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Stability of Direct and Inverse Eigenvalue Problems for Schrödinger Operators on Finite Intervals

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We consider the inverse eigenvalue problem for Schrödinger operators on finite intervals. Among others, we show that if the potential is in L_p, then the perturbation of the potentials can be estimated by the $l_{p'}$-norm of the sequence of the eigenvalue differences only if $p \geq 2$. As a consequence, we give estimates if only finite number of eigenvalues are known with an error $< \varepsilon$.

1 Introduction

Consider the eigenvalue problem

$$-y'' + q(x)y = \lambda y \quad \text{on } [0, \pi], \tag{1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(\pi)\cos\beta + y'(\pi)\sin\beta = 0. \tag{1.2}$$

The set of eigenvalues is denoted by $\sigma(\alpha, \beta)$ or $\sigma(\alpha, \beta, q)$. The inverse Sturm–Liouville problem aims to identify the operator from a set of eigenvalues. Since the fundamental work of Borg [2], we know that in most cases two complete spectra are needed for the unique recovery of the operator (i.e., of the potential q). Later it became clear that a sufficiently large part of more than two spectra also implies uniqueness; a brief account

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of this topic is given, for example, in [10]. The idea of taking the eigenvalues from infinitely many spectra appears first in Gesztesy, del Rio, and Simon [5]. In Horváth [10], a necessary and sufficient condition for the uniqueness is given, which covers most of the former results. The aim of the present paper is to study the stability of the operator reconstruction using eigenvalues from infinitely many spectra. However, many of our results are new also in the classical case where two complete spectra are known. Finally, we present stability estimates if only finitely many noisy data are available.

We will use the following notations throughout: $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, q and q^* are two real valued potentials, $q, q^* \in L^1(0, \pi)$. Sometimes we assume that $\|q\|_1, \|q^*\|_1 \le D$; $c(D)$ always means a positive constant, possibly different in different occurrences, depending only on D. The eigenvalues corresponding to the potential q and q^* are denoted by λ_n and λ_n^*, respectively. If $\lambda_n^* \in \sigma(\alpha_n, 0, q^*)$, λ_n is the corresponding element of $\sigma(\alpha_n, 0, q)$, that is, their indices inside the spectrum $\sigma(\alpha_n, 0)$ are the same (we can relax this assumption, see the remark after Theorem 5.9).

In this paper, we investigate relationships between the L^p-norm of the potential perturbation

$$\|\Delta q\|_p = \|q - q^*\|_{L^p}$$

and the $l^{p'}$-norm of the perturbation of eigenvalues

$$\|\Delta\lambda\|_{p'} = \|\lambda_n - \lambda_n^*\|_{l^{p'}}$$

$\frac{1}{p} + \frac{1}{p'} = 1$, $1 \le p \le \infty$. Roughly speaking, we will prove that

$$\|\Delta\lambda\|_{p'} \le c\|\Delta q\|_p \quad \text{for } p \le 2 \text{ but not for } p > 2$$

related to the stability of the direct problem of determining the eigenvalues corresponding to a potential and

$$\|\Delta q\|_{p'} \le c\|\Delta\lambda\|_p \quad \text{for } p \le 2 \text{ but not for } p > 2$$

concerning the stability of the inverse problem of reconstructing the potential from a set of eigenvalues. More details are given below. Unless $p = 2$, these results are new also in the classical situation where the λ_n's run over $\sigma(0, 0) \cup \sigma(\frac{\pi}{2}, 0)$.

The above stability estimates for the nonlinear relation between the potential and the eigenvalues are reduced to simple continuity properties of the linear mapping (1.4) below, see Theorems 1.1–1.2. The main tool in proving stability results is Lemma 6.1. Finally, we extend two results of Marletta and Weikard [19] concerning the situation where only finitely many eigenvalues are known with errors.

1.1 General results

We present two general theorems, which will essentially imply the subsequent more specialized results.

Consider the system

$$C(\Lambda) = \{\varphi_n : n \geq 0\}, \ \varphi_0 = 1, \quad \varphi_n = \cos 2\sqrt{\lambda_n^*}x : n \geq 1 \tag{1.3}$$

and the mapping

$$T: \quad b \mapsto (\langle b, \varphi_n \rangle) \qquad (n \geq 0) \qquad b \in L^1[0, \pi]. \tag{1.4}$$

In what follows, L_0^p denotes the subset of $L^p = L_p(0, \pi)$ in which $\int_0^\pi b = 0$, l^p denotes the infinite sequences beginning from the 0th index with the usual p-norm, while l_0^p consists of sequences whose 0th coordinate is zero. If q^* is fixed, let $\Delta\Lambda(q)$ denote the sequence $\left(\int_0^\pi (q - q^*), (\lambda_n - \lambda_n^*)_{n\geq 1}\right)$.

Theorem 1.1. Let $q^* \in L^1[0, \pi]$, $q - q^* \in L_0^1$ and consider the eigenvalues $0 \neq \lambda_n^* \to \infty$, $\lambda_n^* \in \sigma(\alpha_n, 0; q^*)$. Suppose that $\lambda_n \in \sigma(\alpha_n, 0; q)$ are the eigenvalues corresponding to λ_n^*. The (nonlinear) mapping

$$q - q^* \mapsto \Delta\Lambda(q) \tag{1.5}$$

is continuous from L_0^r to l^s ($1 \leq r, s \leq \infty$) at $q = q^*$ if and only if the (linear) mapping (1.4) restricted to L_0^r is $L^r \to l^s$ continuous, that is,

$$\left(\sum |\langle h, \varphi_n \rangle|^s\right)^{1/s} \leq C\|h\|_r, \quad h \in L_0^r \tag{1.6}$$

(with the usual modification for $s = \infty$).

Moreover, if the mapping (1.4) from $L^r \to l^s$ $(1 \le r, s \le \infty)$ is bounded by a constant C, then

$$\left(\sum_n |\lambda_n - \lambda_n^*|^s\right)^{\frac{1}{s}} \le c(D)C\|q - q^*\|_r, \tag{1.7}$$

provided that $\|q\|_1, \|q^*\|_1 \le D$ and $\lambda_n^* \ge -D$. □

The general answer to the question of the stability of the inverse problem depends on whether the inverse of the mapping defined in (1.4) is bounded.

Theorem 1.2. Let $q^* \in L^1[0,\pi]$, $q - q^* \in L_0^1$ and consider the eigenvalues $0 \ne \lambda_n^* \to \infty$, $\lambda_n^* \in \sigma(\alpha_n, 0; q^*)$. Suppose that $\lambda_n \in \sigma(\alpha_n, 0; q)$ are the eigenvalues corresponding to λ_n^*. The following statements are equivalent:

A) The (possibly multivalued) inverse of the mapping (1.5)

$$\Delta\Lambda(q) \mapsto q - q^* \tag{1.8}$$

with domain $\{(c_n) \in l_0^s | \exists q \in L^1 : q - q^* \in L_0^r \text{ and } c_n = \lambda_n - \lambda_n^* \ \forall n > 0\}$ is a (nonlinear) $l^s \to L^r$ continuous mapping at $q = q^*$, that is, $\lambda_n(q) - \lambda_n^* \to 0$ in l^s implies $q - q^* \to 0$ in L^r.

B) The inverse of (1.4) with domain $\{(c_n) \in l_0^s | \exists h \in L_0^r : \ \forall n\, c_n = \langle h, \varphi_n\rangle\}$ is a (linear) $l^s \to L^r$ continuous mapping $(1 \le r, s \le \infty)$, that is,

$$\|h\|_r \le C\left(\sum |\langle h, \varphi_n\rangle|^s\right)^{1/s}, \quad h \in L_0^r \tag{1.9}$$

with obvious modification for $s = \infty$. The right-hand side is allowed to be infinite.

Moreover, in this case

$$\|q - q^*\|_r \le c(D)C\left(\sum_n |\lambda_n - \lambda_n^*|^s\right)^{\frac{1}{s}}, \tag{1.10}$$

provided that $\|q\|_1$, $\|q^*\|_1 \le D$, $q - q^* \in L_0^r$ and $\lambda_n^* \ge -D$. The upper bound in (1.10) can again be infinite. □

Remark. If the restriction of the mapping (1.4) to L^r_0 is continuous, then it is continuous on the entire L^r. However, their bounds can be different. Similarly, if (1.9) holds, then the same inequality hold for all $h \in L^r$ with a different constant, see Lemma 8.1. □

Remark. Instead of the condition $\lambda^*_n \neq 0$, we could require $\lambda^*_n \neq \mu$ for any $\mu \in \mathbb{C}$. In that case, we would have defined $\varphi_0(x) = \cos 2\mu x$ and we would have restricted the mapping (1.4) to the subspace orthogonal to $\cos 2\mu x$ (instead of L^r_0), and for the estimates we would have had to require $|\mu| \leq D$. □

Remark. The estimate (1.10) shows that there are no different potentials q with the same eigenvalues $\lambda_n \in \sigma(\alpha_n, 0, q)$. This (and more) has been previously known: in Horváth [10] it is proved that the completeness of the system (1.3) is necessary and sufficient for the unique recovery of q from the eigenvalues. By definition, the completeness means that no nontrivial L^r-function can be orthogonal to all the elements of (1.3), that is, the mapping $b \mapsto (\langle b, \varphi_n \rangle)$ is injective. To ensure stability of the recovery of the potential we need more, namely that the inverse mapping is bounded. □

1.2 Frames

The system $\{\varphi_n\}$ in a separable Hilbert space H is a *frame* if there exist two constants $0 < m, M < \infty$ such that

$$m\|h\|^2 \leq \sum |\langle h, \varphi_n \rangle|^2 \leq M\|h\|^2 \quad h \in H. \tag{1.11}$$

We know that (see [3] and references therein) in this case for all $h \in H$ the series

$$Fh = \sum_n \langle h, \varphi_n \rangle \varphi_n \tag{1.12}$$

converges in norm and the operator F—the frame operator—is a bounded bijection of H. The left inverse of (1.4) is then given by

$$T^{-1}: \qquad (c_n) \mapsto \sum_n c_n F^{-1} \varphi_n \qquad (n \geq 0). \tag{1.13}$$

By left inverse, we mean that $T^{-1}T = \mathrm{id}_H$, while $TT^{-1}(c_n) = (c_n)$ if $(c_n) \in \mathrm{Ran}T$. By F, we can express the constants in (1.11) as follows:

$$m = \frac{1}{\|F^{-1}\|}, \qquad M = \|F\|. \tag{1.14}$$

The system $\{F^{-1}\varphi_n\}$ is known to be another frame, the inverse frame in H.

If only the first inequality holds in (1.11), then we say that $\{\varphi_n\}$ satisfies the *lower frame condition* with constant m in H. If only the second inequality holds in (1.11), then $\{\varphi_n\}$ is called *Bessel system* in H.

The system $\{\varphi_n\}$ is called a *Riesz basis* in H if it is the image of an orthonormal basis under an isomorphism of H. The system ψ_n is *biorthogonal* to the Riesz basis $\{\varphi_n\}$ if

$$\langle \varphi_n, \psi_k \rangle = \delta_{n,k}.$$

It is known that every Riesz basis is a frame and every minimal frame is a Riesz basis. We see from the first inequality (1.11) that a frame is necessarily complete, so it is either a Riesz basis or it contains "superfluous" terms (which are in the closed linear hull of the others). In a Riesz basis, the biorthogonal system is the inverse frame, that is, $\psi_n = F^{-1}\varphi_n$. Consequently, if the system (1.3) is a Riesz basis in $L_2(0,\pi)$ then the inverse of (1.4) as an $L^2 \to l^2$ operator exists and has the form (1.13).

1.3 Positive results

The following theorems require almost the same set of assumptions. For the convenience of the reader, we collect them in the condition (C):

(C) $\|q\|_1, \|q^*\|_1 \le D,\ 0 \ne \lambda_n^* \to \infty,\ -D \le \lambda_n^* \in \sigma(\alpha_n, 0, q^*),$ and $\lambda_n \in \sigma(\alpha_n, 0, q)$ are the eigenvalues corresponding to λ_n^* and $\lim_{n\to\infty} |\lambda_n^* - \lambda_n| = 0$.

The latter condition turns out to be equivalent to $\int_0^\pi (q - q^*) = 0$, see Lemma 6.3.

