INEQUALITIES BETWEEN THE FIXED-ENERGY PHASE SHIFTS

MIKLÓS HORVÁTH

Abstract. Consider the fixed-energy inverse scattering problem with spherically symmetrical, compactly supported potentials $q(r)$. We give infinitely many inequalities between the phase shifts if $q(r) \leq 1$.

1. Introduction

The fixed-energy inverse scattering problem with spherically symmetrical potential $q(r)$ can be described by the system

$$\varphi_n''(r) - \frac{n(n+1)}{r^2}\varphi_n(r) + (1 - q(r))\varphi_n(r) = 0 \quad r \geq 0, \tag{1.1}$$

$$\varphi_n(r) = \gamma_n r^{n+1}(1 + \mathbf{o}(1)) \quad r \to 0+, \tag{1.2}$$

$$\varphi_n(r) = \sin(r - n\pi/2 + \delta_n) + \mathbf{o}(1) \quad r \to +\infty. \tag{1.3}$$

The definition of phase shifts can be naturally extended: for $\Re\lambda > 0$ let

$$\varphi"(r,\lambda) - \frac{\lambda^2 - 1/4}{r^2}\varphi(r,\lambda) + (1 - q(r))\varphi(r,\lambda) = 0 \quad \lambda \leq 0, \tag{1.4}$$

$$\varphi(r,\lambda) = \gamma(\lambda) r^{\lambda+1/2}(1 + \mathbf{o}(1)) \quad r \to 0+, \tag{1.5}$$

$$\varphi(r,\lambda) = \sin(r - \pi/2(\lambda - 1/2) + \delta(\lambda)) + \mathbf{o}(1) \quad r \to +\infty. \tag{1.6}$$

Then $\delta_n = \delta(n + 1/2)$. Throughout the paper we will assume that

$$rq(r) \in L_1(0,\infty).$$

Concerning the distribution of the phase shifts δ_n very little is known and these are mostly connected with some derivatives of the phase

---

Research supported by the Hungarian NSF Grants OTKA T 32374, T 37491, T 47035 and T 61311.
2000. Math. Subject Classification: Primary 34A55, 34B20, 81U40; Secondary 34L40, 47A75
Key words and phrases: Inverse scattering, phase shifts, Weyl-Titchmarsh m-function.

shifts. Mention first the formula of Regge [12]

$$\frac{d\delta(\lambda)}{d\lambda} = \frac{\pi}{2} - 2\lambda \int_0^\infty \frac{\varphi^2(r,\lambda)}{r^2} dr = 2\lambda \int_0^\infty \frac{\varphi_0^2(r,\lambda) - \varphi^2(r,\lambda)}{r^2} dr \tag{1.7}$$

(where φ_0 is the function φ of the zero potential $q = 0$) and its trivial corollary

$$\delta_{n+1} - \delta_n < \frac{\pi}{2}. \tag{1.8}$$

The functional derivative of $\delta(\lambda)$ with respect to the potential can be given by

$$\dot{\delta} = -\int_0^\infty \dot{q}\varphi^2. \tag{1.9}$$

It is proved in [6], for the physical phase shifts δ_n it appeared first in [2] and [3]. Consequently $q_1 \leq (\geq) q_2$ implies $\delta(\lambda; q_1) \geq (\leq) \delta(\lambda, q_2)$; in particular nonnegative potentials have nonpositive phase shifts. We know that if $\lambda > 0$ and

$$\left[\pi \int_0^\infty r|q(r)|\, dr\right]^4 + \frac{1}{16} < \lambda^2. \tag{1.10}$$

then for all $r \geq 0$

$$|\varphi(r,\lambda)| \leq \sqrt{2\pi r}, \quad |\varphi(r,\lambda)| \leq \frac{2\sqrt{2\pi}}{2^\lambda \Gamma(\lambda+1)} r^{\lambda+1/2}, \tag{1.11}$$

see [4]. Consequently if (1.10) holds for q_1 and q_2 then

$$|\delta(\lambda; q_1) - \delta(\lambda, q_2)| \leq 2\pi \int_0^\infty r|q_1(r) - q_2(r)|\, dr, \tag{1.12}$$

$$|\delta(\lambda; q_1) - \delta(\lambda, q_2)| \leq c \left(\frac{1}{\lambda} + \int_{2\lambda/e}^\infty r|q_1(r) - q_2(r)|\, dr \right). \tag{1.13}$$

