AUTHOR QUERY FORM

ELSEVIER	**Journal:** YJMAA **Article Number:** 15410	**Please e-mail or fax your responses and any corrections to:** **E-mail: corrections.essd@elsevier.vtex.lt** **Fax: +1 61 9699 6735**

Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list.

For correction or revision of any artwork, please consult *http://www.elsevier.com/artworkinstructions*

No queries have arisen during the processing of your article.

Thank you for your assistance.

Contents lists available at ScienceDirect
Journal of Mathematical Analysis and Applications
www.elsevier.com/locate/jmaa

Partial identification of the potential from phase shifts ☆

Miklós Horváth *

Department of Analysis, Institute of Mathematics, Budapest University of Technology and Economics, Műegyetem rkp. 3-9, H 1111 Budapest, Hungary

ARTICLE INFO

Article history:
Received 5 August 2010
Available online xxxx
Submitted by S. Fulling

Keywords:
Inverse scattering
Phase shifts
Scattering amplitude

ABSTRACT

We consider the three-dimensional inverse scattering with fixed energy in the spherically symmetrical case. We give a characterization of the sequences of phase shifts for two potentials which can be different only in a ball of radius a. In other words we study how the large distance interaction influences the asymptotical behavior of the phase shifts. We also characterize the tail of the potential by the growth order of the scattering amplitude $F(t)$ for large t.

1. Introduction

It is known (see e.g. [5] or [6]) that the three-dimensional inverse scattering with fixed energy in the case when the potential is spherically symmetrical, is described by the following system of equations

$$\varphi_n''(r) - \frac{n(n+1)}{r^2}\varphi_n(r) - q(r)\varphi_n(r) + k^2\varphi_n(r) = 0, \quad r \geqslant 0 \tag{1.1}$$

with real-valued potential function $q(r)$, $rq(r) \in L_1(0,\infty)$ and fixed energy $k^2 = 1$. It is known that there exists a unique solution of (1.1) with

$$\varphi_n(r) = \gamma_n r^{n+1}\big(1 + \mathbf{o}(1)\big), \quad r \to 0+ \tag{1.2}$$

and

$$\varphi_n(r) = \sin(r - n\pi/2 + \delta_n) + \mathbf{o}(1), \quad r \to +\infty. \tag{1.3}$$

The quantities δ_n are called *phase shifts*.

The inverse scattering problem investigated here consists of the recovery of the potential q from the phase shifts δ_n.

Sometimes it is useful to extend the system (1.1) to noninteger λ, $\Re\lambda > 0$ as follows:

$$\varphi''(r,\lambda) - \frac{\lambda^2 - 1/4}{r^2}\varphi(r,\lambda) + \big(1 - q(r)\big)\varphi(r,\lambda) = 0, \quad r \geqslant 0, \tag{1.4}$$

$$\varphi(r,\lambda) = \gamma(\lambda) r^{\lambda+1/2}\big(1 + \mathbf{o}(1)\big), \quad r \to 0+, \tag{1.5}$$

$$\varphi(r,\lambda) = \sin\big(r - \pi/2(\lambda - 1/2) + \delta(\lambda)\big) + \mathbf{o}(1), \quad r \to +\infty. \tag{1.6}$$

Then $\delta_n = \delta(n + 1/2)$ and $\gamma_n = \gamma(n + 1/2)$ for $n \geqslant 0$.

☆ Research supported by the Hungarian NSF Grant OTKA T 61311.
* Fax: +361 463 3172.
E-mail address: horvath@math.bme.hu.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.10.071

The scattering amplitude $F(t)$ can be expressed by the phase shifts:

$$F(t)=\sum_{n=0}^{\infty}(2n+1)F_nP_n(t),\qquad F_n=e^{i\delta_n}\cdot\sin\delta_n. \tag{1.7}$$

Here the functions $P_n(t)$ are the Legendre polynomials, see (2.16) below. The scattering amplitude has a physical interpretation only for $t\in[-1,1]$, however the formula (1.7) can be extended to any $t\in\mathbf{C}$ in case of convergence. For example if the potential is compactly supported then the phase shifts have a more than exponential decay, hence (1.7) defines an entire function $F(t)$. In the next statement, which is the main result of this paper we characterize the knowledge of the tail of the potential by the asymptotical behavior of the scattering amplitude.

Theorem 1.1. *Let $rq(r), rq^*(r)\in L_1(0,\infty)$ and let $0<a<\infty$.*

(a) *If $q=q^*$ a.e. on (a,∞) then $F(t)-F^*(t)$ is entire and*

$$\bigl|F(t)-F^*(t)\bigr|\leqslant c\bigl(1+|t|\bigr)\exp\bigl(a\sqrt{2|t|}\bigr). \tag{1.8}$$

(b) *Conversely, if q and q^* have compact support, $F(t)-F^*(t)$ is an entire function and*

$$F(t)-F^*(t)=\mathbf{O}\bigl(\exp\bigl(a_1\sqrt{2|t|}\bigr)\bigr) \tag{1.9}$$

holds for all $a_1>a$, then $q=q^*$ a.e. on (a,∞).