Theorem 1.3. Assume condition (C) and suppose that $\{\varphi_n\}$ is a Bessel system in $L^2(0,\pi)$, that is,

$$\sum |\langle h, \varphi_n \rangle|^2 \le M\|h\|_2^2 \quad h \in L_2.$$

Then for $1 \le p \le 2$,

$$\left(\sum_n |\lambda_n - \lambda_n^*|^{p'}\right)^{\frac{1}{p'}} \le c(D) M^{\frac{1}{p'}} \|q - q^*\|_p. \tag{1.15}$$

□

Theorem 1.4. Assume (C) and suppose that the system φ_n satisfies the lower frame condition with a constant m in $L^2(0,\pi)$. Then

$$m\|q - q^*\|_2^2 \le c(D) \sum_n |\lambda_n - \lambda_n^*|^2. \tag{1.16}$$

□

Remark. The first estimate of type (1.16) is implicitly given in a proof in Borg's fundamental paper [2] where the λ_n run over $\sigma(0,0) \cup \sigma(0,\beta)$ and if the right-hand side of (1.16) is small. A similar estimate with explicit constants is proved in Hald [6] for the Dirichlet spectrum of a symmetric potential but only in a small neighborhood of the zero potential. More precisely, let $q \in L^1$ be symmetrical, that is, $q(x) = q(\pi - x)$ a.e. and denote $\lambda_n = \lambda_n(q)$ the Dirichlet eigenvalues. Hald verified $\|q - q^*\|_2^2 \le c \sum |\lambda_n(q) - \lambda_n(q^*)|^2$ with an explicit constant if $\sum |\lambda_n(q) - \lambda_n(0)|^2$ and $\sum |\lambda_n(q^*) - \lambda_n(0)|^2$ are small. It is easy to see that the Dirichlet eigenvalues of a symmetric potential run over $\sigma(0,0) \cup \sigma(\pi/2, 0)$ if the operator is defined on the half-interval $[0, \pi/2]$. So Hald's result is equivalent to a local version of (1.16) for nonsymmetrical potentials. Ryabushko [23] then proved (1.16) in full generality for the set of eigenvalues $\lambda_n^* \in \sigma(\pi/2, 0, q) \cup \sigma(0, 0, q)$. □

Theorem 1.5. Assume (C). Suppose that the system φ_n is a frame in $L^2(0,\pi)$ with frame operator F and assume there exists a constant C such that for the elements of the inverse frame

$$\|F^{-1}\varphi_n\|_\infty \le C \qquad (n \ge 0). \tag{1.17}$$

Then for $1 \le p \le 2$

$$\|q - q^*\|_{p'} \le c(D) \left(C^{p'-2} \|F^{-1}\|\right)^{\frac{1}{p'}} \left(\sum_n |\lambda_n - \lambda_n^*|^p\right)^{\frac{1}{p}}. \tag{1.18}$$

□

Remark. In [2], Borg proved among others the following statement. Let $q \in L^1$ be symmetrical and denote $\lambda_n = \lambda_n(q)$ the Dirichlet eigenvalues. If $\sum |\lambda_n(q) - \lambda_n(0)|^p < \infty$ for some $1 < p \le 2$, then $q \in L^{p'}$. By the last remark, this result of Borg is almost the same as Theorem 1.5 in the special case where $q^* = 0$ and the eigenvalues are $\lambda_n \in \sigma(0, 0) \cup \sigma(\pi/2, 0)$. For $p = 1$, several special cases have been known. About symmetric non-Dirichlet boundary conditions and symmetric potential, see Theorems 2 and 3 in Hald [7], see also Hochstadt [8]. A local version for nonsymmetric potential with two spectra $\sigma(\alpha, 0) \cup \sigma(\alpha, \pi/2)$ is given in part 8.1 in Yurko [26]. □

Remark. If we consider a spectrum defined by $y'(0) - hy(0) = 0 = y'(\pi) + Hy(\pi)$ and the norming constants $\alpha_n = \int_0^\pi y(x, \lambda_n)^2 \, dx$, where $y(x, \lambda)$ is the solution started from $y(0) = 1$, $y'(0) = h$, the following local result is given in Mizutani [21]: if

$$A = \sum \left(|\lambda_n - \tilde{\lambda}_n| + \sqrt{|\lambda_n|}|\alpha_n - \tilde{\alpha}_n| \right)$$

is small enough then $\|q - \tilde{q}\|_\infty \le cA$. For the Dirichlet spectrum, an analogous statement can be found in McLaughlin [20]: if

$$A = \sqrt{\sum \left(|\lambda_n - \tilde{\lambda}_n|^2 + n^6 |\alpha_n - \tilde{\alpha}_n|^2 \right)}$$

is small then $\|q - \tilde{q}\|_2 \le cA$. For the stability of reconstruction from the spectral function, see also Marchenko and Maslov [18]. If we have to reconstruct the potential from one spectrum and from the constants $|y(\pi, \lambda_n)/y(0, \lambda_n)|$, stability is proved in Isaacson and Trubowitz [12], Pöschel and Trubowitz [22], and Chelkak and Korotyaev [4]. □

Theorem 1.6. Suppose (C) and let the system φ_n be a frame in $L^2(0, \pi)$ with frame operator F. Assume that there exists a constant C such that for the elements of the inverse frame

$$\|F^{-1}\varphi_n\|_{p'} \le C \qquad (n \ge 1). \tag{1.19}$$

Then for $p' \ge 2$

$$\|q - q^*\|_{p'} \le c(D) C \sum_n |\lambda_n - \lambda_n^*|. \tag{1.20}$$

□

Remark. Comparing with Theorem 1.5, we get here from a weaker assumption ((1.19) instead of (1.17)) a weaker statement (since the l^p-norms are decreasing in p). $\square$

1.4 Negative results

Theorem 1.7. Suppose (C), let the system φ_n be a frame in $L^2(0,\pi)$ with frame operator F and assume there exists a constant C such that the elements of the inverse frame satisfy (1.17). Then $\|q-q^*\|_{p'} \to 0$ does not imply $\sum_n |\lambda_n - \lambda_n^*|^p \to 0$ if $1 \le p < 2$. That is, the mapping (1.4) from $L_0^{p'}$ to l_0^p is not continuous at $q = q^*$. $\square$

Theorem 1.8. Suppose (C) and that the system φ_n is a frame in $L^2(0,\pi)$. If the elements of the inverse frame are not bounded in $L^{p'}$, then either $\sum_n |\lambda_n - \lambda_n^*| < \infty$ does not imply $\|q-q^*\|_{p'} < \infty$ or $\sum_n |\lambda_n - \lambda_n^*| \to 0$ does not imply $\|q-q^*\|_{p'} \to 0$ if $p' > 2$. $\square$

Theorem 1.9. Suppose (C) and that the system $\{\varphi_n\}$ is a Bessel system in $L^2(0,\pi)$. Then $\sum_n |\lambda_n - \lambda_n^*|^{p'} \to 0$ does not imply $\|q-q^*\|_p \to 0$ if $1 \le p < 2$. $\square$

1.5 Results in L^2

Consider the system

$$e(\Lambda) = \left\{1, e^{\pm 2i\sqrt{\lambda_n^*}x} : n \ge 1\right\}. \tag{1.21}$$

It is easy to see that if $e(\Lambda)$ is a frame or a Riesz basis in $L_2(-\pi,\pi)$, then also is the system (1.3) in $L^2(0,\pi)$, with similar constants (see Lemma 8.3). Using this observation, for $p = 2$ the following useful stability result (a special case of Theorems 1.3 and 1.4) can be stated:

Theorem 1.10. Let $\|q^*\|_1 \le D$, $-D \le \lambda_n^* \in \sigma(\alpha_n, 0, q^*)$ $(n \ge 1)$ are given such that $\lambda_n^* \ne 0$, $\lim_{n\to\infty} \lambda_n^* = +\infty$. If $\|q\|_1 \le D$, $\int_0^\pi (q-q^*) = 0$, λ_n are the corresponding elements of $\sigma(\alpha_n, 0, q)$ and the system (1.21) is a frame in $L_2(-\pi,\pi)$, then

$$c(D)m\|q-q^*\|_2^2 \le \sum |\lambda_n - \lambda_n^*|^2 \le c(D)M\|q-q^*\|_2^2 \tag{1.22}$$

with the constants in (1.14). $\square$

1.6 Special eigenvalues

For efficient Riesz basis tests of $e(\Lambda)$, see for example Hruscev, Nikolskii, and Pavlov [11]. We recall the following useful result: if $\delta_n \in \mathbf{C}$, $\delta_n \to 0$ for $|n| \to \infty$ and if $(n+\delta_n)$ is separated, that is, $\inf_{n\neq m}|(n+\delta_n)-(m+\delta_m)|>0$ then the system $\{e^{i(n+\delta_n)x}\}$ is Riesz basis in $L_2(-\pi,\pi)$. Now let $\{\lambda_{1,n}: n\geq 1\} = \sigma(\pi/2,0)$, $\{\lambda_{2,n}: n\geq 1\}=\sigma(0,0)$ and list in a common sequence $\{\lambda_n : n\geq 1\}$ the union of the two spectra. We know that $2\sqrt{\lambda_{1,n}} = 2n-1+\mathbf{o}(1)$ and $2\sqrt{\lambda_{2,n}} = 2n+\mathbf{o}(1)$, see for example [15]. Since $\sigma(\pi/2,0)\cap\sigma(0,0)=\emptyset$, we see that the exponents $0, \pm\sqrt{\lambda_n}$ are separated and give a $\mathbf{o}(1)$-perturbation of $\mathbb{Z}$. Consequently, (1.21) is a Riesz basis in $L_2(-\pi,\pi)$ and the following statement is a special case of Theorem 1.10:

Corollary 1.11. Define $\{\lambda_n : n\geq 1\} = \sigma(\pi/2,0,q)\cup\sigma(0,0,q)$ and analogously for q^* and λ_n^*. Then (1.22) holds in the same sense as in Theorem 1.10. □

Using Theorem 1.5, Theorem 1.9 and Theorem 1.15 , we can state

Corollary 1.12. Let $\|q\|_1, \|q^*\|_1 \leq D$, $1 < p \leq \infty$. Define $\{\lambda_n : n\geq 1\} = \sigma(\pi/2,0,q)\cup \sigma(0,0,q)$ and analogously for q^* and λ_n^*. Then the value of

$$\frac{\|q-q^*\|_p}{\left(\sum_n |\lambda_n-\lambda_n^*|^{p'}\right)^{\frac{1}{p'}}} \tag{1.23}$$

is locally bounded for $p\geq 2$, while for $p<2$ it can be arbitrarily large even if $\|q-q^*\|_p \to 0$. □

1.7 Finitely many known eigenvalues

For a practical point of view, one can measure only finitely many eigenvalues, hence we need a theorem which gives an estimate tending to zero if an increasing number of eigenvalues are equal.

Theorem 1.13. Suppose condition (C). If the system (1.21) is a frame in $L^2(-\pi,\pi)$ and the L^∞-norm of the elements of the inverse frame is bounded by C, then

$$\sup_{0\leq x\leq\pi} \left|\int_0^x (q-q^*)\right| \leq C\sum_n \frac{c(D)}{\sqrt{|\lambda_n^*|}}|\lambda_n-\lambda_n^*|. \tag{1.24}$$

□

The previous theorem has an immediate consequence:

Theorem 1.14. Assume (C) and that $\|q-q^*\|_2 \le D$. Suppose further that $|\lambda_n - \lambda_n^*| < \varepsilon$ if $1 \le n \le N$, for a given $\varepsilon > 0$. If the system (1.21) is a frame in $L^2(-\pi,\pi)$ with frame operator F and the L^∞-norm of the elements of the biorthogonal system is bounded by C, then

$$\sup_{0\le x\le\pi} \left|\int_0^x (q-q^*)\right| \le C c(D)\varepsilon \sum_{n=1}^{N} \frac{1}{\sqrt{|\lambda_n^*|}} + C c(D)\|F\|^{\frac{1}{2}} \left(\sum_{n=N+1}^{\infty} \frac{1}{|\lambda_n^*|}\right)^{\frac{1}{2}}. \tag{1.25}$$

□

If, for example, the first N Dirichlet eigenvalues and the first N Dirichlet–Neumann eigenvalues are given (i.e., there are $2N$ pair of eigenvalues and $\sqrt{\lambda_n^*} = \frac{1}{2}n + \mathbf{o}(1)$), then $\|F\|$ and C depend only on D, and this estimate gives $c(D)(\varepsilon \log N + N^{-\frac{1}{2}})$, which is the main result of Marletta and Weikard [19]. Remark that for the operators defined on the half-line, an analogous estimate is obtained in Marchenko and Maslov [18]. Theorem 1.13 also contains the corresponding result in Marletta and Weikard [19]. To verify it, we need that the system biorthogonal to (1.19) is uniformly bounded if the λ_n^* run over $\sigma(0,0) \cup \sigma(\pi/2,0)$. We prove more:

Theorem 1.15. Let $0 \ne \mu_n = n^2 + \mathbf{O}(1)$, $n \ge 1$ be arbitrary different real or complex numbers. Then the system $\{1, e^{\pm i\sqrt{\mu_n}x} : n \ge 1\}$ is a Riesz basis in $L^2(-\pi,\pi)$ and its biorthogonal system is uniformly bounded in $L^\infty(-\pi,\pi)$. □

2 Structure of the Proofs

First, we summarize the proof of Theorems 1.1–1.2. It consists of three comparisons between the norms

$$\|q-q^*\| \leftrightarrow \|A_q(q-q^*)\| \leftrightarrow \|(\langle A_q(q-q^*), \cos 2\sqrt{\lambda_n^*}x\rangle)\| \leftrightarrow \|\Delta\Lambda(q)\|, \tag{2.1}$$

(where the operators A_q are defined in (4.5)). The first comparison is based on the fact that the operators A_q are of Volterra type. This is verified in Section 4. The third one is contained in Lemma 6.2, which is a consequence of a series of estimates in Section 5. Thus, we reduce the comparison between $\|\Delta q\|$ and $\|\Delta\lambda\|$ to a comparison between $h = A_q(q-q^*)$ and $\langle h, \cos 2\sqrt{\lambda_n^*}x\rangle)$. The continuity of the mapping (1.4) or its inverse guarantees the stability of the direct or the inverse problem, respectively. In order to show that this continuity is an equivalent condition to the stability, we need to prove that the range of $A_q(q-q^*)$ contains a common part of a ball and a dense set. This is also found in Section 4.