In [6] the following exact bounds are given: if $rq(r) \in L_1(0,a)$ and $q = 0$ for $r > a$ then

$$\arctan \frac{J_\lambda(a)}{Y_\lambda(a)} - k\pi < \delta(\lambda) < \infty$$

where J_λ and Y_λ are the usual Bessel functions and k is the number of zeros of Y_λ on $(0,a)$. In particular this means that

$$-a < \delta_0 < \infty, \quad -a + \arctan a < \delta_1 < \infty. \tag{1.14}$$

Finally remark that Loeffel [9] found a rather complicated description of the sequences of phase shifts with compactly supported potentials.

In the present paper a sequence of inequalities on the phase shifts is proved for compactly supported potentials. Introduce the function

$$f(\lambda) = \lambda - a\frac{J'_\lambda(a) - \tan\delta(\lambda)Y'_\lambda(a)}{J_\lambda(a) - \tan\delta(\lambda)Y_\lambda(a)} \tag{1.15}$$

and its divided differences defined inductively by

$$f(\lambda_0,\dots,\lambda_k) = \frac{f(\lambda_1,\dots,\lambda_k) - f(\lambda_0,\dots,\lambda_{k-1})}{\lambda_k - \lambda_0}. \tag{1.16}$$

Then we have the following exact inequalities:

Theorem 1.1. *Let $rq(r) \in L_1(0,a)$ and $q = 0$ for $r > a$. If*

$$q(r) \le 1 \text{ a.e. and } \frac{1}{2}\int_0^a r(1-q(r))\,dr < \lambda_0 < \dots < \lambda_k \tag{1.17}$$

then

$$(-1)^k f(\lambda_0,\dots,\lambda_k) \ge 0 \qquad \forall n \ge 0. \tag{1.18}$$

If in (1.18) equality occurs for some k and $\lambda_0,\dots,\lambda_k$ then $q = 1$ a.e. on $(0,a)$.

One might think that the converse is also true, namely that all the inequalities (1.18) imply $q \le 1$, but this is not the case. The details are given below in the remark after the proof of Theorem 1.1.

Concerning the physically relevant phase shifts the following special case can be formulated. Introduce the quantities

$$\mu_n = f(n + 1/2). \tag{1.19}$$

The backward differences are defined by $\Delta^0\mu_n = \mu_n$, $\Delta^{k+1}\mu_n = \Delta^k\mu_n - \Delta^k\mu_{n+1}$.

Corollary 1.2. *Let $rq(r) \in L_1(0,a)$ and $q = 0$ for $r > a$. If*

$$q(r) \le 1 \text{ a.e. and } \int_0^a r(1-q(r))\,dr < 2n_0 + 1 \tag{1.20}$$

then

$$\Delta^k \mu_n \geq 0 \quad \forall n \geq n_0,\ \forall k \geq 0. \tag{1.21}$$

If equality occurs for some k and n then $q = 1$ a.e. on $(0, a)$.

