ON A THEOREM OF AMBARZUMIAN

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Abstract. We consider the eigenvalue problem for the one-dimensional (stationary) Dirac operator with some boundary conditions. We prove that if the spectrum is the same as the spectrum belonging to the zero potential then the potential is actually zero. The analogous statement for Schrödinger operator is due to Ambarzumian. The proof is based on the fact that the (generalized) moments of a function cannot have alternating signs unless the moments are zero (see Section 2 below).

1. Setting of the problem

Consider the eigenvalue problem

$$\begin{aligned} u_2' + (V(x)+m)\,u_1 &= \lambda u_1 \\ -u_1' + (V(x)-m)\,u_2 &= \lambda u_2 \end{aligned} \qquad \text{on} \quad (0,\pi) \tag{1.1}$$

with the boundary condition

$$u_1(0) = u_1(\pi) = 0. \tag{1.2}$$

The differential operator on the left of (1.1) is called the stationary Dirac-operator, the functions $u_1(x)$, $u_2(x)$ are defined on $[0,\, \pi]$, $m > 0$ denotes the mass of the particle, V is the potential of the system described by (1.1). In what follows we suppose that $V \in C[0,\pi]$. In case $V \equiv 0$ we can easily count the eigenvalues and eigenfunctions of (1.1), (1.2). After straightforward calculation we get

$$\begin{aligned} &\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sin kx \\ -\frac{k}{\lambda+m}\cos kx \end{pmatrix}, \quad \lambda = \pm\sqrt{m^2+k^2}, \quad k = 1,\, 2, \dots \\ &\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lambda = -m. \end{aligned} \tag{1.3}$$

The main result of this paper reads as follows.

Theorem 1.1. *Let $0 < m \le 1/2$ and $V \in C[0,\pi]$. Suppose that (1.1) and (1.2) produce the same spectrum as $V \equiv 0$ in (1.3) i.e. that the*

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eigenvalues are $-m$ *and* $\pm\sqrt{m^2+k^2}$, $k = 1, 2, \ldots$ *Then* $V \equiv 0$ *on* $[0, \pi]$.

Remark 1.2. In fact we will prove more, see in Section 4.

Remark 1.3. From the physical perspective the natural boundary conditions for the Dirac equation are periodic boundary conditions. However the exceptional situation described by the Ambarzumian theorem requires very special boundary conditions. So the above Theorem has no direct physical interpretation.

The analogous statement for the Schrödinger operator with Neumann boundary conditions has been proved by V. Ambarzumian [1] in 1929. This was the starting point of the inverse spectral theory aiming to reconstruct the potential from the spectrum (or spectra). Later it became clear that the case investigated by Ambarzumian was exceptional; in general two spectra is needed to determine the potential, see G. Borg [2]. After the m-function of Weyl-Titchmarsh has been proved to be useful in inverse spectral problems; we mention the fundamental result of V. A. Marchenko [16] stating that the m-function determines the potential. Other important contributions are due to I. M. Gelfand, B. M. Levitan and others, see e.g. in Levitan, Sargsjan [14]. Recently the theory has started to develop intensively. In a series of papers, motivated by the results [10] of Hochstadt and Lieberman, B. Simon and F. Gesztesy analyzed the idea that a spectrum determines the half of the potential and gave a new approach for the inverse spectral theory , we mention only the papers [6], [7], [8], [5] and [17]. The Ambarzumian theorem has been generalized in many directions. Ambarzumian's original proof is based on a careful analysis of the dependence of the eigenvalues on a parameter put into the potential. Borg [2] found a simplified proof based on the fact that the eigenfunction corresponding to the first eigenvalue has no zeros. Later on, Levitan and Gasymov [15] gave another short proof, using the following well-known extremal property of the smallest eigenvalue: the functional

$$\int_0^\pi ((\varphi')^2 + q\varphi^2)$$

attains its minimum at the first eigenfunction and the minimal value is the first eigenvalue. Based on this idea, Chakravarty and Acharyya [3] extended the Ambarzumian's theorem to 2×2 matrix-valued potential. Recently Chern and Shen [4] proved the more general statement with $n \times n$ matrix potential. In the multi-variable case I mention the paper of Kuznetsov [13] for the case of two- or three-dimensional Schrödinger operator on a bounded domain with smooth boundary and

Neumann boundary conditions. Harrell [12] found similar statements for Schrödinger operators defined on symmetric n-dimensional spaces like spheres. In all of these papers the existence of a smallest eigenvalue and its special properties are heavily used, so these ideas seem to be ineffective in case of the Dirac operator when the sequence of eigenvalues goes from $-\infty$ to $+\infty$. The present paper aims to fill in this gap.