Remark. There are some straightforward extensions of the above listed stability and instability results. In Theorems 1.4, 1.5, 1.6, 1.7, 1.13, and 1.14, we can suppose that the set of eigenvalues λ_n^* contains a subsystem with the desired properties while in Theorems 1.3 and 1.9 we can suppose that $\{\lambda_n^*\}$ can be extended to a system with the prescribed properties. Secondly, since the L^p-norms are increasing (apart from constant factors) and the l^p-norms are decreasing, a simple corollary of Theorems 1.5 and 1.9 is that $\|\Delta q\|_r \le c\|\Delta\lambda\|_s$ holds for $1 \le r \le 2,\ 1 \le s \le r/(r-1)$ and does not hold for $r > 2$, $s \ge r/(r-1)$. Analogously, we infer from Theorems 1.3 and 1.7 that $\|\Delta\lambda\|_s \le c\|\Delta q\|_r$ is true for $r \ge 2,\ s \ge r/(r-1)$ and is not true for $1 \le r < 2,\ 1 \le s \le r/(r-1)$.

3 Extensions to Complex-valued Potentials

If we allow the potentials to take complex values, the main difficulty is the appearance of algebraically multiple eigenvalues. More precisely, let q and q^* be two potentials in L_p and $q_s(x) = sq^*(x) + (1-s)q(x)$ be the linear deformation of the potential from q to q^*. Introduce the (characteristic) function

$$F(\lambda, s) = \cos\alpha\, y_2(\pi, \lambda; q_s) + \sin\alpha\, y_2'(\pi, \lambda; q_s)$$

where the solution y_2 is defined at the beginning of Section 5. Clearly, $\lambda \in \sigma(0, \alpha; q_s)$ if and only if $F(\lambda, s) = 0$. It is known that if $\lambda = \lambda(0) \in \sigma(0, \alpha; q_0)$ then there exists a continuous branch of eigenvalues $\lambda(s) \in \sigma(0, \alpha; q_s)$. Suppose, for example, that $\lambda(0)$ is an algebraically double eigenvalue, that is,

$$F_\lambda'(\lambda(0), 0) = 0.$$

Expanding $F(\lambda, s)$ around $(\lambda(0), 0)$ at the point $(\lambda(s), s)$, we see that for small s, $(\lambda(s) - \lambda(0))^2$ is proportional to s (if $F_s'(\lambda(0), 0) \neq 0$), consequently no inequality of type $|\lambda^* - \lambda| \le c\|q^* - q\|_p$ can be true if λ is a multiple eigenvalue. Thus, no direct stability estimate $\|\Delta\lambda\|_{p'} \le c\|\Delta q\|_p$ holds in this setting. The negative inverse stability results obviously remain valid. The positive inverse stability results listed in Section 1 can be proved if all the values λ_n^* are different. We need the following interpretation of the corresponding eigenvalues. If we are given eigenvalues $\lambda_n^* \in \sigma(\alpha_n, 0; q^*)$, $n \ge 0$, we say that $\lambda_n \in \sigma(\alpha_n, 0; q)$

is the eigenvalue corresponding to λ_n^* if λ_n^* can be continuously shifted to λ_n by the linear deformation of the potentials, that is, if there exists a continuous function $\lambda(s)$ with $\lambda(0) = \lambda_n^*$, $\lambda(1) = \lambda_n$, and $\lambda(s) \in \sigma(\alpha_n, 0; q_s)$. We choose a correspondence $\lambda_n^* \mapsto \lambda_n$ such that every value λ_n occurs in the sequence no more than its multiplicity. The condition (C) is substituted by

(CC) $\|q\|_1, \|q^*\|_1 \leq D$, $q - q^* \in L_0^1$, $0 \neq \lambda_n^* \in \sigma(\alpha_n, 0, q^*)$ are different values, $-D \leq \Re\lambda_n^* \to \infty$ and $\lambda_n \in \sigma(\alpha_n, 0, q)$ are the eigenvalues corresponding to λ_n^* in the above defined sense.

The following inverse results hold for complex potentials:

Theorem 3.1. Let $q^* \in L_1(0, \pi)$ and consider the different eigenvalues $0 \neq \lambda_n^* \in \sigma(\alpha_n, 0, q^*)$, $-D \leq \Re\lambda_n^* \to \infty$. Then the implication **B** $\Rightarrow$ **A** of Theorem 1.2 holds and **B** implies (1.10) if $\|q\|_1 \leq D$, $\|q^*\|_1 \leq D$, and $q - q^* \in L_0^r$. □

Theorem 3.2. The statements of Theorems 1.4, 1.5, 1.6, and 1.13 hold if (C) is substituted by (CC). □

The counterpart of Theorem 1.14 is as follows:

Theorem 3.3. Assume (CC) and suppose that $|\lambda_n - \lambda_n^*| < \varepsilon$ if $1 \leq n \leq N$, for a given $\varepsilon > 0$. If the system (1.21) is a frame in $L^2(-\pi, \pi)$ with frame operator F and the L^∞-norm of the elements of the biorthogonal system is bounded by C, then

$$\sup_{0\leq x\leq\pi} \left| \int_0^x (q - q^*) \right| \leq C c(D) \varepsilon \sum_{n=1}^{N} \frac{1}{\sqrt{|\lambda_n^*|}} \tag{3.1}$$

$$+ C c(D) \left(\sum_{n=N+1}^{\infty} \frac{1}{|\lambda_n^*|} \right)^{\frac{1}{2}} \left(\sum_n |\lambda_n^* - \lambda_n|^2 \right)^{\frac{1}{2}}. \qquad \square$$

Unfortunately, this statement does not contain the Marletta–Weikard result if the union $\sigma(0, 0) \cup \sigma(\pi/2, 0)$ contains multiple eigenvalues.

In the proofs, we cannot use the variational calculus: the formula $\dot{\lambda}_n = |g_n^2|$ corresponding to (5.8) holds only for simple eigenvalues. Instead, we apply the representation (5.6) and some Lipschitz properties of its kernel function. More details are given at the end of the paper.

4 Integral Operators

As before, $c(D)$ denotes constants, depending only on D, possibly different in each occurrence. Let $\lambda \in \mathbb{C}$, $z = \sqrt{\lambda}$. Introduce the function $v(x, \lambda)$ as the solution of (1.1) with the initial conditions

$$v(\pi, \lambda) = 0, \quad v'(\pi, \lambda) = -1.$$

We need the following lemmas:

Lemma 4.1. (Lemma 5.2 of Horváth [10]) Let $\|q\|_1, \|q^*\|_1 \le D$. Then there exists a continuous kernel function M_1 such that

$$1 - 2z^2 v(\pi - x, \lambda) v^*(\pi - x, \lambda) = \cos 2xz + \int_0^x \cos 2tz M_1(x, t, q, q^*)\, \mathrm{d}t, \tag{4.1}$$

and

$$|M_1(x, t, q, q^*)| \le c(D), \tag{4.2}$$

$$|M_1(x, t, q_1, q^*) - M_1(x, t, q_2, q^*)| \le c(D)\|q_1 - q_2\|_1. \tag{4.3}$$

□

Corollary 4.2. Let $h \in L^p(0, \pi)$. Then

$$\int_0^\pi h - 2z^2 \int_0^\pi h(x) v(x, \lambda) v^*(x, \lambda)\, \mathrm{d}x = \int_0^\pi A_q\, (h(\pi - x)) \cos 2xz\, \mathrm{d}x, \tag{4.4}$$

where

$$(A_q h)(x) = h(x) + \int_x^\pi M(x, t) h(t)\, \mathrm{d}t \qquad h \in L_1(0, \pi). \tag{4.5}$$

with $M(x, t) = M_1(t, x)$. □

Proof. Multiplying (4.1) by $h(\pi - x)$, integrating from 0 to π and changing the order of integrations give the formula (4.4). ■

Substituting $z = 0$ to (4.4) gives

Corollary 4.3.

$$\int_0^\pi A_q h = \int_0^\pi h, \tag{4.6}$$

□

and using $1 - \cos 2xz = 2\sin^2 xz$ we get the following:

Corollary 4.4. Let $h \in L^p(0, \pi)$. Then

$$\int_0^\pi h(x)v(x,\lambda)v^*(x,\lambda)\,\mathrm{d}x = \int_0^\pi A_q\,(h(\pi - x))\left(\frac{\sin xz}{z}\right)^2 \mathrm{d}x. \tag{4.7}$$

□

Substituting $h(x) = q(\pi - x) - q^*(\pi - x)$ to (4.4) gives

Corollary 4.5.

$$\begin{aligned}\int_0^\pi (q - q^*) - 2z^2 \int_0^\pi (q(x) - q^*(x))v(x,\lambda)v^*(x,\lambda)\,\mathrm{d}x \\ = \int_0^\pi A_q\left(q(\pi - x) - q^*(\pi - x)\right)\cos 2xz\,\mathrm{d}x.\end{aligned} \tag{4.8}$$

Lemma 4.6. Let $q^* \in L_1[0, \pi]$ be fixed. If $\|q\|_1 \le D$, $\|q^*\|_1 \le D$, then $A_q : L_0^p \to L_0^p$ is continuous, linear and invertible, and both of A_q and A_q^{-1} are bounded with a bound depending only on D. Moreover,

$$\|A_{q_1} - A_{q_2}\|_p \le c(D)\|q_1 - q_2\|_1 \le c(D)\|q_1 - q_2\|_p. \tag{4.9}$$

□

Proof. Let $\|h\|_1 \le D$. By the first estimate on M_1 (and M),

$$\frac{1}{\pi}\|(I - A_q)^{n+1}h\|_p \le \|(I - A_q)^{n+1}h\|_\infty \le \frac{c(D)^n}{n!}\|h\|_1 \le \frac{c(D)^n}{n!}\|h\|_p, \tag{4.10}$$

which implies the continuity of A_q and (by the convergence of the Neumann series) of A_q^{-1} from L^p to L^p with a norm depending only on D. By (4.6), A_q restricted to L_0^p is also an isomorphism with the same norms. Finally, the second estimate on M_1 implies (4.9). ■

Lemma 4.7. [10] Let B_1 and B_2 be Banach spaces, and let for all $q \in B_1$

$$A_q : B_1 \to B_2$$

be a continuous, linear operator. If

(i) for some fixed $q_0 \in B_1$ A_{q_0} is one-to-one and its inverse is continuous,
(ii) the mapping $q \to A_q$ has the following property: for all $h \in B_1$

$$\|(A_{q_1} - A_{q_2})h\| \leq c(q_0)\|q_1 - q_2\|\|h\|, \text{ if } \|q_1\|, \|q_2\| \leq \|q_0\| + 1, \tag{4.11}$$

where $c(q_0)$ is a constant independent of q,q^* and of h,

then the set $\{A_q(q - q_0) : q \in B_1\}$ contains a ball in B_2, centered at the origin. □

For the sake of completeness, we provide a (new) proof with a lower bound for the radius of the ball.

