. Then by a result of Wielandt [18] all Hall σ'-subgroups of G are abelian, and we are done.

Step 3: Conclusion of the proof.
Let $L \in Hall_{\sigma'}(G)$ contained in $H \cap K$. Since H and K are both supersolvable, $N_H(L) = C_H(L)$ and $N_K(L) = C_K(L)$. We will prove that $N_G(L) = C_G(L)$. Let $x \in N_G(L)$ be fixed. Since $G = HK$, $x = yz$, where $y \in H$ and $z \in K$. Thus $L^y = L^{z^{-1}}$ and so L and L^y are Hall σ'-subgroups of $H \cap K$. Since L is abelian, the cited result of Wielandt implies that $L^y = L^w$ for suitable $w \in H \cap K$. Hence $yw^{-1} \in N_H(L) = C_H(L)$ and $z^{-1}w^{-1} \in N_K(L) = C_K(L)$. Thus $x = yz = yw^{-1}(z^{-1}w^{-1})^{-1} \in C_G(L)$ and we have that $L \leq Z(N_G(L))$. By the same argument as in Burside's transfer theorem it follows that G has a normal Hall σ-subgroup. Since $|\sigma| = 2$ the Hall σ- subgroup is also solvable. Thus we get again a solvable normal subgroup and we are done. □

Before formulating the next result, we introduce the following notation. For any prime $p \in \pi = \pi(G)$, let μ_p be the set defined by $\mu_p = \{u \in \pi | u > p.\}$ In particular $\mu_r = \emptyset$.

Theorem 4.2. *Let G be a finite group and let H and K be non-conjugate supersolvable subgroups of G of prime indices $|G : H| = p$, $|G : K| = q$, with $p \geq q$. Then G has a normal, supersolvable Hall μ_p-subgroup D such that $\overline{G} = G/D$ is supersolvable. G is supersolvable iff the following condition $\beta)$ holds:*
$\beta)$ For every maximal subgroup M of G which contains the normalizer of some Hall u'-subgroup of G for suitable prime $u \in \mu_p$, $|G : M| = u$.

Proof. By Theorem 4.1 G is solvable. Both H and K contain supersolvable normal Hall μ_p-subgroups D_1, D_2, being Hall μ_p-subgroups of G, as well. Since these contain distinguished Sylow towers, these subgroups are conjugate in G. Since H and K are not conjugate, we get that $D_1 = D_2 = D$ is normal in G. We want to prove that $\overline{G} = G/D$ is supersolvable. Taking $\overline{G}$ instead of G, we may assume that $p = r$ and $\mu_p = \emptyset$. If $q < p = r$ then let $R \in Syl_r(K)$. Then $R \in Syl_r(G)$. Since $K \leq N_G(R)$, $|G : K| = q$ and $|G : N_G(R)| \equiv 1 \bmod q$, R has to be normal in G. Now by Lemma 2.2 applied to the supersolvable subgroup H of index r, we have that G is supersolvable. If $q = p = r$, we note that H and K cannot be at the same time Hall r'-subgroups of G, since they are not conjugate. Let $R \in Syl_r(G)$, then say $H \cap R \neq 1$. We may assume that $R \cap H$ is normal in H and since this subgroup is of index r in R, also R is normal in G. Applying Lemma 2.2 for H again we have that G is supersolvable.
So in any case we have that G/D is supersolvable. Assume $\beta)$. Since G is solvable,

every maximal subgroup is of prime power index. Let $u \in \mu_p$. Then G is also u-solvable. By condition β) and Theorem 3.4 G is u-supersolvable. So every maximal subgroup of u-power index is already of index u in G. Let $u \in \pi$ be a prime but $u \notin \mu_p$. If M is a maximal subgroup of G of u-power index in G then $D \leq M$ and $\overline{M} = M/D$ is a maximal subgroup of $\overline{G} = G/D$. Since $\overline{G}$ is supersolvable, $|\overline{G} : \overline{M}| = u$, and hence $|G : M| = u$. Thus every maximal subgroup in G has prime index, and by Huppert's theorem we obtain that G is supersolvable. □