Concerning the characterization by the decay of $\delta_n-\delta_n^*$ we have

Theorem 1.2. *Let $rq(r), rq^*(r)\in L_1(0,\infty)$.*

(a) *If $q=q^*$ a.e. on (a,∞), then for all sufficiently large n*

$$\bigl|\delta_n-\delta_n^*\bigr|\leqslant\frac{c}{n^2}\left(\frac{ae}{2n}\right)^{2n}. \tag{1.10}$$

(b) *Conversely, if q and q^* have compact support and*

$$\delta_n-\delta_n^*=\mathbf{O}\left(\left(\frac{a_1e}{2n}\right)^{2n}\right),\quad n\to\infty \tag{1.11}$$

holds for all $a_1>a$, then $q=q^*$ a.e. on (a,∞).

Remark that in the special case $q^*=0$ a statement similar to part (a) is given in Ramm et al. [16]; they proved that if $q(r)=0$ for a.e. $r>a$ and q has constant sign in some interval $(a-\varepsilon,a)$ then

$$\lim_{n\to\infty}n|\delta_n|^{1/(2n)}=\frac{ae}{2}.$$

Remark. The reconstruction of the potential from the phase shifts may be not unique. For example there are nontrivial potentials, oscillating and of order $r^{-3/2}$ at infinity for which all the phase shifts vanish, see e.g. Newton [10], Chapter 20.4. However, for potentials of compact support, uniqueness is already proved in the paper of Loeffel [9] in 1968. In this case very sparse subsequences of δ_n are enough for the unique reconstruction of the potentials, see Ramm [17]. If the potentials are not spherically symmetrical, uniqueness is given in Ramm [14] for the case of compact support and in Novikov [11] for bounded potentials with some exponential decay.

The inverse scattering is only weakly stable. Examples for very different stepfunction potentials with almost the same phase shifts are given e.g. in [13]. The idea that the error in the output can be estimated by the reciprocal of the logarithm of the input error, appears in Alessandrini [2] for the stability of the inverse conductivity problem. Similar results are obtained by Stefanov [19] for the inverse scattering with fixed energy. A logarithmic bound for the Fourier transform of the potential perturbation is given in Ramm [15]. If only finitely many phase shifts are available with some error, a logarithmic estimate can be found in Horváth and Kiss [8].

Please cite this article in press as: M. Horváth, Partial identification of the potential from phase shifts, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.10.071

2. Preliminaries

In this section we prove three results we need in the later parts of the paper. The first one is a collection of uniform estimates of $\varphi(r,\lambda)$ for large λ. The second one is a characterization of even entire functions $F(z^2)$ of exponential type $\leqslant A$ through the coefficients of the expansion of $F(z)$ by the Legendre polynomials; the third one is a "discrete" uniqueness result for the Laplace transform of a function in $L_1(0,a)$.

It is known that for $q=0$ we have $\varphi(r,\lambda)=u(r,\lambda)$, $u(r,\lambda)=\sqrt{\frac{\pi r}{2}}J_\lambda(r)$. Another solution of (1.4) in case $q=0$ is the function $w(r,\lambda)=-i\sqrt{\frac{\pi r}{2}}H_\lambda^{(1)}(r)$, see e.g. [5]. The following uniform estimates are necessary for later purposes:

Lemma 2.1. *Suppose that $\lambda>0$ is sufficiently large to satisfy*

$$\left[\pi\int_0^\infty r\left|q(r)\right|dr\right]^4+\frac{1}{16}<\lambda^2. \tag{2.1}$$

Then

$$\left|\varphi(r,\lambda)\right|\leqslant\sqrt{2\pi r}, \tag{2.2}$$

$$\left|\varphi(r,\lambda)w\left(r^*,\lambda\right)\right|\leqslant\pi\sqrt{rr^*}\left(\lambda^2-\frac{1}{16}\right)^{-1/4},\quad \textit{if } 0\leqslant r\leqslant r^*<\infty, \tag{2.3}$$

$$\left|\varphi(r,\lambda)\right|\leqslant\frac{2\sqrt{2\pi}}{2^\lambda\Gamma(\lambda+1)}r^{\lambda+1/2}. \tag{2.4}$$

Proof. We know e.g. from Alfaro and Regge [3, p. 84] that

$$\varphi(r,\lambda)=e^{-i\delta(\lambda)}u(r,\lambda)+\int_0^\infty K\left(r,r'\right)q\left(r'\right)\varphi\left(r',\lambda\right)dr' \tag{2.5}$$

with kernel function

$$K\left(r,r'\right)=u(r_<,\lambda)w(r_>,\lambda),\quad r_<=\min\left(r,r'\right),\ r_>=\max\left(r,r'\right). \tag{2.6}$$