Proof. Suppose that

$$\|A_{q_0}\|, \|A_{q_0}^{-1}\| \leq K \text{ and } c(q_0) \leq K$$

for some $K > 1$, where $c(q_0)$ is the constant from (4.11). Consider two vectors $q, \tilde{q} \in B_1$ with

$$\|q - q_0\|, \|\tilde{q} - q_0\| \leq \frac{1}{2K^2}.$$

Then

$$\|A_{q_0}^{-1}(A_q - A_{q_0})\| \leq K^2\|q - q_0\| \leq \frac{1}{2}$$

and

$$\|A_q^{-1}\| = \|\left(I + A_{q_0}^{-1}(A_q - A_{q_0})\right)^{-1} A_{q_0}^{-1}\| \leq 2K.$$

Finally,

$$\|A_{\tilde{q}}^{-1} - A_q^{-1}\| \leq \|A_{\tilde{q}}^{-1}\|\|A_{\tilde{q}} - A_q\|\|A_q^{-1}\| \leq 4K^3\|\tilde{q} - q\|.$$

Let $q_2 \in B_2$ with $\|q_2\| \le \frac{1}{8K^3}$. Define $F(q)$ by

$$A_q(F(q) - q_0) = q_2,$$

that is, $F(q) = q_0 + A_q^{-1} q_2$. Then F is defined on the ball $\|q - q_0\| \le \frac{1}{2K^2}$ and maps into this ball since $\|A_q^{-1} q_2\| \le \frac{1}{4K^2}$. F is a contraction on this ball since

$$\|F(\tilde{q}) - F(q)\| \le \|A_{\tilde{q}}^{-1} - A_q^{-1}\| \|q_2\| \le 4K^3 \|\tilde{q} - q\| \|q_2\| \le \frac{1}{2} \|\tilde{q} - q\|.$$

By the Banach fixed point theorem, F has a fixed point q, that is, the equation $A_q(q - q_0) = q_2$ has a solution q for every $\|q_2\| \le \frac{1}{8K^3}$. ■

By the previous lemma, we obtain the following special cases:

Corollary 4.8. Fix $q^* \in L_1$. Define the operators A_q as in Corollary 4.2 and let $\|q\|_1$, $\|q^*\|_1 \le D$. Then the set $\{A_q(q - q^*) : q - q^* \in L_0^1\}$ contains a ball of radius $\ge c(D) > 0$ around the origin in L_0^1. □

Corollary 4.9. Fix $q^* \in L_1$. Define the operators A_q as in Corollary 4.2 and let $\|q\|_1$, $\|q^*\|_1 \le D$. Let X be a function space on $[0, \pi]$ equipped with such a norm $\|\cdot\|_X$ that the operators A_q are isomorphism on $(X \cap L_0^1, \|\cdot\|_X)$ with a bound depending only on D. Then the set $\{A_q(q - q^*) : q - q^* \in X \cap L_0^1\}$ contains the intersection of X and a ball of radius $\ge c(D) > 0$ around the origin in L_0^1. □

Proof. We know from the previous corollary that the set $\{A_q(q - q^*) : q - q^* \in L_0^1\}$ contains a ball of radius $\ge c(D) > 0$ around the origin in L_0^1. Let h be an element of this ball, then $h = A_q(q - q^*)$ for some $q - q^* \in L_0^1$. If also $h \in X$, then $(q - q^*) \in X$ for A_q is an isomorphism on $X \cap L_0^1$. ■

5 Derivative with Respect to the Potential

Let y_1 and y_2 are two solutions of (1.1) such that $\begin{pmatrix} y_1(0,\lambda) \\ y_1'(0,\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} y_2(0,\lambda) \\ y_2'(0,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Statement 5.1. [13, 22, 27] For $q \in L^1(0, \pi)$

$$\frac{\partial y_j}{\partial q}(x) = y_j(t)[y_1(t)y_2(x) - y_2(t)y_1(x)]\chi_{(0,x)}(t) \qquad j = 1, 2, \tag{5.1}$$

$$\frac{\partial y_j'}{\partial q}(x) = y_j(t)[y_1(t)y_2'(x) - y_2(t)y_1'(x)]\chi_{(0,x)}(t) \qquad j = 1, 2. \tag{5.2}$$

□

Lemma 5.2. Let $\lambda = z^2 \geq -D$, $\|q\|_1 \leq D$. Then

$$|y_1(x, \lambda)| \leq c(D), \tag{5.3}$$

$$|y_2(x, \lambda)| \leq \frac{c(D)}{1 + |z|}. \tag{5.4}$$

□

Proof. By a well-known representation

$$y_1(x, \lambda) = \cos xz + \int_0^x K_1(x, t) \cos tz \, \mathrm{d}t, \tag{5.5}$$

$$y_2(x, \lambda) = \frac{\sin xz}{z} + \int_0^x K_2(x, t) \frac{\sin tz}{z} \, \mathrm{d}t, \tag{5.6}$$

where the kernel K_1, K_2 are continuous and $|K_i(x, t)| \leq c(D)$, see [10], Lemma 5.1. Thus, $|\cos xz| \leq c(D)$ and $\frac{\sin xz}{z} \leq \frac{c(D)}{1+|z|}$ gives the statement. ■

Using this lemma, an elementary estimate of the supremum of the derivative yields that

Corollary 5.3.

$$|y_2(x, \lambda, q) - y_2(x, \lambda, q^*)| \leq \frac{c(D)}{1 + |\lambda|} \|q - q^*\|_1. \tag{5.7}$$

□

Statement 5.4. [13, 22, 27] Let $q \in L_1(0, \pi)$, $\lambda_n = \lambda_n(q) \in \sigma(\alpha, \beta, q)$. Then λ_n is an analytic function of q and

$$\frac{\partial \lambda_n}{\partial q} = g_n^2(t), \tag{5.8}$$

where g_n is the normed eigenfunction corresponding to λ_n. □

Lemma 5.5. Suppose $\|q\|_1 \le D$. Then

$$\int_0^\pi y_2^2(x,\lambda,q)\,\mathrm{d}x < c(D)\int_0^\pi y_2^2(x,\lambda,0)\,\mathrm{d}x, \tag{5.9}$$
$$\int_0^\pi y_2^2(x,\lambda,q)\,\mathrm{d}x > \varepsilon(D)\int_0^\pi y_2^2(x,\lambda,0)\,\mathrm{d}x, \tag{5.10}$$

for some numbers $c(D)$ and $\epsilon(D) > 0$, independent of q and of λ. □

Proof. Taking into account that $v(x,q) = y_2(\pi - x, q(\pi - x))$, we can substitute $h(x) = 1$ and $q(\pi - x)$ instead of q and q^* into (4.7) and using the $L^\infty \to L^\infty$ continuity of the operator A_q:

$$\int_0^\pi y_2^2(x,\lambda)\,\mathrm{d}x \le \|A_q\|\int_0^\pi \left(\frac{\sin xz}{z}\right)^2 \mathrm{d}x. \tag{5.11}$$

Since $A_q : L^\infty \to L^\infty$ is an isomorphism bounded by a constant $c(D)$, there exists $h \in L^\infty$ with $A_{q(\pi-x)}h(\pi - x) = 1$. Substituting it into (4.7) gives

$$\int_0^\pi \left(\frac{\sin xz}{z}\right)^2 \mathrm{d}x \le \|A_q^{-1}\|\int_0^\pi y_2^2(x,\lambda)\,\mathrm{d}x, \tag{5.12}$$

which proves the lemma. ■

Corollary 5.6. Let $\lambda \in \mathbb{R}$, $\|q\|_1 \le D$. Then

$$\frac{|y_2(x,\lambda,q)|^2}{\|y_2(x,\lambda,q)\|_2^2} < \begin{cases} c(D), & \text{if } \lambda \ge 0, \\ c(D)(1+|z|), & \text{if } \lambda < 0. \end{cases} \tag{5.13}$$

□

Proof.

$$\frac{|y_2(x,\lambda,q)|}{\|y_2(x,\lambda,q)\|_2} \le \frac{c(D)}{\varepsilon(D)}\frac{\|\sin xz\|_\infty}{\|\sin xz\|_2}, \qquad x \in [0,\pi], \tag{5.14}$$

which is bounded if $\lambda \to 0$ or $\lambda \to +\infty$, and $O(|z|^{\frac{1}{2}})$ in the case of $\lambda \to -\infty$. ■

By $v(x, \lambda, q) = y_2(\pi - x, \lambda, q(\pi - x))$ and by $\left|\frac{\sin xz}{z}\right| \le \frac{c(D)}{1+|z|}$ for $\lambda \ge -D$, we have:

Corollary 5.7. For $\lambda \ge -D$, $\|q\|_1 \le D$

$$\frac{\varepsilon(D)}{1+|\lambda|} < \int_0^\pi v^2(x, \lambda, q)\, \mathrm{d}x < \frac{c(D)}{1+|\lambda|}. \tag{5.15}$$

□

Theorem 5.8. Denote by λ_n the nth element of $\sigma(\alpha, 0)$. If $p, q \in L^1$, $||p||_1, ||q||_1 \le D$, $N \ge 0$ and $\lambda_n(q + t(p-q)) \ge -N^2$ for every $t \in (0, 1)$, then

$$|\lambda_n(p) - \lambda_n(q)| < c(D)(N+1)||p-q||_1, \tag{5.16}$$

where $c(D)$ is independent of α, n, N, p, and q. □

Proof.

$$|\lambda_n(p) - \lambda_n(q)| \le \int_0^1 |\frac{\mathrm{d}}{\mathrm{d}t}\lambda_n(q + t(p-q))|\, \mathrm{d}t, \tag{5.17}$$

and taking into account the bound of Corollary 5.6

$$\begin{aligned} |\frac{\mathrm{d}}{\mathrm{d}t}\lambda_n(q + t(p-q))| &= |\int_0^\pi g_n^2(x, q + t(p-q))(p-q)\, \mathrm{d}x| \\ &< c(D)(N+1)||p-q||_1, \end{aligned}$$

which implies the statement. ■

Theorem 5.9. Denote by λ_n the nth element of $\sigma(\alpha, 0)$. If $\lambda_n(q) \ge -D$, $p, q \in L^1$ and $||p||_1$, $||q||_1 \le D$ then

$$|\lambda_n(p) - \lambda_n(q)| < c(D)||p-q||_1, \tag{5.18}$$

where $c(D)$ is independent of α, p, q and n. □

Proof. Choose $N = N(D)$ so large that $2Dc(D)(N+1) < N^2 - D$ for the constant $c(D)$ appearing in Theorem 5.8. We claim that $\lambda_n(p) > -N^2$ at every point $\|p\|_1 \le D$ (and then Theorem 5.9 follows from Theorem 5.8). Suppose indirectly that there is a point p for which $\lambda_n(p) = -N^2$. If there are points in the segment (q, p) for which $\lambda_n = -N^2$, we redefine p as the point closest to q with this property. Thus, we can suppose that $\lambda_n > -N^2$ on (q, p). Applying Theorem 5.8, we get $|\lambda_n(p) - \lambda_n(q)| \leq c(D)(N+1)\|p-q\|_1 \leq 2Dc(D)(N+1) \leq N^2 - D$. But this contradicts to $\lambda_n(q) \geq -D$ and $\lambda_n(p) = -N^2$. Thus, the statement follows by contraposition. ∎

Remark. We will not use directly that the indices of λ_n and λ_n^* are equal, only the result of Theorem 5.9, that is, $|\lambda_n - \lambda_n^*| < c(D)||q - q^*||_1$. □

Corollary 5.10. Let $\lambda_n^* \in \sigma(\alpha, 0, q^*)$ and let λ_n be the corresponding element of $\sigma(\alpha, 0, q)$. If $\lambda_n^* \geq -D$, $||q||_1, ||q^*||_1 \leq D$ then $0 < \varepsilon(D) \leq \frac{1+|\lambda_n|}{1+|\lambda_n^*|} \leq c(D)$. □

Proof.

$$\frac{1+|\lambda_n|}{1+|\lambda_n^*|} \leq 1 + \frac{|\lambda_n - \lambda_n^*|}{1+|\lambda_n^*|} \leq c(D), \tag{5.19}$$

and similarly $\frac{1+|\lambda_n^*|}{1+|\lambda_n|}$ is also bounded. ∎

Lemma 5.11. Let $\lambda,\ \lambda^* \geq -c(D)$, $\lambda = z^2$, $\lambda^* = z^{*2}$, $q^* = q$, and $\|q\|_1 \leq D$. Then

$$|v(x,\lambda) - v(x,\lambda^*)| \leq |\lambda - \lambda^*| \frac{c(D)}{1+\min(|\lambda|, |\lambda^*|)}. \tag{5.20}$$