We will need the following

Definition 4.3. *Let G be a finite group, let H and K be non-conjugate maximal subgroups of G. We say that the pair (H, K) is balanced, if $H \cap K$ is a maximal subgroup both in H and K. We say that (H, K) is balanced with respect to H, if $H \cap K$ is a maximal subgroup of H.*

Lemma 4.4. *Let G be a finite solvable group, let A and B be maximal subgroups of G such that A is a supersolvable group and the pair (A, B) is balanced with respect to A. Then $|G : B|$ is a prime number and for every $x \in G$ the pair (A, B^x) is balanced with respect to A.*

Proof. Let us fix $x \in G$. Then $|B| = |B^x|$ and B^x is a maximal subgroup of G. Since the pair (A, B) is balanced with respect to A, A and B are not conjugate in G. Thus A and B^x are not conjugate in G, either. Since G is solvable,by a result of Ore [10, II. Satz 3.9], $G = AB = AB^x$. Hence $|A : A \cap B^x| = |G : B^x| = |G : B| = |A : A \cap B|$. Since A is supersolvable and (A, B) is balanced with respect to A, $|A : A \cap B|$ must be a prime. Thus $A \cap B^x$ is also maximal in A. So the pair (A, B^x) is balanced with respect to A, too. □

Theorem 4.5. *Let G be a finite group that contains a balanced pair (H, K) of maximal subgroups such that both H and K are supersolvable groups and both of them contain the normalizer of a Sylow-complement in G, say $N_G(U) \leq H$ and $N_G(V) \leq K$, where $U \in Hall_{p'}(G)$ and $V \in Hall_{q'}(G)$ for some primes p and q with $p \geq q$. Assume that whenever M is an arbitrary maximal subgroup of G that contains the normalizer of a Hall u'-subgroup for some $u \in \mu_p$, then for some $Y_M \in \{H, K\}$ the pair (Y_M, M) is balanced with respect to Y_M. Then G is a supersolvable group.*

Proof. By our conditions H and K are non-conjugate maximal subgroups of G. They are both supersolvable and both are of prime power index in G. Thus by Theorem 4.1 G is a solvable group. By Lemma 4.4 we have that $|G : H| = p$ and $|G : K| = q$. Let now M be an arbitrary maximal subgroup of G that contains the normalizer of a Hall u'-subgroup of G for some $u \in \mu_p$. Since by assumption Y_M and M are not conjugate, by the above theorem of Ore, $G = Y_M M$. Since the pair (Y_M, M) is balanced with respect to Y_M, we have that $|G : M| = |Y_M : M \cap Y_M| = u$. Thus by 4.2 G is a supersolvable group. □

For the next result we will need the following

Definition 4.6. *Let G be a finite group, let H, K and L be given maximal subgroups of G. The ordered triple (H, K, L) will be called regular if the following conditions are satisfied:*

(i) *H and K are supersolvable groups and the group L is solvable.*

(ii) *Every pair* (X, Y) *with* $\{X, Y\} \subseteq \{H, K, L\}$ *is either balanced or* X *and* Y *are conjugate in* G.
(iii) *The pair* (H, K) *is balanced.*
(iv) *The index* $|G : L|$ *is a prime.*

Remark 4.7. *If* G *is a supersolvable group, then every maximal subgroup of* G *is a supersolvable group of prime index. If* H *and* K *are non-conjugate maximal subgroups of* G *then the pair* (H, K) *is balanced. Thus for each choice of maximal subgroups* H,K *and* L, *if they are not all conjugate in* G, *then the triple* (H, K, L) *is regular if and only if with a suitable choice of the notation some permuted triple of* (H, K, L) *is regular.*

First we prove

Lemma 4.8. *If the finite group* G *has a regular triple* (H, K, L) *then* G *is solvable.*