Finally mention the paper [11] of the author where it is shown that the convergence and divergence properties of the eigenfunction expansion corresponding to the Dirac operator are similar to those of the trigonometric Fourier series.

The paper is organized as follows. In Section 2. we prove a statement about generalized moments with alternating signs. In Section 3. this result is used to prove our Theorem. The proof is completely different from that of Ambarzumian; our starting point is an idea of Borg [2]. Finally Section 4. is devoted to some remarks about the necessity of the assumptions we made throughout the paper. We also discuss briefly some pathological properties of the Dirac eigenfunctions.

2. Moment theorems

In this part we prove the following theorem on the generalized moments with alternating sign. We borrow some ideas of Hausdorff [9].

Lemma 2.1. *Let $0 \leqslant f \in C[a,b]$ and $g \in L_1(a,b)$ be arbitrary functions in the finite segment $[a,b]$. Suppose that the generalized moments*

$$\mu_k = \int_a^b f^k g \qquad k = 0, 1, \dots \tag{2.1}$$

have alternating signs; for example $(-1)^k \mu_k \geqslant 0$. Then

$$\mu_k = \int_a^b f^k g = 0 \quad k = 1, 2, \dots$$

Proof. After dividing (2.1) by $(\max_{[a,b]} f)^k$ we can suppose that $0 \leqslant f \leqslant 1$. Recall the well-known formula

$$\Delta^n \mu_k = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \mu_{k+i} \tag{2.2}$$

Substituting here the moments we get

$$\Delta^n \mu_k = \int_a^b f^k g \sum_{i=0}^{n} \binom{n}{i} (-1)^i f^i = \int_a^b f^k (1-f)^n g. \tag{2.3}$$

From here it follows that

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} \mu_k| \leqslant \int_a^b \sum_{k=0}^{n} \binom{n}{k} f^k (1-f)^{n-k} |g| = \int_a^b |g|, \tag{2.4}$$

a bound independent of n. Finally, since the sign of μ_k is alternating, in the sum (2.2) every summand have the same sign , so

$$|\Delta^n \mu_k| = \sum_{i=0}^{n} \binom{n}{i} |\mu_{k+i}|. \tag{2.5}$$

Putting this into (2.4) we obtain that the sums

$$\sum_{k=0}^{n} \binom{n}{k} |\Delta^{n-k} \mu_k| = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} |\mu_{k+i}|$$

are bounded in n. Choosing $i = 0$ in the inner sum, the quantities $\binom{n}{k} |\mu_k|$ must have a bound independent of n and k. Consequently $\mu_k = 0$ for $k = 1, \ldots, n-1$. Since n is arbitrary, this finishes the proof. $\square$

Since the arguments applied above have a linear character, the same proof works in the following more general setting:

Lemma 2.2. *Let $0 \leqslant f_j \in C[a_j, b_j]$, $g_j \in L_1(a_j, b_j)$ for $j = 1, ..., d$. If the quantities*

$$\mu_k = \sum_{j=1}^{d} \int_{a_j}^{b_j} f_j^k g_j \qquad k = 0, 1, ...$$

have alternating signs then $\mu_k = 0$ for $k \geqslant 1$. $\square$

3. The proof

We prove the Theorem through a series of Lemmas.

Lemma 3.1. *If the potential V has the spectrum (1.3) then*

$$\int_0^\pi V = d\pi \quad \textit{with an integral } d. \tag{3.1}$$

Proof. We need the following asymptotic formula for the eigenvalues, valid for all continuous potentials:

$$\lambda_n = n + \frac{\eta}{\pi} + \mathbf{O}(1/n), \qquad \eta = \int_0^\pi V$$

see [14], Ch.VII, §2. In our case, on the other hand

$$\lambda_n = \pm\sqrt{m^2+n^2} = n + \mathbf{O}(1/n).$$

Since the numbering of the eigenvalues in the above two formulae is not necessarily the same, we can only conclude that $1/\pi \int_0^\pi V$ is integer, as we asserted. □