In [3, Appendix D] the estimate

$$\left|K\left(r,r'\right)\right|\leqslant\frac{\pi}{2}\sqrt{rr'}\left(\lambda^2-1/16\right)^{-1/4} \tag{2.7}$$

is proved. Putting this into (2.5) gives (we omit the second components λ)

$$\left|\varphi(r)\right|\leqslant\left|u(r)\right|+\frac{\pi}{2}\left(\lambda^2-1/16\right)^{-1/4}\int_0^\infty\sqrt{rr'}\left|q\left(r'\right)\right|\left|\varphi\left(r'\right)\right|dr'. \tag{2.8}$$

The estimate $|J_\lambda(r)|\leqslant 1$ (see [1, 9.1.60]) implies $|u(r)|\leqslant\sqrt{\pi/2}\sqrt{r}$. We are looking for a similar estimate $|\varphi(r)|\leqslant c\sqrt{r}$. For fixed λ such a constant c exists, see (1.5) and (1.6). To check that c is independent also of λ return to (2.8), we obtain

$$\left|\varphi(r)\right|\leqslant\sqrt{\pi r/2}+\frac{\pi}{2}\sqrt{r}\left(\lambda^2-1/16\right)^{-1/4}\int_0^\infty\sqrt{r'}\left|q\left(r'\right)\right|c\sqrt{r'}\,dr' \tag{2.9}$$

or

$$\left|\varphi(r)\right|/\sqrt{r}\leqslant\sqrt{\pi/2}+c\frac{\pi}{2}\left(\lambda^2-1/16\right)^{-1/4}\int_0^\infty r'\left|q\left(r'\right)\right|dr'. \tag{2.10}$$

If $c=\sup(|\varphi(r)|/\sqrt{r})$ then

$$c\leqslant\sqrt{\pi/2}+c\frac{\pi}{2}\left(\lambda^2-1/16\right)^{-1/4}\int_0^\infty r'\left|q\left(r'\right)\right|dr'\leqslant\sqrt{\pi/2}+c/2. \tag{2.11}$$

Please cite this article in press as: M. Horváth, Partial identification of the potential from phase shifts, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.10.071

That is, $c \leqslant \sqrt{2\pi}$ and this proves (2.2). To check (2.3) define

$$c = \sup\bigl(\bigl|\varphi(r)w\bigl(r^*\bigr)\bigr|/\sqrt{rr^*}\bigr) \quad \text{if } 0 \leqslant r \leqslant r^* < \infty. \tag{2.12}$$

This constant is finite: near the origin $\varphi/\sqrt{r}$ and $w/\sqrt{r^*}$ have the order r^λ and $(r^*)^{-\lambda}$, so their product is bounded; if $r^* \to \infty$ then $w(r^*)$ and $\varphi(r)$ are bounded. So c is finite indeed. Multiplying (2.5) by $w(r^*)$ gives

$$\varphi(r)w\bigl(r^*\bigr) = e^{-i\delta(\lambda)}u(r)w\bigl(r^*\bigr) + \int_0^r u\bigl(r'\bigr)w(r)q\bigl(r'\bigr)\varphi\bigl(r'\bigr)w\bigl(r^*\bigr)\,dr' + \int_r^\infty u(r)w\bigl(r^*\bigr)q\bigl(r'\bigr)\varphi\bigl(r'\bigr)w\bigl(r'\bigr)\,dr'. \tag{2.13}$$

Applying here (2.7) three times and (2.12) two times we obtain

$$\bigl|\varphi(r)w\bigl(r^*\bigr)\bigr| \leqslant c_0\sqrt{rr^*} + \int_0^r c_0\sqrt{r'r}\bigl|q\bigl(r'\bigr)\bigr|c\sqrt{r'r^*}\,dr' + \int_r^\infty c_0\sqrt{rr^*}\bigl|q\bigl(r'\bigr)\bigr|cr'\,dr' \tag{2.14}$$

with $c_0 = \pi/2(\lambda^2 - 1/16)^{-1/4}$. Thus, dividing by $\sqrt{rr^*}$ we finally get

$$c \leqslant c_0\left[1 + c\int_0^\infty r\bigl|q(r)\bigr|\,dr\right] \leqslant c_0 + c/2 \tag{2.15}$$

which means $c \leqslant 2c_0$ and this proves (2.3). Finally let

$$c = \sup\bigl(\varphi(r)/r^{\lambda+1/2}\bigr).$$

From [1, 9.1.62] we infer

$$\bigl|u(r)\bigr| \leqslant c_0 r^{\lambda+1/2}, \quad c_0 = \sqrt{\frac{\pi}{2}}\frac{1}{2^\lambda\Gamma(\lambda+1)}.$$