Proof. Consider first the case when L is conjugate to H or to K. We may assume that H and L are conjugate. Then $|G : H| = |G : L| = t$, where t is a prime. Since the pair (H, K) is balanced, H and K are not conjugate and $H \cap K$ is a maximal subgroup of H and also of K. Since H and K are supersolvable, $|H : H \cap K|$ and $|K : H \cap K|$ are prime numbers. These imply that $|G : K| = |G : H|\frac{|H:H\cap K|}{|K:H\cap K|}$ is a prime. Hence G is solvable by Theorem 4.1. Thus we may assume that for any pair (X, Y) with $\{X, Y\} \subseteq \{H, K, L\}$, the pair (X, Y) is balanced. Let $C = Core_G(L)$. Then C is a solvable normal subgroup of G. If $C \neq 1$ and $C \nsubseteq H \cap K \cap L$, then the solvability of G follows directly. Assume that $C \subseteq H \cap K \cap L$. Let $\overline{G} = G/C$. Denote by $\overline{H}$, $\overline{K}$ and $\overline{L}$ the images of H, K and L, respectively. Then $(\overline{H}, \overline{K}, \overline{L})$ is a regular triple of $\overline{G}$. So we may assume by induction that $\overline{G}$ is solvable, thus G is also solvable. Thus we may assume that $C = 1$. This in turn implies that $|G : L| = r$ and that $|G| \not\equiv 0 \bmod r^2$. Denote by p the greatest prime dividing $|H|$, and let q be the greatest prime dividing $|K|$. Assume that $p \geq q$. Let $P \in Syl_p(H)$ and let $Q \in Syl_q(K)$. Then by the supersolvability of H and K the group $H \subseteq N_G(P)$ and $K \subseteq N_G(Q)$. We want to prove that in both cases we may assume that equality holds. Since $C = 1$, it follows for H if $p \neq r$, and for K if $q < r$. So $p = r$ can be assumed. Let $R \in Syl_r(G)$, then recall $G = RL$, $|R| = r = |G : L|$, $L \cap R = 1$. If R is a normal subgroup of G then $G/R \simeq L$, thus G is solvable. So we may assume that $H = N_G(P)$ and $K = N_G(Q)$. Thus $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Since (H, K) is a balanced pair, H and K are not conjugate in G. Thus $p > q$. We distinguish between Case (i) $r > p$ and Case (ii) $r = p$.
(i) If $r > p$ then both $|H|$ and $|K|$ are divisors of $|L|$. Since $|G : K| = |G : L||L|/|K|$, $|G : L| = r$ and p divides $|G : K|$, we have that p divides $|L|/|K|$, hence also $|L : K \cap L|$. Since the pair (K, L) is balanced, $K \cap L$ is a maximal subgroup of L. Solvability of L gives that $|L : K \cap L|$ is a power of p. Then $L = P_0(K \cap L)$, where $P_0 \in Syl_p(L)$ and $P_0 \cap K \cap L = 1$. Then $|G : K \cap L| = |G : L||L : K \cap L| = r|P_0|$. Since neither p nor r divides $|K|$, we have that $K = K \cap L$, a contradiction.
(ii) If $r = p$ then $(|G : H|, |G : L|) = 1$, thus $G = HL$, $H = R(H \cap L)$. We show that H is a Frobenius group (with kernel R and complement $H \cap L$). For let $D = L \cap C_G(R)$; then $D^G = D^{RL} = D^L \leq L$ yields $D = 1$. It follows that $H \cap L$ is cyclic. It is a maximal subgroup in the solvable group L, so $|L : H \cap L| = t^m$ is a prime power. Similarly $H \cap K$ is maximal in H, therefore it has prime index in H. In fact, as r does not divide $|K|$, $|H : H \cap K| = r$, hence $|H \cap K| = |H \cap L|$. It follows that $H \cap K = (H \cap L)^y$ for some $y \in H$. We also have that $|K|$ divides $|L|$, so

$|K : K\cap H| = t$. As $R \lhd G$ implies solvability of G, we can assume that $H = N_G(R)$. Recalling $G = HL$, we obtain that $t^m = |L : H \cap L| = |G : H| = |G : N_G(R)| \equiv 1 \pmod r$.

Let $\tau = \{r, t\}$. Since $|G : H \cap L| = rt^m$, and $H \cap L$ is cyclic, $H \cap L$ contains a unique Hall τ'-subgroup T, that is a Hall τ'-subgroup of G as well.