□

Proof. Again by the well-known representation

$$v(\pi - x, \lambda) = \frac{\sin xz}{z} + \int_0^x K(x,t) \frac{\sin tz}{z} \mathrm{d}t, \tag{5.21}$$

where the kernel K is continuous and $|K(x,t)| \leq c(D)$. Indeed,

$$\left|\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\sin xz}{z}\right| = \left|\frac{x\cos xz - \frac{\sin xz}{z}}{2\lambda}\right| \leq \pi \frac{e^{|\Im z|\pi}}{1+|\lambda|} \leq \frac{c(D)}{1+|\lambda|},$$

$$\left|\frac{\sin xz}{z} - \frac{\sin xz^*}{z^*}\right| \leq |\lambda - \lambda^*| \frac{c(D)}{1+\min(|\lambda|, |\lambda^*|)},$$

which, using (5.21), leads to (5.20). ∎

6 The Main Tool

All estimates in the previous section were made in order to prove the following:

Lemma 6.1. If $\|q\|_p, \|q^*\|_p \le D$, $\lambda_n^* \ge -D$, and λ_n are corresponding elements of $\sigma(\alpha, 0, q)$, then

$$\left|\int_0^\pi (q(x) - q^*(x))v(x, \lambda_n^*)v^*(x, \lambda_n^*)\,\mathrm{d}x\right| \le \frac{c(D)}{1 + |\lambda_n^*|}|\lambda_n - \lambda_n^*|. \tag{6.1}$$

There exist further constants depending only on D such that if $\frac{\|q-q^*\|_p}{1+|z_n^*|} \le c(D)$, then

$$\left|\int_0^\pi (q(x) - q^*(x))v(x, \lambda_n^*)v^*(x, \lambda_n^*)\,\mathrm{d}x\right| \ge \frac{c(D)}{1 + |\lambda_n^*|}|\lambda_n - \lambda_n^*|. \tag{6.2}$$

Especially, this inequality holds either if $U = \|q - q^*\|_p$ is appropriately small, or if $z_n^* \ge c(D)$. □

Proof. If (and only if) λ_n and λ_n^* are both in the spectrum $\sigma(\alpha, 0)$,

$$\begin{aligned} 0 &= \int_0^\pi \frac{\mathrm{d}}{\mathrm{d}x}\left[v'(x, \lambda_n)v^*(x, \lambda_n^*) - v(x, \lambda_n)v^{*\prime}(x, \lambda_n^*)\right]\,\mathrm{d}x \\ &= \int_0^\pi (\lambda_n - \lambda_n^* + q^*(x) - q(x))v(x, \lambda_n)v^*(x, \lambda_n^*)\,\mathrm{d}x, \end{aligned}$$

hence

$$\begin{aligned} &\int_0^\pi (q(x) - q^*(x))v(x, \lambda_n^*)v^*(x, \lambda_n^*)\,\mathrm{d}x \\ &\quad = \int_0^\pi (q(x) - q^*(x))v^*(x, \lambda_n^*)[v(x, \lambda_n^*) - v(x, \lambda_n)]\,\mathrm{d}x \\ &\qquad + (\lambda_n - \lambda_n^*)\int_0^\pi v^*(x, \lambda_n^*)[v(x, \lambda_n) - v(x, \lambda_n^*)]\,\mathrm{d}x \\ &\qquad + (\lambda_n - \lambda_n^*)\int_0^\pi v(x, \lambda_n^*)[v^*(x, \lambda_n^*) - v(x, \lambda_n^*)]\,\mathrm{d}x \\ &\qquad + (\lambda_n - \lambda_n^*)\int_0^\pi v^2(x, \lambda_n^*)\,\mathrm{d}x = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We will give estimates for all of these terms. By Corollary 5.10, dividing each of $1 + |z_n|$ and $1 + |z_n^*|$ leads to the same estimate.

By Lemma 5.2 and Lemma 5.11

$$|I_1| \le \frac{c(D)}{1+|\lambda_n^*|}|\lambda_n - \lambda_n^*| \frac{\|q(x)-q^*(x)\|_1}{1+|z_n^*|} \le \frac{c(D)}{1+|\lambda_n^*|}|\lambda_n - \lambda_n^*| \frac{\|q(x)-q^*(x)\|_p}{1+|z_n^*|}. \tag{6.3}$$

By that again and by Theorem 5.9

$$|I_2| \le \frac{c(D)}{1+|z_n^*|^3}|\lambda_n - \lambda_n^*|^2 \le \frac{c(D)}{1+|\lambda_n^*|}|\lambda_n - \lambda_n^*| \frac{\|q(x)-q^*(x)\|_p}{1+|z_n^*|}. \tag{6.4}$$

By Lemma 5.2 and Corollary 5.3

$$|I_3| \le \frac{c(D)}{1+|\lambda_n^*|}|\lambda_n - \lambda_n^*| \frac{\|q(x)-q^*(x)\|_1}{1+|z_n^*|} \le \frac{c(D)}{1+|\lambda_n^*|}|\lambda_n - \lambda_n^*| \frac{\|q(x)-q^*(x)\|_p}{1+|z_n^*|}. \tag{6.5}$$

Finally, according to Corollary 5.7

$$\frac{c(D)}{1+|\lambda_n^*|}|\lambda_n - \lambda_n^*| \le |I_4| \le \frac{c(D)}{1+|\lambda_n^*|}|\lambda_n - \lambda_n^*|, \tag{6.6}$$

which implies the formulated estimates. ■

Corollary 6.2. Let $\lambda_n^* \in \sigma(\alpha_n, 0, q^*)$ and let λ_n be the corresponding element of $\sigma(\alpha_n, 0, q)$. If $||q||_1, ||q^*||_1 \le D$, $0 \ne \lambda_n^* \ge -D$, $\lim_{n\to\infty} \lambda_n^* = +\infty$, and $\lim_{n\to\infty} |\lambda_n^* - \lambda_n| = 0$, then

$$\left| \int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x) \right) \cos 2z_n^* x \, \mathrm{d}x \right| \le c(D)|\lambda_n - \lambda_n^*|.$$

There exist further constants depending only on D such that if $\frac{\|q-q^*\|_p}{|z_n^*|} \le c(D)$, then

$$\left| \int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x) \right) \cos 2z_n^* x \, \mathrm{d}x \right| \ge c(D)|\lambda_n - \lambda_n^*|. \tag{6.7}$$

Especially, this inequality holds either if $z_n^* \ne 0$ and $U = \|q - q^*\|_p$ is appropriately small, or if $z_n^* \ge c(D)$. □

Proof. According to Riemann's Lemma,

$$\int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x) \right) \cos 2z_n^* x \, \mathrm{d}x \to 0.$$

Then, by (4.8), $\int_0^\pi (q-q^*)=0$ and

$$\begin{aligned}\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2z_n^* x\,\mathrm{d}x \\ = -2z_n^{*2}\int_0^\pi (q(x)-q^*(x))v(x,\lambda_n^*)v^*(x,\lambda_n^*)\,\mathrm{d}x,\end{aligned} \tag{6.8}$$

thus the formulated estimates follow from the previous lemma. ■

Lemma 6.3. Let $q,\ q^*\in L_1(0,\pi)$, $\lambda_n^*\to\infty$, where $\lambda_n^*\in\sigma(\alpha_n,0,q^*)$ and let λ_n be the corresponding element of $\sigma(\alpha_n,0,q)$. Then

$$\lambda_n-\lambda_n^*\to 0\ \ (n\to\infty)\quad\Leftrightarrow\quad\int\limits_0^\pi (q-q^*)=0. \tag{6.9}$$

□

Proof. By (4.8)

$$\begin{aligned}\int_0^\pi (q-q^*)-2\lambda_n^*\int_0^\pi (q(x)-q^*(x))v(x,\lambda_n^*)v^*(x,\lambda_n^*)\,\mathrm{d}x \\ = \int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2\sqrt{\lambda_n^*}x\,\mathrm{d}x.\end{aligned} \tag{6.10}$$

Here the right-hand side tends to zero by the Riemann lemma and the second summand on the left has the exact order $\lambda_n-\lambda_n^*$ for large n. This proves Lemma 6.3. ■

7 Proof of the General Results

Proof of Theorem 1.2. By (4.6), $A_q\,(q(\pi-x)-q^*(\pi-x))\in L_0^r$. If the inverse of (1.4) is continuous, then

$$\begin{aligned}\|q-q^*\|_r &\le c(D)\|A_q\left(q(\pi-x)-q^*(\pi-x)\right)\|_r \\ &\le c(D)C\left(\sum_n\left|\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2\sqrt{\lambda_n^*}x\,\mathrm{d}x\right|^s\right)^{\frac{1}{s}} \\ &\le c(D)C\left(\sum_n|\lambda_n-\lambda_n^*|^s\right)^{\frac{1}{s}},\end{aligned} \tag{7.1}$$

which implies the continuity of (1.8) at $q = q^*$. In contrast, if the inverse mapping is not bounded, we can choose a sequence $h_k \in L_0^r$ such that $\|h_k\|_r = 1$ but

$$\lim_{k\to+\infty}\left(\sum_n\left|\int_0^\pi h_k(x)\cos 2\sqrt{\lambda_n^*}x\,\mathrm{d}x\right|^s\right)^{\frac{1}{s}} = 0. \tag{7.2}$$

Now Corollary 4.9 implies that for appropriately small $\gamma = \gamma(D) > 0$ there exist potentials $q_k \in L^1$, $q_k - q^* \in L_0^r$ such that

$$A_{q_k}\left(q_k(\pi - x) - q^*(\pi - x)\right) = \gamma h_k(x). \tag{7.3}$$

We can choose $\gamma(D) > 0$ so small that (6.7) holds for all q_k. Then

$$\lim_{k\to+\infty}\left(\sum_n|\lambda_{k,n} - \lambda_n^*|^s\right)^{\frac{1}{s}} \le c(D,U)\lim_{k\to+\infty}\left(\sum_n\left|\int_0^\pi \gamma h_k(x)\cos 2\sqrt{\lambda_n^*}x\,\mathrm{d}x\right|^s\right)^{\frac{1}{s}} = 0, \tag{7.4}$$

but

$$\|q_k - q^*\|_r \ge c(D)\|A_q\left(q_k(\pi - x) - q^*(\pi - x)\right)\|_r = \gamma c(D) > 0, \tag{7.5}$$

thus (1.8) is not continuous. ■

Proof of Theorem 1.1. For $s = \infty$ the statement follows from Theorem 5.9. Assume now $s < \infty$. If the mapping (1.4) is continuous, then

$$\begin{aligned}\left(\sum_n|\lambda_n - \lambda_n^*|^s\right)^{\frac{1}{s}} &\le \left(\sum_{|z_n^*|\le c(D)}|\lambda_n - \lambda_n^*|^s\right)^{\frac{1}{s}} + \left(\sum_{z_n^*\ge c(D)}|\lambda_n - \lambda_n^*|^s\right)^{\frac{1}{s}}\\ &\le c(D)\left(\#\{|z_n^*| \le c(D)\}\right)^{\frac{1}{s}}\|q - q^*\|_1 +\\ &\quad + \left(\sum_{z_n^*\ge c(D)}\left|\int_0^\pi A_q\left(q(\pi - x) - q^*(\pi - x)\right)\cos 2\sqrt{\lambda_n^*}x\,\mathrm{d}x\right|^s\right)^{\frac{1}{s}}\\ &\le c(D)\left(\#\{|z_n^*| \le c(D)\}\right)^{\frac{1}{s}}\|q - q^*\|_1 +\\ &\quad + c(D)C\|A_q\left(q(\pi - x) - q^*(\pi - x)\right)\|_r\\ &\le c(D)\left(\#\{|z_n^*| \le c(D)\}\right)^{\frac{1}{s}}\|q - q^*\|_r +\\ &\quad + c(D)C\|q - q^*\|_r. \end{aligned} \tag{7.6}$$

In the second sum, we applied Corollary 6.2 and Lemma 6.3.