From now on we consider the eigenfunction (u_1, u_2) corresponding to $\lambda = -m$. This means that

$$u_1' = Vu_2, \qquad u_2' = -(V+2m)u_1. \tag{3.2}$$

Lemma 3.2. *Let $[a,b] \subset (0,\pi)$ be a closed subinterval where u_2 has no zeros. Then for $n \geqslant 1$*

(3.3)

$$\int_a^b V = \sum_{k=1}^n \frac{(-1)^{k+1}}{2k-1}\left[\left(\frac{u_1}{u_2}\right)^{2k-1}\right]_a^b + 2m\int_a^b \sum_{k=1}^n (-1)^k \left(\frac{u_1}{u_2}\right)^{2k} + (-1)^n \int_a^b V\left(\frac{u_1}{u_2}\right)^{2n}.$$

Proof. By (3.2) we have

$$\int_a^b V = \int_a^b \frac{u_1'}{u_2} = \left[\frac{u_1}{u_2}\right]_a^b + \int_a^b \frac{u_1 u_2'}{u_2^2} = \left[\frac{u_1}{u_2}\right]_a^b - \int_a^b (V+2m)\frac{u_1^2}{u_2^2}$$

which proves (3.3) for $n=1$. By the same argument we can complete an induction on n; the step from n to $n+1$ is the formula

$$\int_a^b V\left(\frac{u_1}{u_2}\right)^{2n} = \int_a^b \frac{u_1' u_1^{2n}}{u_2^{2n+1}} = \left[\frac{1}{2n+1}\left(\frac{u_1}{u_2}\right)^{2n+1}\right]_a^b - \int_a^b (V+2m)\left(\frac{u_1}{u_2}\right)^{2n+2}$$

and we are ready. □

Lemma 3.3. *If (a,b) is a maximal subinterval where u_2 has no zeros, then*

(3.4)

$$\int_a^b \left(V + 2m\frac{u_1^2}{u_1^2+u_2^2}\right) = \begin{cases} 0 \ \textit{or} \ \pm\pi & \textit{if} \quad 0<a<b<\pi \\ \pm\pi/2 & \textit{if} \quad a=0 \ \textit{or} \ b=\pi \ \textit{(not both)} \\ 0 & \textit{if} \quad a=0 \quad \textit{and} \quad b=\pi. \end{cases}$$

Proof. In order to handle with the first sum on the right of (3.3) we observe that

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} y^{2k-1} = \int_0^y \sum_{k=1}^{n} (-1)^{k-1} t^{2k-2}\, dt = \int_0^y \frac{1-(-t^2)^n}{1+t^2}\, dt.$$

Hence (3.3) can be rewritten as

$$\int_{a+\delta}^{b-\delta} V = \int_{\frac{u_1(a+\delta)}{u_2(a+\delta)}}^{\frac{u_1(b-\delta)}{u_2(b-\delta)}} \frac{1+(-1)^{n+1}t^{2n}}{1+t^2}\, dt$$
$$- 2m \int_{a+\delta}^{b-\delta} \frac{u_1^2}{u_2^2} \frac{1-(-1)^n \left(\frac{u_1}{u_2}\right)^{2n}}{1+\frac{u_1^2}{u_2^2}} + (-1)^n \int_{a+\delta}^{b-\delta} V \left(\frac{u_1}{u_2}\right)^{2n},$$

or

$$\int_{a+\delta}^{b-\delta} \left(V + 2m\frac{u_1^2}{u_1^2+u_2^2}\right) - \int_{\frac{u_1(a+\delta)}{u_2(a+\delta)}}^{\frac{u_1(b-\delta)}{u_2(b-\delta)}} \frac{dt}{1+t^2}$$
$$= (-1)^n \left[\int_{\frac{u_1(a+\delta)}{u_2(a+\delta)}}^{\frac{u_1(b-\delta)}{u_2(b-\delta)}} t^{2n} \frac{-1}{1+t^2}\, dt + \int_{a+\delta}^{b-\delta} \left(\frac{u_1}{u_2}\right)^{2n} \left(V + 2m\frac{u_1^2}{u_1^2+u_2^2}\right)\right]$$
$$= (-1)^n \mu_n.$$