We can assume that $t \neq q$. For otherwise every prime in τ' is smaller than q hence $N_G(T) = C_G(T)$. Then by Bunside's transfer theorem (e.g. [10, 2.6 Hauptsatz], p. 419.) there is a normal Hall τ-subgroup A of G. Since $|\tau| = 2$, A is solvable. Now A has a characteristic subgroup N of prime power order. Since $Core_G(L) = 1$, $N = R$. But then $G = RL$ gives the solvability of G.

Let $y \in H$ with $H\cap L = (H\cap K)^{y^{-1}} = H\cap K^{y^{-1}}$. Let $K_1 = K^{y^{-1}}$ and $Q_1 = Q^{y^{-1}}$. Then $K_1 = N_G(Q_1)$. As K_1 and L are non-conjugate maximal subgroups of G, $K_1 \not\leq L$. Thus $K_1 > L \cap K_1 \geq L \cap H$. Since $t = |K : H \cap K| = |K_1 : H \cap K_1| = |K_1 : H \cap L|$,it follows that $L \cap K_1 = L \cap H$. Since $Q_1 \leq T \leq L \cap H = L \cap K_1$, Q_1 is a characteristic subgroup of T and hence $N_G(T) \leq N_G(Q_1) = K_1$. Since $H \cap L$ is cyclic, $H \cap L \leq N_G(T) \leq K_1$ we have either $N_G(T) = K_1$ or $N_G(T) = H \cap L$. If $N_G(T) = H \cap L$, the latter being cyclic $T \leq Z(N_G(T))$ gives a nontrivial solvable normal subgroup, i.e. solvability of G; so we may assume that $N_G(T) = K_1$. As K_1/T is a t-group, and since $|K_1 : H \cap L| = t$, $H \cap L$ is normal in K_1. Since $H \cap L$ is cyclic, every subgroup $U \leq H \cap L$ is normal in K_1. This implies that if for an element $x \in G$ $T^x \neq T$, i.e. $x \in G\backslash K_1$, then $T \cap T^x = 1$, otherwise $T \cap T^x$ would be normal in K_1 and in K_1^x, thus also in G.

Let $S \in Syl_t(G)$ be contained in L. Since T is a Hall t'-subgroup of L and $N_L(T) = L \cap N_G(T) = L \cap K_1 = L \cap H$ shows that $T \leq Z(N_L(T))$, so $S \lhd L$. We want to prove that $T = H \cap L$. Otherwise, since $(|L : H \cap L|, |L : S|) = 1$, $L = S(H \cap L)$. Let $U = S \cap (H \cap L) = S \cap H$, then $U \neq 1$ and $U \in Syl_t(H \cap L)$. Since $N_S(U) > U$ and $H \cap L$ is maximal in L, U would be normal in L. So $U = 1$, $T = H \cap L$ and $|L : T| = t^m$. Thus L is a Frobenius group with complement T. To exclude this possibility we need a combinatorial result of K. Corrádi, [3] see also [11, Problem 13.13].

Lemma 4.9. *Ley A be a finite set and let $B_1, B_2, ..., B_m$ given subsets of A with*

(i) $|B_i| = r$, $1 \leq i \leq m$

(ii) $|B_i \cap B_j| \leq k$, *if* $i \neq j$.

Then $|A| \geq \frac{mr^2}{r+k(m-1)}$.

(Conclusion of the proof of Theorem 4.8) Let $R = \langle z\rangle$. Let us define the set $A = \{T^x | x \in G\}$, $B_i = \{T^{wz^{i-1}} | w \in L\}$, $1 \leq i \leq r.\}$ Observe that $|A| = |G : K| = r|L|/|K| = rt^{m-1}$, $|B_i| = |L : N_L(T)| = |L : L \cap K_1| = |L : L \cap H| = t^m$ for $1 \leq i \leq r$. Since the elements of B_i are all maximal subgroups in $L^{z^{i-1}}$, and any two of them generate $L^{z^{i-1}}$, $|B_i \cap B_j| \leq 1$ if $i \neq j$. Hence we deduce: $rt^{m-1} \geq \frac{rt^{2m}}{t^m+(r-1)}$. Thus $t^m \leq t^m(t-1) \leq r-1$. This contradicts to $t^m \equiv 1 \mod r$. □