Another series of inequalities are given in

Theorem 1.3. *Define $g(z) = f(\sqrt{-z}) - \sqrt{-z}$. If $q(r) = 0$ for $r > a$ and*

$$z_0 < \cdots < z_k < -\frac{1}{4}\left(\int_0^a r|1 - q(r)|\, dr\right)^2 \tag{1.22}$$

then

$$g(z_0, \ldots, z_k) > 0 \qquad \forall k \geq 1. \tag{1.23}$$

Analogous statements are valid for the distribution of the values of the Weyl-Titchmarsh m-function. Consider the Schrödinger equation on the half-line

$$-y" + Q(x)y = zy, \qquad 0 \leq x < \infty \quad \text{with } Q \in L_1(0, \infty). \tag{1.24}$$

The operator is in the limit point case at infinity, so the solution $0 \neq y(x, z)$, $y \in L_2(0, \infty)$ of (1.24) is unique up to a constant factor. The m-function is defined by

$$m(z) = \frac{y'(0, z)}{y(0, z)}. \tag{1.25}$$

Now we have the following analogies of Theorems 1.1 and 1.3

Theorem 1.4. *Let $h(\tau) = \tau + m(-\tau^2)$. Now if*

$$Q \leq 0 \quad \text{and} \quad \frac{1}{2}\int_0^\infty |Q| < \tau_0 < \cdots < \tau_k \tag{1.26}$$

then

$$(-1)^k h(\tau_0, \ldots, \tau_k) \geq 0. \tag{1.27}$$

If equality occurs for some k and $\tau_0, \ldots, \tau_k$ then $Q = 0$ a.e.

Theorem 1.5. *If $Q \in L_1(0, \infty)$ and*

$$z_0 < \cdots < z_k < -\frac{1}{4}\|Q\|_1^2 \tag{1.28}$$

then

$$m(z_0, \ldots, z_k) > 0 \qquad n \geq 1. \tag{1.29}$$

Some simple special cases of the above inequalities:
a. If $\|Q\|_1 < \tau_0 < \tau_1$ then $m(-\tau_0^2) > m(-\tau_1^2)$; if moreover $Q \le 0$ then the stronger inequality $m(-\tau_0^2) \ge \tau_1 - \tau_0 + m(-\tau_1^2)$ holds with equality if and only if $Q = 0$;
b. if $\int_0^a |1 - q(r)|\, dr < 1$ and $q = 0$ for $r > a$ then

$$\frac{a^2}{1 - a\cot(\delta_1 + a)} > 1 + a\cot(\delta_0 + a), \tag{1.30}$$

if, moreover, $q \le 1$ then

$$\frac{a^2}{1 - a\cot(\delta_1 + a)} \ge 2 + a\cot(\delta_0 + a) \tag{1.31}$$

with equality if and only if $q = 1$ a.e. on $(0, a)$.

Finally we provide some improvements of the inequalities (1.14).

Proposition 1.6. *Let* $q \le 1$ *on* $(0, a)$, $q = 0$ *on* (a, ∞). *Then*

$$\int_0^a r(1 - q(r))\, dr < 1 \Rightarrow \arctan a - a \le \delta_0 < \pi - a, \tag{1.32}$$

$$\int_0^a r(1 - q(r))\, dr < 3 \Rightarrow \frac{\pi}{2} - a + \arctan\frac{a^2 - 3}{3a} \le \delta_1 < \pi - a + \arctan a. \tag{1.33}$$

The proof of the main results are based on the notion of the A-function, introduced in Simon [11]. Remling [10] found a (local) description of the A-function. In the last Remark of this paper we discuss the close connection between Remling's results and a classical characterization of the spectral function given by Levitan and Gasymov [7].

2. Proofs

Proof of Theorems 1.3 and 1.5
Consider the spectral function $\varrho(\lambda)$ of the potential $Q \in L_1(0, \infty)$ subject to the boundary condition $y(0) = 0$. It is connected with the m-function by the known formula

$$m(z) = \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) d\varrho(\lambda) + c, \quad c = \Re m(i). \tag{2.1}$$

for nonreal z. Since $d\varrho(\lambda) = 0$ for $\lambda < -1/4\|Q\|_1^2$, (2.1) extends to $z < -1/4\|Q\|_1^2$. It is known that for smooth functions $F(z)$, $F(z_0, \ldots, z_k) =$