Putting all the above considered inequalities into (2.13) gives

$$\bigl|\varphi(r)\bigr| \leqslant c_0 r^{\lambda+1/2} + \int_0^r \pi/2\bigl(\lambda^2 - 1/16\bigr)^{-1/4}\sqrt{rr'}\bigl|q\bigl(r'\bigr)\bigr|cr^\lambda\sqrt{r'}\,dr' + \int_r^\infty c_0 r^{\lambda+1/2}\bigl|q\bigl(r'\bigr)\bigr|\pi r'\bigl(\lambda^2 - 1/16\bigr)^{-1/4}\,dr'.$$

After dividing by $r^{\lambda+1/2}$ and extending both integrals from zero to infinity it follows that $c \leqslant c_0 + c/2 + c_0$, that is $c \leqslant 4c_0$ which proves (2.4). □

Our next statement gives a characterization of even entire functions $F(z^2)$ of exponential type $\leqslant A$ in terms of the coefficients of the expansion of $F(z)$ with respect to the Legendre polynomials

$$P_n(z) = \frac{1}{2^n n!}\bigl[\bigl(z^2 - 1\bigr)^n\bigr]^{(n)}. \tag{2.16}$$

Introduce the Legendre functions

$$Q_n(z) = \frac{1}{2}\int_{-1}^1 P_n(y)\frac{dy}{z-y}. \tag{2.17}$$

If $F(z)$ is an entire function, it can be expanded into a series

$$F(z) = \sum_{n=0}^\infty a_n P_n(z), \quad a_n = \frac{2n+1}{2\pi i}\oint_G F(t)Q_n(t)\,dt,$$

where G is any ellipse with foci ± 1, see [20, 15.4].

Please cite this article in press as: M. Horváth, Partial identification of the potential from phase shifts, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.10.071

Lemma 2.2.

(a) *If $F(z)$ is an entire function satisfying*

$$\left|F(z)\right| \leqslant c e^{A\sqrt{|z|}}, \quad z \in \mathbf{C} \tag{2.18}$$

then

$$F(z) = \sum_{n=0}^{\infty} a_n P_n(z) \quad \text{with } |a_n| \leqslant c\sqrt{n}\left(\frac{eA}{2\sqrt{2n}}\right)^{2n} \quad \text{for } n \geqslant 1. \tag{2.19}$$

(b) *Conversely, if* (2.19) *holds then $F(z)$ is entire and*

$$\left|F(z)\right| \leqslant c\left(1+\sqrt{|z|}\right)e^{A\sqrt{|z|}}. \tag{2.20}$$

The constants c can be different in different occurrences.

Proof. Substituting (2.16) into (2.17) gives after n integrations by parts that

$$Q_n(z) = \frac{1}{2^{n+1}n!}\int_{-1}^{1}\left[\left(y^2-1\right)^n\right]^{(n)}\frac{dy}{z-y} = \frac{1}{2^{n+1}}\int_{-1}^{1}\left(1-y^2\right)^n\frac{dy}{(z-y)^{n+1}}.$$

If the parameter of the ellipse G is $2R+2$ then $|z-y| \geqslant R$ for $z \in G$ and $y \in [-1,1]$, hence

$$\left|Q_n(z)\right| \leqslant \frac{1}{(2R)^{n+1}}\int_{-1}^{1}\left(1-y^2\right)^n dy \leqslant \frac{c}{\sqrt{n}}\frac{1}{(2R)^{n+1}}. \tag{2.21}$$

Now if (2.18) holds then

$$|a_n| \leqslant cn \oint_{|t-1|+|t+1|=2R+2}\left|F(t)\right|\left|Q_n(t)\right|dt \leqslant cn\oint e^{A\sqrt{R+1}}\frac{1}{\sqrt{n}(2R)^{n+1}}\,dt \leqslant c\sqrt{n}e^{A\sqrt{R}}\frac{1}{(2R)^n}.$$

The right-hand side is minimal at $\sqrt{R} = 2n/A$ and this gives (2.19). To prove the converse we will use the bound [12, 8.21]

$$\left|P_n(z)\right| \leqslant \frac{c(\delta)}{\sqrt{n}}|z|^{-1/2}\left(|z|+\sqrt{\left|z^2-1\right|}\right)^{n+1/2}, \quad \operatorname{dist}\left(z,[-1,1]\right) > \delta \tag{2.22}$$