Since $(-1)^n\mu_n$ is constant, μ_n has alternating signs for $n \geqslant 1$, hence by Lemma 2.2 $\mu_n = 0$ for $n \geqslant 2$. In other words,

$$\int_{a+\delta}^{b-\delta} \left(V + 2m\frac{u_1^2}{u_1^2+u_2^2}\right) = \int_{\frac{u_1(a+\delta)}{u_2(a+\delta)}}^{\frac{u_1(b-\delta)}{u_2(b-\delta)}} \frac{dt}{1+t^2}.$$

Now take the limit $\delta \to 0+$. Since by definition $u_1(a)u_2(a) = u_1(b)u_2(b) = 0$, the statement (3.4) follows. □

Lemma 3.4.

$$\int_0^\pi \left(V + 2m\frac{u_1^2}{u_1^2+u_2^2}\right) = d_1\pi \quad \textit{for some} \quad d_1 \in \mathbb{Z}. \tag{3.5}$$

Proof. Denote

$$G = \{x \in (0,\pi) : \quad u_2(x) \neq 0\}, \quad K = [0,\pi] \setminus G.$$

It may happen that K has positive measure (see the remarks below), so our first task is to prove that

$$\int_K \left(V + 2m\frac{u_1^2}{u_1^2+u_2^2}\right) = 0. \tag{3.6}$$

Indeed, let x^* be any accumulation point of K. Then $u_2(x^*) = 0$ and $u_2'(x^*) = 0$ hold true. By the unicity of the solution of initial value problems $u_1(x^*) \neq 0$, hence $(V + 2m)u_1 = -u_2' = 0$ implies $V(x^*) = -2m$. Consequently $V + 2m\frac{u_1^2}{u_1^2+u_2^2} = 0$ in all accumulation points (hence a.e.) in K, so (3.6) follows.
The open set G is the disjoint union of open intervals (the number of which can be infinite, see the remarks below). On every such intervals the integral is a multiple of $\pi/2$ by Lemma 3.3, so all integrals with finitely many exceptions should actually vanish. If u_2 has no zeros at all, the integral (3.5) is zero by Lemma 3.3; if u_2 has zeros then $u_2(0) \neq 0, \quad u_2(\pi) \neq 0$ implies that among the intervals of G there is one of the form $(0, b)$ and another of the form (a, π). On the union of these two intervals the integral is zero (mod π), just like on any of the other intervals of G. This proves (3.5). □

Proof of the Theorem From Lemmas 3.1 and 3.4 we know that

$$2m \int_0^{\pi} \frac{u_1^2}{u_1^2 + u_2^2} \tag{3.7}$$

must be an integer times π . Since $0 < m \leqslant 1/2$, we see that this integer can be 0 or 1. If (3.7) is π then $m = 1/2$ and $u_2 \equiv 0$, in contradiction with $u_1(0) = 0$. So (3.7) is zero and then $u_1 \equiv 0$. From (3.2) we finally obtain $u_2 = const \neq 0$, so $V = \frac{u_1'}{u_2} \equiv 0$. The Theorem is proved. □

4. Remarks

This final part of the paper is devoted to some concluding remarks about the above proofs and a brief discussion of the distribution of zeros of the eigenfunctions.

Remark 4.1. In fact we also proved the following statements (with $0 < m \leqslant 1/2$ and continuous V):
a.) If infinitely many eigenvalues (including $-m$) of (1.1), (1.2) are taken from (1.3) then $V \equiv 0$.
b.) If $\lambda = -m$ is an eigenvalue and if there are large eigenvalues arbitrarily near to integers (i.e. if there are infinitely many n with $\lambda_n = n + \mathbf{o}(1)$) then $V \equiv 0$.
c.) If $\int_0^{\pi} V = 0$ and if $\lambda = -m$ is an eigenvalue of (1.1), (1.2) then $V \equiv 0$.
d.) If $\lambda^* = -m + 1/\pi \int_0^{\pi} V$ is an eigenvalue then $V \equiv \lambda^* + m$ is constant.

Remark 4.2. The condition $m > 0$ is necessary: if $m = 0$, then the eigenvalues are determined by $\int_0^\pi V$. Indeed, in this case

$$u_1(x) = \cos\omega(x), \qquad u_2(x) = \sin\omega(x)$$

with

$$\omega(x) = \lambda x - \int_0^x V + \frac{\pi}{2}$$

and the eigenvalues are the solutions of $\omega(\pi) = (k + 1/2)\pi$ i.e.

$$\lambda_k = k + \frac{1}{\pi}\int_0^\pi V.$$

So in case $m = 0$ many potentials can produce the same spectrum as $V \equiv 0$. However, the necessity of the condition $m \leqslant 1/2$ is not clear.