To complete the first part, we need only the following

Statement 7.1.

$$\left(\#\{|z_n^*| \leq c(D)\}\right)^{\frac{1}{s}} \leq c(D)C, \tag{7.7}$$

where C is the norm of the mapping (1.4). ■

Proof. Observe that if k is an integer and $|2z_n^* - k| \leq \frac{1}{2}$ then

$$\begin{aligned} \left|\int_0^\pi \cos kx \cos 2xz_n^* \,\mathrm{d}x\right| &= \frac{1}{2}\left|\frac{1}{2z_n^*+k} + \frac{1}{2z_n^*-k}\right| |\sin \pi(2z_n^*-k)| \\ &\geq \frac{1}{4}\frac{|\sin \pi(2z_n^*-k)|}{|2z_n^*-k|} \geq \frac{1}{2} \end{aligned} \tag{7.8}$$

■

by the concavity of the sine function on $[0, \frac{\pi}{2}]$. For fixed k

$$\begin{aligned} \left(\#\left\{|2z_n^*-k| \leq \frac{1}{2}\right\}\right)^{\frac{1}{s}} &\leq 2\left(\sum_{|2z_n^*-k|\leq\frac{1}{2}} |\langle \cos kx, \varphi_n\rangle|^s\right)^{\frac{1}{s}} \\ &\leq 2\left(\sum_n |\langle \cos kx, \varphi_n\rangle|^s\right)^{\frac{1}{s}} \leq 2C\|\cos kx\|_r \leq 4C, \end{aligned}$$

hence

$$\begin{aligned} \#\{|z_n^*| \leq c(D)\} &\leq \sum_{|k|\leq c(D)} \#\left\{|2z_n^*-k| \leq \frac{1}{2}\right\} \\ &\leq c(D)(4C)^s \leq (c(D)C)^s. \end{aligned}$$

To prove the second part, if the mapping (1.4) is not bounded, there is a sequence $h_k \in L_0^r$ such that $\lim_{k\to\infty} \|h_k\|_r = 0$ but

$$\left(\sum_n \left|\int_0^\pi h_k(x)\cos 2\sqrt{\lambda_n^*}x \,\mathrm{d}x\right|^s\right)^{\frac{1}{s}} \geq 1 \tag{7.9}$$

holds. If $\|h_k\|_r$ (and then $\|h_k\|_1$) is small enough, by Corollary 4.9 there are potentials $q_k \in L^1$, $q_k - q^* \in L_0^r$ such that

$$A_{q_k}\left(q_k(\pi - x) - q^*(\pi - x)\right) = h_k(x). \tag{7.10}$$

Then by Corollary 6.2 and Lemma 6.3,

$$\begin{aligned}
&\left(\sum_n |\lambda_{k,n} - \lambda_n^*|^s\right)^{\frac{1}{s}} \\
&\geq \frac{1}{c(D)}\left(\sum_n \left|\int_0^\pi A_{q_k}\left(q_k(\pi - x) - q^*(\pi - x)\right)\cos 2\sqrt{\lambda_n^*}x \, \mathrm{d}x\right|^s\right)^{\frac{1}{s}} && (7.11)\\
&\geq \frac{1}{c(D)} && (7.12)
\end{aligned}$$

but

$$\lim_{k\to+\infty} \|q_k - q^*\|_r \geq c(D)\| A_{q_k}\left(q_k(\pi - x) - q^*(\pi - x)\right)\|_r = 0, \tag{7.13}$$

thus (1.5) is not continuous. ■

8 The Proof of Theorems 1.3–1.10

Proof of Theorem 1.3. The linear operator (1.4) is bounded by $c(D)$ as an $L^1 \to l^\infty$ mapping, and by the Bessel system property of $C(\Lambda)$, is bounded by $M^{\frac{1}{2}}$ as an $L^2 \to l^2$ mapping. From the M. Riesz convexity theorem [24], it follows that the operator has to be bounded by $c(D)^{\frac{1}{p}} M^{\frac{1}{p'}}$ as an $L^p \to l^{p'}$ mapping, if $1 \leq p \leq 2$. Hence, the statement follows from Theorem 1.1. ■

Proof of Theorem 1.4. The lower frame condition with constant m on (1.3) ensures that statement **B** of Theorem 1.2 holds with $r = s = 2$ and $C = m$. Thus, the theorem follows from (1.10). ■

Proof of Theorem 1.5. Condition (1.17) ensures that (1.13) is bounded by C from l^1 to L^∞, while the frame property of (1.3) gives its continuity from l^2 to L^2, with bound $\frac{1}{m}$. Then by the M. Riesz convexity theorem, (1.13) is $l^p \to L^{p'}$ continuous. Thus, the statement follows from Theorem 1.2. ■

Proof of Theorem 1.6. Condition (1.19) ensures the boundedness of $F^{-1}\varphi_n$'s in $L^{p'}$, which is equivalent to the continuity of the inverse of (1.4) from l_0^1 to $L^{p'}$, hence Theorem 1.2 gives the norm estimate. ■

Proof of Theorem 1.7. By the frame property of (1.3), the operator T^{-1} in (1.13) is continuous from l^2 to L^2; the uniform boundedness of the inverse frame implies its continuity from l^1 to L^∞. By interpolation (1.13) is continuous from l^p to $L^{p'}$.

Suppose that $\|q - q^*\|_{p'} \to 0$ implies $\sum |\lambda_n - \lambda_n^*|^p \to 0$. Then by Theorem 1.1, the operator T is bounded from $L^{p'}$ to l^p and hence T is an isomorphism of $L^{p'}$ on to a (closed) subspace of l^p. But this is impossible if $1 < p < 2$, see the monography of Banach [1] concerning the linear dimension. It is impossible for $p = 1$, for l^1 is separable, while L^∞ is not. ■

Proof of Theorem 1.8. This theorem is not a formal consequence of Theorem 1.2, but we can prove it with similar arguments. Assume that $\sum |\langle h, \varphi_n\rangle| < \infty$ for some $h \in L_0^1$. Corollary 4.9 implies that for appropriately small $\gamma = \gamma(D) > 0$ there exists a potential $q \in L^1$, $q - q^* \in L_0^1$ such that

$$A_q\left(q(\pi - x) - q^*(\pi - x)\right) = \gamma h(x). \tag{8.1}$$

We can choose $\gamma(D) > 0$ so small that (6.7) holds for q. Then by Corollary 6.2 and Lemma 6.3,

$$\|(\Delta\Lambda(q))\|_1 \le c(D, U)\gamma \left\| \left(\int_0^\pi h(x) \cos 2\sqrt{\lambda_n^*}x \, dx \right) \right\|_1 < \infty.$$

If $h \notin L^{p'}$, then $q - q^* \in L^{p'}$ either, thus $\sum_n |\lambda_n - \lambda_n^*| < \infty$ does not imply $q - q^* \in L^{p'}$. Otherwise, if the finiteness of $\sum |\langle h, \varphi_n\rangle|$ implies that $h \in L_0^{p'}$, then, in particular, the elements of the dual basis for $n \ge 1$ are also in $h \in L_0^{p'}$. Assume in contrast with the statement of the theorem that $\sum_n |\lambda_n - \lambda_n^*| \to 0$ implies $\|q - q^*\|_{p'} \to 0$. This is statement **A** of Theorem 1.2 with $r = p'$ and $s = 1$, thus also statement **B** holds. If we substitute the elements of the dual basis to (1.9), we get these bounded, in contrast with the conditions of the theorem. ■

Lemma 8.1. Suppose that

$$\|h\|_r \le c\left(\sum |\langle h, \varphi_n\rangle|^s\right)^{\frac{1}{s}} \qquad h \in L_0^r. \tag{8.2}$$

Then (with a possibly different constant) the same inequality holds for every $h \in L^r$. □

Proof. Let $h = h_0 + \gamma$ with $h_0 \in L_0^r$ and $\gamma \in \mathbb{R}$. Now $\langle h, \varphi_0 \rangle = \pi\gamma$ implies $|\gamma| \le 1/\pi \|Th\|_s$. This implies $\|Th_0\|_s \le \|Th\|_s + |\gamma| \|T1\|_s \le c\|Th\|_s$ and hence $\|h\|_r \le \|h_0\|_r + \|\gamma\|_r \le c(\|Th_0\|_s + |\gamma|) \le c\|Th\|_s$. Thus, (8.2) holds for all $h \in L^r$. ■

Proof of Theorem 1.9. From the Bessel property of φ_n we infer that the operator T in (1.4) is continuous from L^2 to l^2. The L^1 to l^∞ continuity being immediate, we get by interpolation that T is L^p to $l^{p'}$ continuous.

Suppose that $\sum_n |\lambda_n - \lambda_n^*|^{p'} \to 0$ does imply $\|q - q^*\|_p \to 0$. Then by Theorem 1.2

$$\|h\|_p \le c\left(\sum |\langle h, \varphi_n\rangle|^{p'}\right)^{1/p'} \qquad h \in L_0^p. \tag{8.3}$$

By Lemma 8.1, the same inequality holds for every $h \in L^p$. This implies that T is an isomorphism of L^p on to a (closed) subspace of $l^{p'}$. But this is impossible if $1 < p < 2$, see Banach [1]. For $p = 1$, the contradiction follows from the next lemma. ■

Lemma 8.2. Let the sequence λ_n^* be bounded from below, $\mu_n = 2\sqrt{\lambda_n^*}$ and $d > 0$. Then $\|h\|_1 \le c\|(\langle h, \cos\mu_n x\rangle)\|_\infty$ could not hold for all $h \in L^1[0, d]$. □

Proof. Suppose $d = 1$, for other values d the same proof works with obvious modifications. We shall construct a sequence $h_n(x) \in L^1$ such that $|\langle h_n, \cos\mu_j x\rangle|$ is bounded but $\|h_n\|_1 \to \infty$. Consider the Rademacher system:

$$R_0(x) = \begin{cases} 1 & \text{if } \; 0 \le x < \frac{1}{2} \\ -1 & \text{if } \; \frac{1}{2} \le x < 1, \end{cases} \quad R_0(x+1) = R_0(x),\, R_n(x) = R_0(2^n x). \tag{8.4}$$

If μ is real, let $2^{k_0} \le \mu < 2^{k_0+1}$. Using $R_k\big(x + \frac{1}{2^{k+1}}\big) = -R_k(x)$,

$$\int_0^1 R_k(x)\cos\mu x \,\mathrm{d}x = \int_0^{\frac{1}{2^{k+1}}} R_k(x)\cos\mu x \,\mathrm{d}x$$
$$- \int_0^{1-\frac{1}{2^{k+1}}} R_k(x)\cos\mu\left(x + \frac{1}{2^{k+1}}\right)\mathrm{d}x,$$

$$
\begin{aligned}
2\left|\int_0^1 R_k(x)\cos\mu x\,\mathrm{d}x\right| &\le \left|\int_0^{\frac{1}{2^{k+1}}} R_k(x)\cos\mu x\,\mathrm{d}x\right| \\
&\quad + \left|\int_{1-\frac{1}{2^{k+1}}}^1 R_k(x)\cos\mu x\,\mathrm{d}x\right| \\
&\quad + \left|\int_0^{1-\frac{1}{2^{k+1}}} R_k(x)\left(\cos\mu x - \cos\mu\left(x+\frac{1}{2^{k+1}}\right)\right)\,\mathrm{d}x\right| \\
&\le \frac{1}{2^k} + \mu\frac{1}{2^{k+1}} \le \frac{2}{2^{k-k_0}}.
\end{aligned}
$$

If $k < k_0$,

$$
\left|\int_0^1 R_k(x)\cos\mu x\,\mathrm{d}x\right| \le \sum_{j=1}^{2^{k+1}}\left|\int_{\frac{j-1}{2^{k+1}}}^{\frac{j}{2^{k+1}}} R_k(x)\cos\mu x\,\mathrm{d}x\right| \le \frac{2^{k+2}}{\mu} \le \frac{4}{2^{k_0-k}}.
$$

Let $h_n(x) = \sum_{k=0}^n R_k(x)$. Then

$$
\left|\int_0^1 h_n(x)\cos\mu x\,\mathrm{d}x\right| \le \sum_{k=0}^{\infty}\left|\int_0^1 R_k(x)\cos\mu x\,\mathrm{d}x\right| \le \sum_{k=0}^{\infty}\frac{5}{2^k} = 10.
$$

For a purely imaginary μ,

$$
\left|\int_0^1 R_k(x)\cos\mu x\,\mathrm{d}x\right| \le \left|\int_{1-\frac{1}{2^k}}^1 R_k(x)\cosh|\mu|x\,\mathrm{d}x\right| \le \frac{\cosh|\mu|}{2^k},
$$

thus

$$
\left|\int_0^1 h_n(x)\cos\mu x\,\mathrm{d}x\right| \le 2\cosh|\mu|,
$$

which is also bounded, if λ_n^* are bounded from below. However, $R_n(x)$'s as random variables, are independent and identically distributed with expectation zero and variance one. According to the central limit theorem,

$$
P\left(\frac{\sum_{k=0}^n R_k}{\sqrt{n+1}} < t\right) \to \Phi(t),
$$

where $\Phi(t)$ is the distribution function of the standard normal distribution. In particular

$$P\left(\sqrt{n+1} \le h_n < 2\sqrt{n+1}\right) \to \Phi(2) - \Phi(1) > 0,$$

hence

$$\begin{aligned}\int_0^1 |h_n(x)|\,\mathrm{d}x &\ge \int_{\{\sqrt{n+1}\le h_n<2\sqrt{n+1}\}} |h_n(x)|\,\mathrm{d}x \\ &\ge \sqrt{n+1}\;P\left(\sqrt{n+1} \le h_n < 2\sqrt{n+1}\right),\end{aligned}$$

which tends to infinity while $n \to \infty$. ■

Proof of Theorem 1.10. The next lemma shows that the system (1.3) is a frame in $L^2[0,\pi]$, and then the statement appears as a special case of Theorems 1.3 and 1.4. ■