to obtain

$$\begin{aligned}
\left|F(z)\right| &\leqslant |a_0| + \sum_{n=1}^{\infty} c\sqrt{n}\left(\frac{eA}{2\sqrt{2n}}\right)^{2n}\cdot\frac{1}{\sqrt{n}}\left(|z|+\sqrt{\left|z^2-1\right|}\right)^{n+1/2} \\
&\leqslant |a_0| + c\sum_{n=1}^{\infty}\left(\frac{e^2A^2}{8n^2}\left(|z|+\sqrt{\left|z^2-1\right|}\right)\right)^n \\
&\leqslant |a_0| + c\sum_{n=1}^{\infty}\sqrt{n}\frac{\left[A\sqrt{\frac{|z|+\sqrt{|z^2-1|}}{2}}\right]^{2n}}{(2n)!} \\
&\leqslant |a_0| + c\sum_{n=1}^{\infty}\sqrt{|z|}\frac{\left[A\sqrt{\frac{|z|+\sqrt{|z^2-1|}}{2}}\right]^{2n-1}}{(2n-1)!} \\
&\leqslant |a_0| + c\sum_{n=1}^{\infty}\sqrt{|z|}\exp\left(A\sqrt{\frac{|z|+\sqrt{|z^2-1|}}{2}}\right) \\
&\leqslant c\left(1+\sqrt{|z|}\right)e^{A\sqrt{|z|}},
\end{aligned}$$

which is (2.20). □

Please cite this article in press as: M. Horváth, Partial identification of the potential from phase shifts, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.10.071

Corollary 2.3. *The even and entire function $F(z^2)$ is of exponential type $\leqslant \sigma$ if and only if*

$$F(z) = \sum_{n=0}^{\infty} a_n P_n(z), \quad \limsup\bigl(n \sqrt[2n]{|a_n|}\bigr) \leqslant \frac{e\sigma}{2\sqrt{2}}. \tag{2.23}$$

The third topic considered is the following uniqueness result concerning the Laplace transform:

Proposition 2.4. *Let $f \in L_1(0,a)$. If for all $\varepsilon > 0$*

$$\int_0^a f(y) e^{-ny}\, dy = \mathbf{O}\bigl(e^{-an(1-\varepsilon)}\bigr), \quad n \to \infty \tag{2.24}$$

then $f = 0$ a.e., in particular in all Lebesgue points of f.

The continuous version, where (2.24) holds also for noninteger n, is proved in Simon [18].

To verify Proposition 2.4, we need

Lemma 2.5. *In all Lebesgue points r of $f \in L_1(0,a)$ we have*

$$\int_0^a \left[\frac{f(y)}{1 - e^{i(\varepsilon - ir) - y}} - \frac{f(y)}{1 - e^{i(-\varepsilon - ir) - y}}\right] dy \to 2\pi i f(r), \quad \varepsilon \to 0+. \tag{2.25}$$

Proof. Remark first that (2.25) is valid for $f = 1$, that is,

$$\int_0^a \left[\frac{1}{1 - e^{i(\varepsilon - ir) - y}} - \frac{1}{1 - e^{i(-\varepsilon - ir) - y}}\right] dy \to 2\pi i, \quad \varepsilon \to 0+. \tag{2.26}$$

Indeed, for $w = \pm\varepsilon - ir$

$$\int_0^a \frac{dy}{1 - e^{iw - y}} = \int_1^{e^a} \frac{dt}{t - e^{iw}} = \log\bigl(e^a - e^{iw}\bigr) - \log\bigl(1 - e^{iw}\bigr) \tag{2.27}$$

and the imaginary part on the right of (2.27) tends to $\pm\pi$ for $w = \pm\varepsilon - ir$. This verifies (2.26). Thus for the proof of (2.25) it is enough to check that

$$\int_0^a \bigl(f(y) - f(r)\bigr)\left[\frac{1}{1 - e^{i(\varepsilon - ir) - y}} - \frac{1}{1 - e^{i(-\varepsilon - ir) - y}}\right] dy \to 0,$$

that is,

$$\int_0^a \bigl(f(y) - f(r)\bigr) \frac{\sin\varepsilon}{\cosh(y - r) - \cos\varepsilon}\, dy \to 0, \quad \varepsilon \to 0+. \tag{2.28}$$

Since

$$\cosh(y - r) - \cos\varepsilon = \frac{(y - r)^2 + \varepsilon^2}{2} + \mathbf{O}\bigl((y - r)^4 + \varepsilon^4\bigr)$$

hence

$$\frac{\sin\varepsilon}{\cosh(y - r) - \cos\varepsilon} = \begin{cases} \mathbf{O}\bigl(\frac{1}{\varepsilon}\bigr) & \text{if } |y - r| < \varepsilon, \\ \mathbf{O}\bigl(\frac{\varepsilon}{(y - r)^2}\bigr) & \text{if } |y - r| > \varepsilon. \end{cases}$$