Definition 4.10. *Let G be a finite group, let $\sigma \subseteq \pi = \pi(G)$. Let $\mathsf{H}_\sigma = \{H \in Hall_\sigma(G)\}$. Let $\sigma' = \pi\backslash\sigma$. As before $\mu_p = \{u \in \pi | u > p\}$.*

We conclude with our main result in section 4

Theorem 4.11. *Let G be a finite group, let (H, K, L) be a regular triple in G. Let p be the greatest prime divisor of $|G : H||G : K|$. Assume that for every* *fixed $u \in \mu_p$ $\mathsf{H}_{u'} \neq \emptyset$ and for some $Z \in \mathsf{H}_{u'}$ for every maximal subgroup M of G containing $N_G(Z)$, there exists an element $X_M \in \{H, K\}$ such that the pair (M, X_M) is balanced with respect to X_M. Then G is supersolvable.*

Proof. G is solvable by 4.8. By Ore's theorem, for every pair U, V of non-conjugate maximal subgroups, $G = UV$. If U and V are both supersolvable, then $|G : U|$ and $|G : V|$ are both prime numbers. This is the case for the pair (H, K). We may assume that $|G : H| = p$ and $|G : K| = q$ for primes p, q with $p \geq q$.Fix a prime $u \in \mu_p$. Then $\mathsf{H}_{u'} \neq \emptyset$. Since G is solvable the elements of $\mathsf{H}_{u'}$ are all conjugate. By assumption, every maximal subgroup M of G that contains the normalizer of an element of $\mathsf{H}_{u'}$ determines a subgroup $X_M \in \{H, K\}$ such that the pair (M, X_M) is balanced with respect to X_M. Since G is solvable and X_M is supersolvable, $|G : M| = |X_M : M \cap X_M|$ is prime. This is by our choice u. By 4.2 G is supersolvable. □

We finish this section by giving a simple group G that satisfies the conditions of 4.8 except for the condition (ii) in 4.6 for which $\mu_p = \emptyset$ for the prime p in 4.11.

Example. *Let $G = A_5$. Let P, Q, S be Sylow subgroups belonging to primes $5, 3, 2$. respectively, let $H = N_G(P), K = N_G(Q), L = N_G(S)$. Then the triple (H, K, L) satisfies (i),(iii) and (iv) in the Definition 4.7, and the maximal prime divisor of $|G : H||G : K|$ is 5, so $\mu_5 = \emptyset$. Note that $|H| = 10, |K| = 6, |L| = 12$. Neither H, K, L can be abelian, because they are maximal subgroups, and every finite group having an abelian maximal subgroup is solvable. So H and K are dihedral and thus they are supersolvable. The pair (H, K) is balanced, since $H \cap K = 1$ would imply $G = HK$ and from this for every $x \in G$, $G = HK^x$. But this is impossible because the involutions of G are all conjugate, hence there would be an $x \in G$ with $|H \cap K^x| = 2$. But then $|G| = |H||K|/|H \cap K^x| = 30$, a contradiction.*

5. Further results

We shall need the following

Definition 5.1. *Let G be a finite group, $p \in \pi(G)$ and $P \in Syl_p(G)$. We shall say that G has a canonical chain that belongs to the pair (p, P), if there are subgroups $M_{p,i}$ $(0 \leq i \leq n_p)$ in G satisfying*
(i) $G = M_{p,n_p}$
(ii) $|M_{p,i+1} : M_{p,i}| = p$, $0 \leq i \leq n_p - 1$
(iii) $P \cap M_{p.0} \leq \Phi(P)$.

One may observe that the existence of a canonical chain is independent of the choice of P. As (iii) implies that for any Sylow p-subgroup P^* in G, $M_{p,0} \cap P^* \leq \Phi(P^*)$ since $G = M_{p,0}P$, and so P can be conjugated to P^* by some element in $M_{p,0}$. So we can speak about a canonical chain belonging to the prime p.