$F^{(k)}(z^*)/k!$ with some z^* between z_0 and z_k. Now if $z_0 < \cdots < z_k < -1/4\|Q\|_1^2$ then

$$(2.2)\qquad m(z_0,\ldots,z_k) = \int\limits_{-1/4\|Q\|_1^2}^{\infty} \frac{1}{(\lambda - z^*(\lambda))^{k+1}} d\varrho(\lambda) > 0 \qquad k \geq 1$$

which is (1.29). To prove Theorem 1.3, apply the variable substitution $x = \log(a/r)$, it transforms $r \in (0,a]$ onto $x \in [0,\infty)$. It is not hard to check that the functions $y(x,-\lambda^2) = r^{-1/2}\varphi(r,\lambda),\ 0 < r \leq a$ satisfy $y \in L_2(0,\infty)$ and $-y" + Q(x)y = -\lambda^2 y$ with the new potential $Q(x) = r^2(q(r)-1)$, $Q \in L_1(0,\infty)$, see [5]. Since $y \in L_2$, we can express the Weyl-Titchmarsh m-fuction by

$$(2.3)\qquad m(-\lambda^2) = \frac{y'(0,-\lambda^2)}{y(0,-\lambda^2)} = \frac{1}{2} - a\frac{\varphi'(a,\lambda)}{\varphi(a,\lambda)}$$
$$= -a\frac{J'_\lambda(a) - \tan\delta(\lambda)Y'_\lambda(a)}{J_\lambda(a) - \tan\delta(\lambda)Y_\lambda(a)} = f(\lambda) - \lambda = g(-\lambda^2).$$

On the other hand,

$$\int\limits_0^\infty |Q(x)|\,dx = \int\limits_0^a \frac{1}{r}|Q(\ln\frac{a}{r})|\,dr = \int\limits_0^a r|1-q(r)|\,dr < \infty.$$

Thus (1.28) is transformed into (1.22) and (1.29) into (1.23). □

Proof of Theorems 1.1 and 1.4 We need the notion of the A-function introduced in Simon [11] where, among others, the following properties are verified: $A - Q$ is continuous, $|A(\alpha) - Q(\alpha)| \leq c\exp(\alpha\|Q\|_1)$ and

$$m(-\tau^2) = -\tau - \int\limits_0^\infty A(\alpha)e^{-2\tau\alpha}d\alpha, \quad \tau > \frac{1}{2}\int\limits_0^\infty |Q|.$$

In [1] it is proved that $|Q_1| \leq -Q_2$ implies $|A_1| \leq -A_2$. In particular this means that $Q \leq 0$ implies $A \leq 0$. Consequently

$$(2.4)\qquad h(\tau) = \int\limits_0^\infty |A(\alpha)|e^{-2\tau\alpha}d\alpha, \quad \tau > \frac{1}{2}\int\limits_0^\infty |Q|.$$

Thus

$$(2.5)\qquad (-1)^k h(\tau_0,\ldots,\tau_k) = \frac{1}{k!}\int\limits_0^\infty |A(\alpha)|(2\alpha)^k e^{-2\tau^*(\alpha)\alpha}d\alpha \geq 0,$$

the convergence of the integral follows from the estimate $|A(\alpha)| \leq |Q(\alpha)| + c\exp(\alpha\|Q\|_1)$. If equality holds in (2.5) then $A = 0$ and then