Introduce the function

$$F(y) = \int_r^y \bigl|f(t) - f(r)\bigr|\, dt,$$

then $F(r+t)-F(r-t)=\mathbf{o}(t)$ if $t\to 0$ since r is a Lebesgue point. In (2.28) the following estimates can be applied:

$$\int\limits_{|y-r|<\varepsilon} = \mathbf{O}\biggl(\frac{1}{\varepsilon}\int\limits_{|y-r|<\varepsilon}\bigl|f(y)-f(r)\bigr|\,dy\biggr)\to 0,\quad \varepsilon\to 0+,$$

$$\int\limits_{|y-r|>\varepsilon} = \mathbf{O}\biggl(\varepsilon\int\limits_{|y-r|>\varepsilon}\frac{|f(y)-f(r)|}{(y-r)^2}\,dy\biggr) = \mathbf{O}\biggl(\varepsilon\int\limits_{|y-r|>\varepsilon}\frac{F'(y)}{(y-r)^2}\,dy\biggr)$$

$$= \mathbf{O}\biggl(\varepsilon\biggl[\frac{F(y)}{(y-r)^2}\biggr]_{y=r+\varepsilon}^{r-\varepsilon} + 2\int\limits_{|y-r|>\varepsilon}\frac{F(y)}{(y-r)^3}\,dy\biggr) = \mathbf{o}(1).$$

This proves Lemma 2.5 □

Proof of Proposition 2.4. Define

$$g_n = \int_0^a f(y)e^{-ny}\,dy.$$

The uniform convergence of the series

$$\sum_{n=0}^{\infty} e^{-n(y-iw)} = \frac{1}{1-e^{iw-y}},\quad \Im w>0$$

in $y\in[0,a]$ implies that

$$h(w) = \sum_{n=0}^{\infty} g_n e^{inw} = \int_0^a f(y)\sum_{n=0}^{\infty} e^{-n(y-iw)}\,dy = \int_0^a \frac{f(y)}{1-e^{iw-y}}\,dy,\quad \Im w>0. \tag{2.29}$$

From (2.24) it follows that the sum in the left-hand side of (2.29) has a regular extension to $\Im w>-a$, while the integral on the right of (2.29) is regular on $w\in\mathbf{C}\setminus[0,-ia]$. Thus the sum and the integral are equal for $\Im w>-a$, $w\notin[0,-ia]$, in particular for $w=\pm\varepsilon-ir$ where r is a Lebesgue point of f. Consequently

$$h(\varepsilon-ir)-h(-\varepsilon-ir) = \int_0^a \biggl[\frac{f(y)}{1-e^{i(\varepsilon-ir)-y}} - \frac{f(y)}{1-e^{i(-\varepsilon-ir)-y}}\biggr]dy.$$

Here the right-hand side tends to $2\pi i f(r)$ by Lemma 2.5 while the left-hand side tends to zero by the continuity of h at $-ir$. Proposition 2.4 is proved. □

3. Proof of the theorems

Consider the Schrödinger operator $Ly=-y''+Q(x)y$ on the half-line $x\in[0,\infty)$ with the potential $Q\in L_1(0,\infty)$. It is known that for $\lambda\in\mathbf{C}\setminus(\beta,\infty)$ the solution $y\in L_2(0,\infty)$ of $-y''+Q(x)y=\lambda y$ is unique up to a constant factor. Using this solution we can define the m-function as

$$m(\lambda) = \frac{y'(0)}{y(0)}.$$

Let $Q^*\in L_1(0,\infty)$ be another potential. In Simon [18] it is proved that $Q=Q^*$ on $(0,a)$ if and only if

$$m\bigl(-\tau^2\bigr)-m^*\bigl(-\tau^2\bigr) = \mathbf{O}\bigl(e^{-2\tau a(1-\varepsilon)}\bigr),\quad \tau\to+\infty$$

holds for all $\varepsilon>0$. We show below that it is enough to verify this condition for the discrete values $\tau=n+1/2$:

Proposition 3.1. *Let Q, $Q^*\in L_1(0,\infty)$. Then $Q=Q^*$ on $(0,a)$ if and only if*

$$m\bigl(-(n+1/2)^2\bigr)-m^*\bigl(-(n+1/2)^2\bigr) = \mathbf{O}\bigl(e^{-2na(1-\varepsilon)}\bigr),\quad n\to\infty \tag{3.1}$$

holds for all $\varepsilon>0$.

Proof. It is known from [18] that

$$m(-\tau^2) = -\tau - \int_0^\infty A(\alpha)e^{-2\tau\alpha}\,d\alpha, \quad \tau > \frac{1}{2}\int_0^\infty |Q|$$

with a function A such that $A - Q$ is continuous and $|A(\alpha) - Q(\alpha)| \leqslant c\exp(\alpha\|Q\|_1)$. The estimate (3.1) means that

$$\int_0^\infty [A(\alpha) - A^*(\alpha)]e^{-2(n+1)\alpha}\,d\alpha = \mathbf{O}(e^{-2na(1-\varepsilon)})$$

for sufficiently large n. Here $\int_0^\infty$ can be substituted by $\int_0^a$, thus by Proposition 2.4 it follows that $A = A^*$ a.e. on $[0,a]$ and then $Q = Q^*$ a.e. on $[0,a]$, see [18]. □