First we prove

Theorem 5.2. *For the supersolvability of a finite group G the existence of a canonical chain for every prime $p \in \pi(G)$ is necessary and sufficient.*

Proof. We need only show that the condition is sufficient. Assume that for every prime $p \in \pi(G)$ the group G has a canonical chain belonging to p. Let s be the smallest prime in $\pi(G)$, and let $M_{s,i}$ $(0 \leq i \leq n_s)$ be the members of a canonical chain belonging to the prime s (with respect to $S \in Syl_s(G)$). By our choice

$M_{s,i} \lhd M_{s,i+1}$ holds for all i $(0 \leq i \leq n_s - 1)$. Therefore $O^s(G) \leq M_{s,0}$. By a result of Tate (see [[10] IV, 4.7 Satz]), since $O^s(G) \cap S \leq \Phi(S)$, $O^s(G)$ has a normal s-complement, that is also a normal s-complement $H_{s'}$ in $M_{s,0}$. We have that $H_{s'} \in Hall_{s'}(G)$ and $H_{s'} = O^s(G) \lhd G$. Let now $p \in \pi(G) \backslash \{s\}$ be given, and let the subgroups $M_{p,i}$ $(0 \leq i \leq n_p)$ be the members of the canonical chain of G belonging to the prime p. Define the subgroups $M^*_{p,i}$ by $M^*_{p,i} = H_{s'} \cap M_{p,i}$ for all i, $0 \leq i \leq n_p$. These subgroups form a canonical chain of $H_{s'}$ belonging to the prime p. Since $p \in \pi(G) \backslash \{s\}$ may be arbitrary, we get by induction that $H_{s'}$ is supersolvable. Let now r be the maximal prime in $\pi(G)$ and let $R \in Syl_r(G)$ that lies in $H_{s'}$. The supersolvability of $H_{s'}$ implies that R is characteristic in $H_{s'}$. Since $H_{s'} \lhd G$, $R \lhd G$ also holds.

We distinguish between two cases.

Case (i): $\Phi(R) = 1$. Consider the canonical chain in G belonging to the prime r; let its members be $M_{r,i}$, $0 \leq i \leq n_r$. Set $H = M_{r,n_r-1}$. Then $|G : H| = r$ and defining for every prime $p \neq r$ the chain $M^{**}_{p,i} = H \cap M_{p,i}$, where $\{M_{p,i} \mid 0 \leq i \leq n_p\}$ is a canonical chain of G belonging to p, we obtain a canonical chain of H for every prime $p \in \pi(H)$. It follows by induction that H is supersolvable. Now R is normal in G, $|G : H| = r$ and the supersolvability of $N_G(R)$ by 2.2 implies that G itself is supersolvable.

Case (ii): $\Phi(R) \neq 1$. Set $\overline{G} = G/\Phi(R)$. Then $\overline{G}$ is supersolvable by induction. Hence $G/\Phi(G)$ is also supersolvable. By Huppert's theorem the supersolvability of G follows. □

Remark 5.3. *We may deduce Theorem 2.3 also from Theorem 5.2.*

As a consequence we have the following

Theorem 5.4. *Let G be a finite p-solvable group. G is p-supersolvable iff*

(i) G' is p-nilpotent, and

(ii) G has a canonical chain that belongs to the prime p.

Proof. We need only show that the conditions are sufficient. It can be assumed that $O_{p'}(G) = 1$. Hence by (i) $G' \leq P \in Syl_p(G)$. Thus $P \lhd G$ and G/P is abelian. So G is q-supersolvable for every prime $q \in \pi(G) \backslash \{p\}$, and for every such prime q the group G has a canonical chain belonging to the prime q. Since the existence of a canonical chain for the prime p is assumed in (ii), all requirements of Theorem 5.2 are satisfied, hence G is supersolvable. □

Remark 5.5. *It is easy to see that condition (ii) alone is not sufficient for the p-supersolvability of G. On the other hand it turns out to be sufficient when the Sylow p-subgroup of G is weakly regular. We also observe that one can prove Theorem 5.4 following the pattern of the proof of Theorem 5.2.*

Definition 5.6. *Let P be a finite p-group. A series P_i $(0 \leq i \leq n)$ in P will be called a canonical chain of P, if*

(i) $P_0 < P_1 < ... < P_n = P$,

(ii) $|P_{i+1} : P_i| = p$, $0 \leq i \leq n - 1$,

(iii) $P_0 \leq \Phi(P)$.