$Q = 0$. This proves Theorem 1.4. Now Theorem 1.1 follows just like in the previous proof, taking into account that $q \le 1$ is equivalent to $Q \le 0$ and that $h = f$. Finally, using the formulae

$$J_{1/2}(r) = \sqrt{\frac{2}{\pi r}} \sin r, \quad Y_{1/2}(r) = -\sqrt{\frac{2}{\pi r}} \cos r$$
$$J_{3/2}(r) = \sqrt{\frac{2}{\pi r}} \left(\frac{\sin r}{r} - \cos r \right), \; Y_{3/2}(r) = \sqrt{\frac{2}{\pi r}} \left(-\frac{\cos r}{r} - \sin r \right)$$

we see from (2.3) that

$$\begin{aligned} m(-1/4) &= \frac{1}{2} - a\frac{\cos a - \tan\delta_0 \sin a}{\sin a + \tan\delta_0 \cos a} = \frac{1}{2} - a\frac{1 - \tan\delta_0 \tan a}{\tan a + \tan\delta_0} \\ &= \frac{1}{2} - a\cot(\delta_0 + a), \end{aligned}$$

and

$$\begin{aligned} m(-9/4) &= \frac{1}{2} - a\frac{\frac{\cos a}{a} - \frac{\sin a}{a^2} + \sin a - \tan\delta_1(\frac{\sin a}{a} + \frac{\cos a}{a^2} - \cos a)}{\frac{\sin a}{a} - \cos a + \tan\delta_1(\frac{\cos a}{a} + \sin a)} \\ &= \frac{1}{2} + 1 - a\frac{\tan a + \tan\delta_1}{(\tan a + \tan\delta_1)/a + \tan a \tan\delta_1 - 1} \\ &= \frac{3}{2} - \frac{a}{1/a - \cot(\delta_1 + a)} = \frac{3}{2} - \frac{a^2}{1 - a\cot(\delta_1 + a)} \end{aligned}$$

and this verifies (1.30) and (1.31). □

Remark The converse of Theorem 1.1 is not true i.e. all the inequalities (1.18) are not enough to ensure $q \le 1$. Indeed, (1.18) for all $\lambda_0, \ldots, \lambda_k$ implies $A \le 0$ (the backward differences of the exponential moments of the measure $-A(\alpha)exp(-(2n_0+1)\alpha)d\alpha$ are nonnegative so the measure is nonnegative by classical Hausdorff moment theorems) but $A \le 0$ does not imply $Q \le 0$ i.e. $q \le 1$. This is illustrated by a Bargmann-type potential. We borrow from [1] the following example: if c, $\kappa > 0$ then the potential

$$Q(x) = -2\frac{d^2}{dx^2} \ln \left(1 + \frac{c}{\kappa^2} \int_0^x \sinh^2(\kappa y)\, dy \right)$$

gives the A-function

$$A(\alpha) = -\frac{2c}{\kappa} \sinh(2\alpha\kappa) < 0.$$

A straightforward calculation shows that $Q \in L_1(0, \infty)$ but $Q(x) > 0$ for large x.

Proof of Proposition 1.6 Consider the box potentials $q(r) = q_0,\ r \in (0,a),\ q(r) = 0,\ r > a$. We know that for $q_0 = 1$

$$\frac{\lambda}{a} = \frac{J'_\lambda(a) - \tan\delta(\lambda; q_0 = 1)Y'_\lambda(a)}{J_\lambda(a) - \tan\delta(\lambda; q_0 = 1)Y_\lambda(a)}, \tag{2.6}$$

see [6]. From here we infer

$$\tan\delta(\lambda; q_0 = 1) = \frac{\lambda J_\lambda(a) - aJ'_\lambda(a)}{\lambda Y_\lambda(a) - aY'_\lambda(a)} = \frac{J_{\lambda+1}(a)}{Y_{\lambda+1}(a)}. \tag{2.7}$$

In particular,

$$\tan\delta_0(q_0 = 1) = \frac{J_{3/2}(a)}{Y_{3/2}(a)} = -\frac{\tan a - a}{1 + a\tan a} = -\tan(a - \arctan a) \tag{2.8}$$