Proof of Theorem 1.2. From the estimate (2.4) it follows that

$$|\varphi_n(r)| \leqslant c\frac{r^{n+1}}{2^n\Gamma(n+3/2)} \leqslant c\frac{r^{n+1}}{2^n(\frac{n+1/2}{e})^{n+1/2}\sqrt{n}} \leqslant c\frac{r}{n}\left(\frac{er}{2n}\right)^n$$

for large n. Recall the variational formula

$$\dot{\delta_n} = -\int_0^\infty \dot{q}\varphi_n^2,$$

see [7]. Using the linear deformation $q(r,t) = tq(r) + (1-t)q^*(r)$ we get

$$\delta_n^* - \delta_n = \int_0^1\int_0^\infty (q^*(r) - q(r))\varphi_n^2(r,t)\,dr\,dt.$$

If $q^* = q$ on (a,∞), this implies

$$|\delta_n^* - \delta_n| \leqslant \frac{c}{n^2}\left(\frac{ea}{2n}\right)^{2n}\int_0^a r|q^*(r) - q(r)|\,dr$$

which proves the part (a) of Theorem 1.2. To verify part (b), fix a number $0 < a < b < \infty$ such that both q and q^* are supported in $[0,b]$. After the variable substitution $x = \log(b/r)$ the new functions $y_n(x) = r^{-1/2}\varphi_n(r)$, $0 < r \leqslant b$ satisfy $y_n \in L_2(0,\infty)$ and $-y_{n''} + Q(x)y_n = -(n+1/2)^2 y_n$ with the new potential $Q(x) = r^2(q(r) - 1)$, $Q \in L_1(0,\infty)$, see [6]. Clearly $q = q^*$ on (a,∞) if and only if $Q = Q^*$ on $(0,\log(b/a))$ if and only if the difference of their m-functions satisfy

$$m(-(n+1/2)^2) - m^*(-(n+1/2)^2) = \mathbf{O}\left(\left(\frac{a}{b}\right)^{2n(1-\varepsilon)}\right), \quad n \to \infty$$

for all $\varepsilon > 0$ by Proposition 3.1. Taking into account the formula

$$m(-(n+1/2)^2) = \frac{y_n'(0)}{y_n(0)} = \frac{1}{2} - b\frac{\varphi_n'(b)}{\varphi_n(b)} = -b\frac{J'_{n+1/2}(b) - \tan\delta_n Y'_{n+1/2}(b)}{J_{n+1/2}(b) - \tan\delta_n Y_{n+1/2}(b)}$$

from [6] or [4] we have to prove that

$$\frac{J'_{n+1/2}(b) - \tan\delta_n^* Y'_{n+1/2}(b)}{J_{n+1/2}(b) - \tan\delta_n^* Y_{n+1/2}(b)} - \frac{J'_{n+1/2}(b) - \tan\delta_n Y'_{n+1/2}(b)}{J_{n+1/2}(b) - \tan\delta_n Y_{n+1/2}(b)} = \mathbf{O}\left(\left(\frac{a}{b}\right)^{2n(1-\varepsilon)}\right), \quad n \to \infty. \tag{3.2}$$

The estimate

$$|\delta_n| \leqslant \frac{c}{n^2}\left(\frac{eb}{2n}\right)^{2n}$$

for large n can be verified as above in part (a). From the known asymptotics [1, 9.3.1]

Please cite this article in press as: M. Horváth, Partial identification of the potential from phase shifts, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.10.071

$$J_{n+1/2}(b) \approx \frac{1}{\sqrt{(2n+1)\pi}}\left(\frac{eb}{2n+1}\right)^{n+1/2},$$
$$Y_{n+1/2}(b) \approx -\sqrt{\frac{2}{(n+1/2)\pi}}\left(\frac{eb}{2n+1}\right)^{-n-1/2}$$

we infer that

$$\left|J_{n+1/2}(b) - \tan\delta_n Y_{n+1/2}(b)\right| \geqslant c\left|J_{n+1/2}(b)\right| \geqslant \frac{c}{n}\left(\frac{eb}{2n}\right)^n.$$

Consequently the derivative of the map

$$H : t \mapsto \frac{J'_{n+1/2}(b) - tY'_{n+1/2}(b)}{J_{n+1/2}(b) - tY_{n+1/2}(b)}$$

for t between $\tan\delta_n$ and $\tan\delta_n^*$ satisfies

$$H'(t) = \frac{J'_{n+1/2}(b)Y_{n+1/2}(b) - J_{n+1/2}(b)Y'_{n+1/2}(b)}{(J_{n+1/2}(b) - tY_{n+1/2}(b))^2} = \mathbf{O}\left(\frac{1}{J^2_{n+1/2}(b)}\right) = \mathbf{O}\left(n^2\left(\frac{2n}{eb}\right)^{2n}\right)$$