The next result will be used later.

Theorem 5.7. *Let G be a finite group that has a Sylow system $\mathcal{S}$ such that each element $P \in \mathcal{S}$ has a canonical chain. If for each $P \in \mathcal{S}$ the chain P_i $(0 \leq i \leq n_p)$ satisfies $P_iQ = QP_i$ for all $0 \leq i \leq n_{p_i}$ and for every $Q \in \mathcal{S}$, then G is supersolvable.*

Proof. Let $p \in \pi(G)$ be fixed and let $P \in \mathcal{S}$ be a Sylow p-subgroup of G. Define $M_{p,i} := (\prod_{Q\in\mathcal{S}\setminus\{P\}} Q)P_i$ for all $0 \le i \le n_p$. One can see that $M_{p,i}$ $(0 \le i \le n_p)$ is a canonical chain of G belonging to the prime p. Since in G such chains exist for all primes $p \in \pi(G)$, 5.2 yields the supersolvability of G. □

Remark 5.8. *One observes at once that in a supersolvable group there are Sylow-systems which satisfy the condition in Thereom 5.7. So this result in fact, is a characterization of supersolvability.*

The main result of this section is the following

Theorem 5.9. *Let G be a finite group that has a Sylow system $\mathcal{S}$ such that for every $P \in \mathcal{S}$, P has a collection $\mathcal{A}_P$ of cyclic subgroups A satisfying*

(i) $P = \langle A | A \in \mathcal{A}_P\rangle$, and

(ii) $AQ = QA$ for all $A \in \mathcal{A}_P$ and for all $Q \in \mathcal{S}$.

Then G is supersolvable.

Remark 5.10. *Theorem 5.9 is a characterization result again.*

Proof. Let q be the minimal prime of $\pi(G)$, and let $Q \in \mathcal{S}$ be a Sylow q-subgroup of G. Fix an $A \in \mathcal{A}_Q$. Then $H_{q'} = \prod_{P\in\mathcal{S}\setminus\{Q\}} P$ is a Hall q'-subgroup of G satisfying $AH_{q'} = H_{q'}A$. So in particular, A is a Sylow q-subgroup of $AH_{q'}$ belonging to the minimal prime of $\pi(AH_{q'})$. Since A is cyclic, $H_{q'}$ is a normal q-complement of $AH_{q'}$. Hence $A \le N_G(H_{q'})$ holds. Since $A \in \mathcal{A}_Q$ was arbitrary, $H_{q'}$ is a normal q-complement of G. Now $H_{q'}$ and $\mathcal{S}^- = \mathcal{S}\setminus\{Q\}$ satisfy the conditions of the theorem, so by induction we have that $H_{q'}$ is supersolvable. This implies that G has a Sylow tower. It follows that for every prime p, the p-length of G is $l_p(G) = 1$. This implies by a result of Huppert, (see [10], 6.11 Satz b), p.694), that for every pair $P, R \in \mathcal{S}$ $\Phi(P)R = R\Phi(P)$ holds. Now fix $P \in \mathcal{S}$, and set $n_P = |P : \Phi(P)|$. Choose elements $A_i \in \mathcal{A}_P$ $(1 \le i \le n_P)$ satisfying $P = \langle A_i | 1 \le i \le n_P\rangle$. Define the subgroups $P_{p,i}$ $(0 \le i \le n_P)$ as $P_{p,0} = \Phi(P)$ and $P_{p,i} = \Phi(P)A_1...A_i$ for all $1 \le i \le n_P$. These subgroups form a canonical chain in P. Since for each $0 \le i \le n_P$ and for all $R \in \mathcal{S}$, $P_{p,i}R = RP_{p,i}$ holds, the conditions of 5.7 are satisfied for each prime p in $\pi(G)$. Thus the supersolvability follows. □

Remark 5.11. *5.9 generalizes a result in* [5].

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