Since δ_0 is a continuous function of a and tends to zero if $a \to 0$, we get

$$\delta_0(q_0 = 1) = \arctan a - a. \tag{2.9}$$

We analogously find that

$$\tan\delta_1(q_0 = 1) = \frac{J_{5/2}(a)}{Y_{5/2}(a)} = -\tan\left(a + \frac{\pi}{2} - \arctan\frac{a^2 - 3}{3a}\right) \tag{2.10}$$

and hence

$$\delta_1(q_0 = 1) = \frac{\pi}{2} - a + \arctan\frac{a^2 - 3}{3a}. \tag{2.11}$$

This verifies the lower estimates of Proposition 1.6. If q diminishes from 1 on the segment $(0,a)$, $\delta_0 + a$ increases from $\arctan a$ and then $1 - a\cot(\delta_0 + a)$ is increasing, too. It can not reach infinity because $m(-1/4)$ must be finite. Thus, $\delta_0 + a < \pi$. Analogously $\delta_1 + a$ and hence $3 - a^2/(1 - a\cot(\delta_1 + a))$ is increasing. The denominator can be infinite but can not take the value 0 since $m(-9/4)$ is finite; therefore we have $1/a \neq \cot(\delta_1 + a)$. Taking (1.14) into account, we see that $\delta_1 + a < \pi + \arctan a$. □

Remark Remling [10] found a property which characterizes the A-functions, see below. In what follows we give a simple verification of the fact that this property holds for every A-functions. Recall first that the A-function is defined formally in [1] by

$$A(\alpha) = -2\int_{-\infty}^{\infty} \frac{\sin 2\alpha\sqrt{\lambda}}{\sqrt{\lambda}}\, d\varrho(\lambda). \tag{2.12}$$

Here ϱ is the spectral function corresponding to the boundary condition $y(0) = 0$. Since the integral may be divergent, the distributional interpretation

(2.13)
$$\int_0^b A(\alpha)f(\alpha)\,d\alpha = -2\int_{-\infty}^{\infty}\int_0^b f(\alpha)\frac{\sin 2\alpha\sqrt{\lambda}}{\sqrt{\lambda}}d\alpha d\varrho(\lambda),\ \ f\in C_0^\infty(0,b)$$

can be used; here $b > 0$ is an arbitrary finite value. If $Q = 0$ then $A = 0$, thus here $d\varrho$ can be substituted by $d\sigma = d\varrho - d\varrho_0$, where $\varrho_0(\lambda) = 2\lambda^{3/2}/(3\pi)$ for $\lambda \geq 0$ and zero for $\lambda < 0$. Remling [10] gave the following description: the function $A \in L_1(0, b)$ is an A-function of a locally integrable potential if and only if

$$I + K_A > 0 \quad \text{in } L_2(0,b) \tag{2.14}$$

where K_A is the integral operator on $L_2(0, b)$ with the kernel

$$K(x,t) = -\frac{1}{2}\int_{|x-t|/2}^{(x+t)/2} A. \tag{2.15}$$

The fact that an A-function satisfies (2.14), follows easily from classical results on the spectral function; this will be verified below. Levitan and Gasymov [7] (see also [8]) proved that an increasing function $\varrho(\lambda)$ is the spectral function of some $Q \in L_1^{loc}[0,\infty)$ under the boundary condition $y(0) = 0$ if and only if

$$\int_{-\infty}^{N}\frac{\cos\sqrt{\lambda}x - 1}{\lambda}d\sigma(\lambda) \to \Phi(x) = \int_{-\infty}^{\infty}\frac{\cos\sqrt{\lambda}x - 1}{\lambda}d\sigma(\lambda) \tag{2.16}$$

with bounded convergence, where the bounds are locally uniform in x, and if

$$\int_{-\infty}^{\infty} E^2\,d\varrho = 0 \Rightarrow f = 0 \text{ a.e.} \tag{2.17}$$

for every $f \in L_2(0,\infty)$ with compact support, where

$$E(\lambda) = \int_0^\infty f(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}dx. \tag{2.18}$$