Remark 4.3. The conclusion of Lemma 2.1 is clearly not true for $k = 0$ if f vanishes on a set of positive measure. On the other hand the following statement seems to be probable for some reasons:

$$\int_{(f>0)} g = 0.$$

Since the present form of Lemma 2.1 is enough for our purposes, we didn't discuss the problem further.

Remark 4.4. As we mentioned above, it may happen that u_2 , corresponding to $\lambda = -m$ has infinitely many isolated zeros or that u_2 vanishes identically on an interval. To provide an example, introduce the Prüfer-type variables $r > 0$ and ω by

$$u_1 = r\sin\omega, \qquad u_2 = r\cos\omega. \tag{4.1}$$

Then (3.2) can be rewritten as

$$r'\sin\omega + \omega' r\cos\omega = Vr\cos\omega, \tag{4.2}$$

$$r'\cos\omega - \omega' r\sin\omega = -(V + 2m)r\sin\omega. \tag{4.3}$$

Solving this linear system we can express ω' and r' :

$$\omega' = V + 2m\sin^2\omega\,, \quad r' = -mr\sin 2\omega. \tag{4.4}$$

It is well-known that the function $x^2\sin(x^{-\alpha})$ is continuously differentiable and has infinitely many isolated zeros if $0 < \alpha < 1$. An appropriately transformed copy of it will produce a function $\omega(x)$, continuously differentiable on $[0, \pi]$ and satisfying the properties
i. $\omega(0) = \omega(\pi) = 0$
ii. ω takes the value $\pi/2$ in infinitely many isolated points
iii. $\omega \equiv \pi/2$ on a subinterval.
Once ω constructed, we determine $V(x)$ and $r(x)$ from (4.4) and then

u_1, u_2 by (4.1). Since (4.4) and (3.2) are equivalent, (u_1, u_2) is an eigenfunction with eigenvalue $-m$ and the zeros of u_2 show the above mentioned irregularities.

Remark 4.5. If the potential V is fixed, the zeros of the eigenfunctions corresponding to large eigenvalues will, however, behave "regularly". To see this, apply Prüfer variables to the system (1.1); we get

$$\omega' = -\lambda + V - m\cos 2\omega, \quad \omega(0) = 0. \tag{4.5}$$

Lemma 4.6. *$\omega(x) = \omega(x, \lambda)$ is strictly decreasing in $\lambda \in \mathbb{R}$ if $0 < x \leqslant \pi$ is fixed.*

Proof. It is very similar to the well-known case of Schrödinger operator; we give it just for the sake of completeness. Let

$$\tilde{\omega} = \frac{\partial \omega}{\partial \lambda};$$

it is enough to show that $\tilde{\omega} < 0$ if $x > 0$. Differentiating (4.5) in λ we get

$$\tilde{\omega}' = -1 + 2m\sin 2\omega \cdot \tilde{\omega}\,, \quad \tilde{\omega}(0) = 0. \tag{4.6}$$

In particular $\tilde{\omega}'(0) < 0$, hence there exists $\delta > 0$ such that $\tilde{\omega} < 0$ on $(0, \delta)$. Take the maximal $\delta \leqslant \pi$ with this property. If $\tilde{\omega}(\delta) = 0$ then $\tilde{\omega}'(\delta) = -1 < 0$, contradicting the fact that on $(0, \delta)$ we have $\tilde{\omega} < 0$. So we must have $\delta = \pi$ and $\tilde{\omega}(\pi) < 0$. □

By this Lemma the eigenvalues are the (only) values $\lambda = \lambda_n$ for which $\omega(\pi) = -n\pi$. Now if $|n|$ is large enough to ensure

$$|\lambda_n| > \|V\|_\infty + m \tag{4.7}$$

then by (4.5) ω becomes strictly monotone in x. This implies the following

Proposition 4.7. *If the numbering of the eigenvalues is made as above and if $|n|$ is large enough to ensure (4.7) then u_1 resp. u_2 has precisely $|n|+1$ resp. $|n|$ zeros and the zeros of u_1 and u_2 separate each other.* □

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