Lemma 8.3. Assume that the system (1.21) is a frame (resp., a Riesz basis) in $L^2[-\pi,\pi]$. Then both (1.3) and the system

$$S(\Lambda) = \left\{ \sin 2\sqrt{\lambda_n^*}x : n \ge 1 \right\} \tag{8.5}$$

are frames (resp., Riesz bases) in $L^2[0,\pi]$. If the elements of the inverse frame of (1.21) are bounded by C in a p-norm, then the elements of the inverse frame of (1.3) and of (8.5) are both bounded by $2C$ in the same norm. □

Proof. Let us denote the elements of $e(\Lambda)$, $C(\Lambda)$, $S(\Lambda)$ by e_n $(n \in \mathbb{Z})$, φ_n $(n \ge 0)$, s_n $(n \ge 1)$. Let the (supposed) frame operators be F_e, F_c, and F_s, respectively. If $h \in L^2[0,\pi]$, let us denote its even and odd extensions by h_e and h_o, that is, let $h_e(x) = h(|x|)$, $h_o(x) = \operatorname{sgn}(x)h(|x|)$. Then

$$F_e(h_e) = \sum_{n\in\mathbb{Z}} \langle h_e, e_n\rangle e_n = 4\left(\sum_{n\ge 0} \langle h, \varphi_n\rangle \varphi_n\right)_e = 4(F_c h)_e,$$

$$F_e(h_o) = \sum_{n\in\mathbb{Z}} \langle h_o, e_n\rangle e_n = -4\left(\sum_{n\ge 1} \langle h, s_n\rangle s_n\right)_o = -4(F_s h)_o,$$

thus

$$F_e^{-1}(h_e) = \frac{1}{4}(F_c^{-1}h)_e, \quad F_e^{-1}(h_o) = -\frac{1}{4}(F_s^{-1}h)_o.$$

This implies that both F_c and F_s are frame operators with

$$\|F_c\| \le \frac{1}{4}\|F_e\|, \quad \|F_c^{-1}\| \le 4\|F_e^{-1}\|,$$
$$\|F_s\| \le \frac{1}{4}\|F_e\|, \quad \|F_s^{-1}\| \le 4\|F_e^{-1}\|.$$

Moreover,

$$\begin{aligned}\|F_c^{-1}\varphi_n\|_p &= 2^{-1/p}\|\left(F_c^{-1}\varphi_n\right)_e\|_p = 2^{2-1/p}\|F_e^{-1}(\varphi_n)_e\|_p \\ &\le 2^{1-1/p}|F_e^{-1}e_n + F_e^{-1}e_{-n}\|_p \le 2^{2-1/p}C,\end{aligned}$$

and similarly,

$$\|F_s^{-1}s_n\|_p \le 2^{2-1/p}C.$$

■

9 Finitely Many Known Eigenvalues

Proof of Theorem 1.13. From $\int_0^\pi A_q(q-q^*) = 0$, we obtain

$$\begin{aligned}&\left|\int_0^\pi \left[\int_0^x A_q(q(\pi-t) - q^*(\pi-t))\,\mathrm{d}t\right]\cdot \sin 2\sqrt{\lambda_n^*}x\,\mathrm{d}x\right| \\ &= \left|\int_0^\pi A_q(q(\pi-x) - q^*(\pi-x))\frac{\cos 2\sqrt{\lambda_n^*}x}{2\sqrt{\lambda_n^*}}\,\mathrm{d}x\right| \le \frac{c(D)}{\sqrt{|\lambda_n^*|}}|\lambda_n - \lambda_n^*|.\end{aligned}$$

Using the fact that (8.5) is a frame and its inverse frame is bounded by C in L^∞,

$$\begin{aligned}\left|\int_0^x A_q(q-q^*)\right| &\le 2C\sum_n\left|\int_0^\pi\int_0^x A_q(q-q^*)\cdot\sin 2\sqrt{\lambda_n^*}x\,\mathrm{d}x\right| \\ &\le C\sum_n\frac{c(D)}{\sqrt{|\lambda_n^*|}}|\lambda_n - \lambda_n^*|.\end{aligned}$$

Comparing that with the next lemma, the proof will be complete. ■

Lemma 9.1. Let $h \in L^1_0[0,\pi]$. Then

$$\frac{1}{c(D)} \sup_{x\in[0,\pi]} \left|\int_0^x h\right| \le \sup_{x\in[0,\pi]} \left|\int_0^x A_q h\right| \le c(D) \sup_{x\in[0,\pi]} \left|\int_0^x h\right|. \tag{9.1}$$

□

Proof. Let us denote $H(x) = \int_0^x h$. By (4.6) $A_{q^*}h \in L^1_0$, hence

$$\begin{aligned}\int_0^{t_0} A_q h &= H(t_0) - \int_{t_0}^{\pi}\int_x^{\pi} M(x,t)h(t)\,\mathrm{d}t\,\mathrm{d}x\\ &= H(t_0) + \int_{t_0}^{\pi} H(t)\left[M(t,t) + \int_{t_0}^{t} M_t(x,t)\,\mathrm{d}x\right]\mathrm{d}t\\ &= (I+B)H(t_0).\end{aligned}$$

It is known that $\int_{t_0}^t M_t(x,t)\,\mathrm{d}x$ is continuous (see Marchenko [16] and Horváth [10]) and then the integral operator B has a kernel uniformly bounded by a constant $c_0(D)$. Using a standard argument, we get by induction on n that

$$\|B^n\|_\infty \le \frac{(\pi c_0(D))^n}{n!}.$$

Consequently, the Neumann series $(I+B)^{-1} = I - B + B^2 - \dots$ converges in the ∞-norm and $\|(I+B)^{-1}\|_\infty \le c(D)$. This proves Lemma 9.1. ■

Consider the case when we know the eigenvalues $\lambda_n^* \in \sigma(\alpha_n, 0, q^*)$, of which the first N may contain an error ε, while the others can contain unknown errors tending to zero.

Proof of Theorem 1.14. By Theorem 1.13 and a Cauchy–Schwartz inequality,

$$\begin{aligned}\sup_{0\le x\le\pi}\left|\int_0^x (q-q^*)\right| &\le C\sum_n \frac{c(D)}{\sqrt{|\lambda_n^*|}}|\lambda_n - \lambda_n^*|\\ &\le C c(D)\varepsilon\sum_{n=1}^N \frac{1}{\sqrt{|\lambda_n^*|}}\\ &\quad + C c(D)\left(\sum_{n=N+1}^\infty \frac{1}{|\lambda_n^*|}\right)^{\frac12}\left(\sum_n |\lambda_n - \lambda_n^*|^2\right)^{\frac12}\\ &\le C c(D)\varepsilon\sum_{n=1}^N \frac{1}{\sqrt{|\lambda_n^*|}} + C c(D)\|F\|^{\frac12}\left(\sum_{n=N+1}^\infty \frac{1}{|\lambda_n^*|}\right)^{\frac12}.\end{aligned}$$

Lemma 9.2. Let $\{e_n\}$ be an orthonormal system in L^2, and $f_n = e_n + \delta_n$, where $\sum_n \|\delta_n\|_2^2 = c^2 < 1$. Then for an arbitrary sequence $\{\alpha_n\} \in l^2$,

$$\sum_n |\alpha_n|^2 \leq \frac{1}{(1-c)^2} \left\| \sum_n \alpha_n f_n \right\|_2^2. \tag{9.2}$$

□

Proof.

$$\left\| \sum_n \alpha_n \delta_n \right\|_2 \leq \sqrt{\sum_n |\alpha_n|^2} \sqrt{\sum_n \|\delta_n\|_2^2},$$

■

hence

$$\begin{aligned} \left\| \sum_n \alpha_n f_n \right\|_2 &\geq \left\| \sum_n \alpha_n e_n \right\|_2 - \left\| \sum_n \alpha_n \delta_n \right\|_2 \\ &= \sqrt{\sum_n |\alpha_n|^2} - \left\| \sum_n \alpha_n \delta_n \right\|_2 \\ &\geq (1-c) \sqrt{\sum_n |\alpha_n|^2}. \end{aligned}$$

Proof of Theorem 1.15. The Riesz basis property of

$$\Phi = \left\{1,\ e^{\pm i\sqrt{\mu_n}x},\ n \geq 1\right\}$$

can be verified as in the beginning of Section 1.6. The remaining part of the proof is decomposed into several steps.

Step 1. The system Φ is not complete in $C[-\pi, \pi]$.

If it were complete, then the system $\{1, e^{\pm inx}, (N > n \geq 1), e^{\pm i\sqrt{\mu_n}x}, (n \geq N)\}$ also would be complete, since the completeness of an exponential system is unaffected if finitely many members are replaced by other exponentials (see Young [25]). Thus every odd function, in particular the function x, could be approximated uniformly by the functions $\{\sin x, \sin 2x, \ldots, \sin(N-1)x, \sin\sqrt{\mu_n}x : n \geq N\}$. Consequently,

$$
\begin{aligned}
\pi \approx \sum_{n\geq N} \alpha_n \sin\sqrt{\mu_n}\pi &\leq \sqrt{\sum_{n\geq N} |\alpha_n|^2}\sqrt{\sum_{n\geq N} O\left(\frac{1}{n^2}\right)} \\
&\leq O\left(\frac{1}{\sqrt{N}}\right)\sqrt{\sum_{n\geq 0}|\alpha_n|^2} \\
&\leq O\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}
$$

Here, we applied the previous lemma to show that the finite sums $\sum|\alpha_n|^2$ have a bound independent of N. The resulting inequality $\pi = O(N^{-1/2})$ is nonsense, the contradiction proves Step 1.

Step 2. *Φ has deficiency 1, that is,*

$$\Phi_1 = \left\{1,\ x,\ e^{\pm i\sqrt{\mu_n}x},\ n \geq 1\right\}$$

is complete in C.

Indeed, the system

$$\left\{\int_0^x f + c:\ f \in L_2,\ c \in \mathbf{C}\right\}$$

is clearly complete in C. Approximating f by $\mathrm{Lin}(\Phi)$ in L^2-norm gives uniform approximation of $\int_0^x f$ by $\mathrm{Lin}(\Phi_1)$. So $\mathrm{Lin}(\Phi_1)$ is indeed dense in C.

Step 3. *There exists a function of bounded variation $\beta \in \mathrm{BV}[-\pi, \pi]$ such that the entire function*

$$0 \neq G(z) = \int_{-\pi}^{\pi} e^{izx}\,\mathrm{d}\beta(x)$$

satisfies $0 = G(0) = G(\pm\sqrt{\mu_n})$.

Indeed, since Φ is not complete, there is a nontrivial functional $0 \neq F \in C^*$ with $0 = F(1) = F(e^{\pm i\sqrt{\mu_n}x})$. By the Riesz representation theorem F has the form $F(f) = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}\beta(x)$ and this verifies Step 3.

Step 4. The functions of the system biorthogonal to Φ *have the form*

$$\frac{1}{iG'(\lambda)} e^{-i\lambda x} \int_{-\pi}^{x} e^{i\lambda t} \mathrm{d}\beta(t)$$

where $\lambda = 0$ *or* $\pm\sqrt{\mu_n}$.

Indeed, $\int_{-\pi}^{\pi} e^{i\lambda t} \mathrm{d}\beta(t) = 0$ implies

$$\begin{aligned}
\frac{G(z)}{z-\lambda} &= \frac{1}{z-\lambda} \int_{-\pi}^{\pi} e^{i(z-\lambda)x} \cdot e^{i\lambda x} \mathrm{d}\beta(x) \\
&= \frac{1}{z-\lambda} \left[e^{i(z-\lambda)x} \int_{-\pi}^{x} e^{i\lambda t} \mathrm{d}\beta(t) \right]_{-\pi}^{\pi} \\
&\quad - i \int_{-\pi}^{\pi} e^{i(z-\lambda)x} \int_{-\pi}^{x} e^{i\lambda t} \mathrm{d}\beta(t)\, \mathrm{d}x \\
&= \int_{-\pi}^{\pi} e^{izx} \cdot \frac{1}{i} e^{-i\lambda x} \int_{-\pi}^{x} e^{i\lambda t} \mathrm{d}\beta(t)\, \mathrm{d}x.
\end{aligned}$$

For $z = \lambda$, the left-hand side is $G(z)/(z-\lambda) = G'(\lambda)$ which proves Step 4.

Step 5. The biorthogonal system is uniformly bounded in C.