(we used the Wronskian [1, 9.1.16]). Thus, using (1.11) the left-hand side of (3.2) can be estimated by

$$\mathbf{O}\left((\tan\delta_n - \tan\delta_n^*)n^2\left(\frac{2n}{eb}\right)^{2n}\right) = \mathbf{O}\left(n^2\left(\frac{a_1}{b}\right)^{2n}\right) = \mathbf{O}\left(\left(\frac{a}{b}\right)^{2n(1-\varepsilon)}\right)$$

for all $\varepsilon > 0$ if a_1 is sufficiently close to a. So (3.2) is verified and then the proof of Theorem 1.2 is complete. □

Proof of Theorem 1.1. We know that

$$F(t) - F^*(t) = \sum_{n=0}^{\infty} a_n P_n(t) \quad \text{with } a_n = (2n+1)\frac{e^{2i\delta_n} - e^{2i\delta_n^*}}{2i}.$$

Now if $q = q^*$ a.e. on (a, ∞) then (1.10) implies

$$|a_n| \leqslant \frac{c}{n}\left(\frac{ea}{2n}\right)^{2n}$$

for large n hence for all $n \geqslant 1$ with another constant. Now Lemma 2.2 says that (1.8) is valid. Conversely if we have (1.9) then Lemma 2.2 implies

$$e^{2i\delta_n} - e^{2i\delta_n^*} = \mathbf{O}\left(\frac{ea_1}{2n}\right)^{2n} \quad \forall a_1 > a$$

and a similar estimate is valid for $\delta_n - \delta_n^*$. Thus by Theorem 1.2 $q = q^*$ a.e. on (a, ∞). □

References

[1] M. Abramowitz, I. Stegun (Eds.), Handbook of Mathematical Functions, Dover Publications, New York, 1972.
[2] S. Alessandrini, Stable determination of conductivity by boundary measurements, Appl. Anal. 27 (1988) 153–172.
[3] V. Alfaro, T. Regge, Potential Scattering, North-Holland, Amsterdam, 1965.
[4] B. Apagyi, M. Horváth, Solution of the inverse scattering problem at fixed energy for potentials being zero beyond a fixed radius, Modern Phys. Lett. B 22 (23) (2008) 2137–2149.
[5] K. Chadan, P.C. Sabatier, Inverse Problems in Quantum Scattering Theory, Springer, 1989.
[6] M. Horváth, Inverse scattering with fixed energy and an inverse eigenvalue problem on the half-line, Trans. Amer. Math. Soc. 358 (11) (2006) 5161–5177.
[7] M. Horváth, Notes on the distribution of phase shifts, Modern Phys. Lett. B 22 (23) (2008) 2163–2175.
[8] M. Horváth, M. Kiss, On the stability of inverse scattering with fixed energy, Inverse Problems 25 (2009) 015011.
[9] J. Loeffel, On an inverse problem in quantum scattering theory, Ann. Inst. H. Poincaré 8 (1968) 339–447.
[10] R.G. Newton, Scattering Theory of Waves and Particles, Dover, 2002.
[11] R.G. Novikov, The inverse scattering problem at fixed energy for the three-dimensional Schrödinger equation with an exponentially decreasing potential, Comm. Math. Phys. 161 (1994) 569–595.
[12] G. Szegő, Orthogonal Polynomials, vol. 23, Amer. Math. Soc. Coll. Publ., New York, 1939.
[13] A. Ramm, R. Airapetyan, A. Smirnova, Example of two different potentials which have practically the same fixed-energy phase shifts, Phys. Lett. A 254 (3–4) (1999) 141–148.
[14] A. Ramm, Recovery of the potential from fixed energy scattering data, Inverse Problems 4 (1988) 877–886.
[15] A. Ramm, Stability estimates in inverse scattering, Acta Appl. Math. 28 (1992) 1–42.

Please cite this article in press as: M. Horváth, Partial identification of the potential from phase shifts, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.10.071

[16] A.G. Ramm, Formula for the radius of the support of the potential in terms of scattering data, J. Phys. A 31 (1998) 39–44.
[17] A.G. Ramm, An inverse scattering problem with part of the fixed-energy phase shifts, Comm. Math. Phys. 207 (1999) 231–247.
[18] B. Simon, A new approach to inverse spectral theory I. Fundamental formalism, Ann. of Math. 150 (1999) 1–29.
[19] P. Stefanov, Stability of the inverse problem in potential scattering at fixed energy, Ann. Inst. Fourier (Grenoble) 40 (4) (1990) 867–884.
[20] E.T. Whittaker, G.N. Watson, Modern Analysis, Cambridge Univ. Press, Cambridge, 1950.

Please cite this article in press as: M. Horváth, Partial identification of the potential from phase shifts, J. Math. Anal. Appl. (2010), doi:10.1016/j.jmaa.2010.10.071