A formal differentiation in (2.16) suggests that

$$A(\alpha) = 2\Phi'(2\alpha). \tag{2.19}$$

The true verification uses (2.13). The bounded convergence of (2.16) gives after an integration by parts that for $f \in C_0^\infty(0,b)$

$$\int_0^b Af = 2\lim_{N\to\infty}\int_{-\infty}^N\int_0^b f'(\alpha)\frac{1-\cos 2\alpha\sqrt{\lambda}}{2\lambda}d\alpha d\varrho(\lambda)$$
$$= -\int_0^b \Phi(2\alpha)f'(\alpha)\,d\alpha$$

thus (2.19) holds in the distributional sense. From $A \in L_1(0,b)$ we infer that Φ is locally absolutely continuous and that (2.19) holds a.e. and then from (2.15)

$$K(x,t) = \frac{\Phi(x-t)-\Phi(x+t)}{2}. \tag{2.20}$$

Now again by the bounded convergence

$$\int_{-\infty}^\infty E^2 d\sigma =$$
$$= \lim_{N\to\infty}\int_{-\infty}^N\int_0^b\int_0^b f(x)f(t)\frac{\cos\sqrt{\lambda}(x-t)-\cos\sqrt{\lambda}(x+t)}{2\lambda}\,dx\,dt\,d\sigma(\lambda)$$
$$= \int_0^b\int_0^b f(x)f(t)\frac{\Phi(x-t)-\Phi(x+t)}{2}\,dx\,dt = (K_Af,f)_{L_2(0,b)}.$$

Since ϱ_0 is the spectral function of the zero potential, $\int_{-\infty}^\infty E^2 d\varrho_0 = (f,f)$, so we finally find that

$$\int_{-\infty}^\infty E^2 d\varrho = ((I+K_A)f,f). \tag{2.21}$$

This proves that the condition (2.14) follows at once from (2.17) and hence (2.14) holds for the A-functions. The other implication of the Remling theorem will not be discussed here.

References

[1] F. Gesztesy and B. Simon, *A new approach to inverse spectral theory II. General real potentials and the connection to the spectral measure*, Ann. of Math. **152** (2000), no. 2, 593–643.

[2] R. Guzman and H. Rabitz, *Forward and inverse functional variations in elastic scattering*, J. Chem. Phys. **86** (1987), no. 3, 1395–1406.

[3] T. Ho and H. Rabitz, *Reconstruction of intermolecular potentials at fixed energy: Functional sensitivity analysis approach*, J. Chem. Phys. **89** (1988), no. 9, 5614–5623.

[4] M. Horváth, *Partial knowledge of the potential from phase shifts*, (manuscript).

[5] ———, *Inverse scattering with fixed energy and an inverse eigenvalue problem on the half-line* , Trans. Amer. Math. Soc. **358** (2006), no. 11, 5161–5177.

[6] ———, *Notes on the distribution of phase shifts*, Modern Physics Letters B **22** (2008), no. 23, 2163–2175.

[7] B. M. Levitan and M. G. Gasymov, *Determination of the differential equation by two spectra (in Russian)*, Uspehi Mat. Nauk **19** (1964), 3–62.

[8] B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac operators (in Russian)*, Nauka, Moscow, 1988.

[9] J. Loeffel, *On an inverse problem in quantum scattering theory*, Ann. Inst. H. Poincaré **8** (1968), 339–447.

[10] Ch. Remling, *Inverse spectral theory for one-dimensional Schrödinger operators: The A function*, Math. Z. **245** (2003), 597–617.

[11] B. Simon, *A new approach to inverse spectral theory I. Fundamental formalism*, Annals of Math **150** (1999), 1–29.

[12] T.Regge, *Introducion to complex orbital momenta*, Nuovo Cimento **14** (1959), no. 5, 951–976.

Department of Analysis
Institute of Mathematics
Budapest University of Technology and Economics
H 1111 Budapest, Műegyetem rkp. 3-9.
E-mail address: horvath (at) math (dot) bme (dot) hu