Indeed, apart from the factors $1/G'(\lambda)$ the biorthogonal system is clearly uniformly bounded; it remains to give a uniform lower estimate $|G'(\lambda)| \geq c > 0$. This is verified by an infinite product representation of G. First of all, the only zeros of G are 0 and $\pm\sqrt{\mu_n}$ and these zeros are simple (otherwise Φ_1 is not complete in C). Since $G(z)$ is bounded along the real axis, the following representation is valid:

$$G(z) = c e^{i\gamma z} \cdot z \prod_{1}^{\infty} \left(1 - \frac{z^2}{\mu_n}\right)$$

with some $\gamma \in \mathbf{R}$ and $c \neq 0$, see Levin [14], Chapter V. From $\mu_n = n^2 + \mathbf{O}(1)$, it follows that

$$\left| \log |G(z) e^{-i\gamma z}| - \log|\sin \pi z| \right|$$

is uniformly bounded apart from the δ-neighborhood of the zeros, that is, for z with $|z \pm n| \geq \delta$, $|z \pm \sqrt{\mu_n}| \geq \delta$, where $\delta > 0$ is an arbitrary constant, see Horváth [9], Lemma 2.6. In particular, if λ is a zero of G then

$$\left|\frac{z-\lambda}{G(z)}e^{i\gamma z}\right| \leq c\left|\frac{z-\lambda}{\sin \pi z}\right|$$

except for the δ-disks around the zeros. Consider a connected neighborhood of $z = \lambda$ in the union of these disks for small δ. On the boundary of this neighborhood, we have

$$\left|\frac{z-\lambda}{G(z)}\right| \leq c$$

independently of the zero λ. This extends by the maximum principle to the point $z = \lambda$ giving that $1/|G'(\lambda)| \leq c$. This finishes the proof. ■

10 **Proofs for Complex Potentials**

Consider again the representation

$$y_2(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x K(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,\mathrm{d}t. \tag{10.1}$$

By standard arguments borrowed from [17] and [10], Lemma 5.1 we easily obtain the following estimates: if $\|q\|_p$, $\|q^*\|_p \leq D$ then

$$|K(x,t)| \leq c(D), \quad |K(x,t) - K^*(x,t)| \leq c(D)\|q-q^*\|_p, \tag{10.2}$$

$$\left|K_x(x,t) - \frac{1}{4}q\left(\frac{x+t}{2}\right) + \frac{1}{4}q\left(\frac{x-t}{2}\right)\right| \leq c(D), \tag{10.3}$$

$$\begin{aligned}\Bigg|K_x(x,t) - K_x^*(x,t) &- \frac{1}{4}q\left(\frac{x+t}{2}\right) + \frac{1}{4}q\left(\frac{x-t}{2}\right)\\ &+ \frac{1}{4}q^*\left(\frac{x+t}{2}\right) - \frac{1}{4}q^*\left(\frac{x-t}{2}\right)\Bigg| \leq c(D)\|q-q^*\|_p,\end{aligned} \tag{10.4}$$

$$\left|K_t(x,t) - \frac{1}{4}q\left(\frac{x+t}{2}\right) - \frac{1}{4}q\left(\frac{x-t}{2}\right)\right| \leq c(D), \tag{10.5}$$

$$\begin{aligned}\Bigg|K_t(x,t) - K_t^*(x,t) &- \frac{1}{4}q\left(\frac{x+t}{2}\right) - \frac{1}{4}q\left(\frac{x-t}{2}\right)\\ &+ \frac{1}{4}q^*\left(\frac{x+t}{2}\right) + \frac{1}{4}q^*\left(\frac{x-t}{2}\right)\Bigg| \leq c(D)\|q-q^*\|_p.\end{aligned} \tag{10.6}$$

Now integrating by parts in (10.1) gives that

$$\left| y_2(x,\lambda) - \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right| \le \frac{c(D)}{1+|\lambda|} e^{|\Im\sqrt{\lambda}|x}, \tag{10.7}$$

$$\left| y_2'(x,\lambda) - \cos\sqrt{\lambda}x \right| \le \frac{c(D)}{1+\sqrt{|\lambda|}} e^{|\Im\sqrt{\lambda}|x}, \tag{10.8}$$

$$|y_2(x,\lambda) - y_2^*(x,\lambda)| \le \frac{c(D)}{1+|\lambda|} e^{|\Im\sqrt{\lambda}|x} \|q - q^*\|_p. \tag{10.9}$$

We have seen in Section 3 that $|\lambda - \lambda^*| \le c(D)\|q - q^*\|_p$ is impossible if λ is a multiple eigenvalue. However, the weaker estimate $|\lambda - \lambda^*| \le c(D)$ is valid:

Lemma 10.1. Let $\|q\|_p$, $\|q^*\|_p \le D$ and $\lambda^* \in \sigma(0,\alpha;q^*)$ is an eigenvalue corresponding to $\lambda \in \sigma(0,\alpha;q)$ (that is, there is a continuous function $\lambda(s) \in \sigma(0,\alpha;q_s = sq^* + (1-s)q)$ with $\lambda(0) = \lambda,\ \lambda(1) = \lambda^*$). Then

$$|\lambda - \lambda^*| \le c(D) \tag{10.10}$$

where $c(D)$ is independent of $\alpha,\ q,\ q^*,\ \lambda,$ and λ^*. □

Proof. Let $F(w) = F(w,s) = \cos\alpha\, y_2(\pi,w;q_s) + \sin\alpha\, y_2'(\pi,w;q_s)$ be the characteristic function whose zeros are the eigenvalues in $\sigma(0,\alpha;q_s)$. Then $F(w) = F_0(w) + R(w)$ with

$$F_0(w) = \sin\alpha\cos(\sqrt{w}\pi) + \cos\alpha\frac{\sin(\sqrt{w}\pi)}{\sqrt{w}}, \tag{10.11}$$

$$R(w) = \sin\alpha\left[K(\pi,\pi)\frac{\sin(\sqrt{w}\pi)}{\sqrt{w}} + \int_0^\pi K_x(\pi,t)\frac{\sin(\sqrt{w}t)}{\sqrt{w}}\,dt\right] + \cos\alpha\int_0^\pi K(\pi,t)\frac{\sin(\sqrt{w}t)}{\sqrt{w}}\,dt. \tag{10.12}$$

By (10.7) and (10.8), the remainder term can be estimated by

$$|R(w)| \le |\sin\alpha|\cdot c_0(D)\frac{\exp(|\Im\sqrt{w}|\pi)}{\sqrt{|w|}} + c_0(D)\frac{\exp(|\Im\sqrt{w}|\pi)}{|w|}. \tag{10.13}$$

The idea is to draw a contour Γ around the eigenvalue λ at a distance of order $c(D)$ such that along Γ, $|F_0(w)| > |R(w)|$ for every potential q_s. Then the eigenvalues corresponding to q_s cannot cross Γ and hence remain in the domain bounded by Γ, which proves the lemma. The construction of the contour needs elementary but tedious considerations, we give only the rough ideas. First choose a number $c(D)$ much larger than $c_0(D)$ in (10.13) in a sense to be specified later and draw a circle around λ with radius R of order $c(D)$ such that $|w| \geq c(D)$ be true along the circle. If the circle crosses the domain

$$\frac{100}{\sqrt{|w|}} > \sin\alpha > \frac{0.01}{\sqrt{|w|}}$$

and if $|\sin\alpha|^{-1} \leq c(D)$ then take a larger radius to avoid this domain. Now if $|\sin\alpha| \geq |w|^{-1/3}$ then

$$|R(w)| \leq c(D)|w|^{-1/2}\exp(|\Im\sqrt{\lambda}|\pi),$$
$$|F_0(w)| \geq c|w|^{-1/3}\exp(|\Im\sqrt{\lambda}|\pi) \text{ if } |\sqrt{w}-n-1/2| > 0.1 \ \ \forall n$$

with a universal constant c, since the first term is dominating in F_0. By small modifications in Γ, we can avoid the domains $|\sqrt{w}-n-1/2| \leq 0.1$ thus for $c(D)$ large enough we have $|F_0(w)| > |R(w)|$. If $|w|^{-1/3} \geq |\sin\alpha| \geq 100|w|^{-1/2}$, then we get similarly

$$|R(w)| \leq c(D)|w|^{-1/2-1/3}\exp(|\Im\sqrt{\lambda}|\pi),$$
$$|F_0(w)| \geq c|w|^{-1/2}\exp(|\Im\sqrt{\lambda}|\pi) \text{ if } |\sqrt{w}-n-1/2| > 0.1 \ \ \forall n.$$

If $100|w|^{-1/2} \geq |\sin\alpha| \geq 0.01|w|^{-1/2}$, then w is large, that is, $|w^{-1/2}-\lambda^{-1/2}| \leq c(D)|w|^{-3/2} \leq |w|^{-1}$ and hence

$$|R(w)| \leq c(D)|w|^{-1}\exp(|\Im\sqrt{\lambda}|\pi),$$
$$|F_0(w) - \sin\alpha\cos(\sqrt{w}\pi) - \cos\alpha\sin(\sqrt{w}\pi)\lambda_{-1/2}| \leq |w|^{-1}\exp(|\Im\sqrt{\lambda}|\pi),$$
$$|\sin\alpha\cos(\sqrt{w}\pi) + \cos\alpha\sin(\sqrt{w}\pi)\lambda^{-1/2}|$$
$$= |\sqrt{\sin^2\alpha + \cos^2\alpha\lambda^{-1}}| \cdot |\sin(\sqrt{w}\pi + \gamma)|$$
$$\geq c|w|^{-1/2}\exp(|\Im\sqrt{\lambda}|\pi) \text{ if } |\sqrt{w}+\gamma/\pi - n| > 0.1 \ \ \forall n.$$

Finally, if $0.01|w|^{-1/2} \geq |\sin\alpha|$ then the second term dominates in F_0 hence

$$|R(w)| \leq c(D)|w|^{-1}\exp(|\Im\sqrt{\lambda}|\pi),$$
$$|F_0(w)| \geq c|w|^{-1/2}\exp(|\Im\sqrt{\lambda}|\pi) \text{ if } |\sqrt{w}-n| > 0.1 \ \ \forall n.$$

The above estimates show that if Γ does not get too close to the zeros of appropriate trigonometric functions then $|F_0| > |R|$ along Γ. So the eigenvalues corresponding to λ cannot cross Γ and the proof is complete. ■

Corollary 10.2. Let $\|q\|_p \le D$ and consider some eigenvalues $\lambda_n \in \sigma(\alpha_n, 0; q)$. If $\Re\lambda_n > -D$ then

$$|\Im\sqrt{\lambda_n}| \le c(D). \qquad \square$$

Proof. Under the linear deformation of q into the zero potential $q^* = 0$, the corresponding eigenvalues $\lambda_n^* \in \sigma(\alpha_n, 0; q^* = 0)$ are real and $\lambda_n^* > -c(D)$ by the previous lemma. Thus, $|\Im\lambda_n| \le c(D)$, $\Re\lambda_n > -D$ and then $|\Im\sqrt{\lambda_n}| \le c(D)$. ■

Lemma 10.3. If $\|q\|_p \le D$, $\lambda \in \sigma(\alpha, 0; q)$, $|\Im\lambda| \le D$, and $|\lambda| \ge c(D)$ with a sufficiently large constant independent of q, α, then λ is a simple eigenvalue. The multiplicity of eigenvalues $|\lambda| \le c(D)$ is bounded by $c_1(D)$. □

Proof. If the characteristic function $F(w)$ has multiple zero at λ, then

$$y_2(\pi)\dot{y}_2'(\pi) - y_2'(\pi)\dot{y}_2(\pi) = 0.$$

From the representation (10.1), we obtain the estimates

$$y_2(\pi) = \frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}} + \mathbf{O}(\lambda^{-1}), \quad y_2'(\pi) = \cos\sqrt{\lambda}\pi + \mathbf{O}(\lambda^{-1/2}),$$
$$\dot{y}_2(\pi) = \frac{\pi\cos\sqrt{\lambda}\pi}{2\lambda} + \mathbf{O}(\lambda^{-3/2}),\ \dot{y}_2'(\pi) = -\frac{\pi\sin\sqrt{\lambda}\pi}{2\sqrt{\lambda}} + \mathbf{O}(\lambda^{-1})$$

with implicit constants $c_0(D)$, thus

$$\left|y_2(\pi)\dot{y}_2'(\pi) - y_2'(\pi)\dot{y}_2(\pi) + \frac{\pi}{2\lambda}\right| \le c_0(D)\lambda^{-3/2}.$$

Consequently, if $|\lambda| \ge c(D)$ is sufficiently large with respect to $c_0(D)$ then λ is a simple eigenvalue. The multiplicity of an eigenvalue $|\lambda| \le c(D)$ can be estimated by drawing a contour around λ as in the previous lemma and by applying the Rouché theorem. ■

Now the estimates (6.1) and (6.2) and the theorems listed in Section 3 can be proved just like for the real potentials.

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