STABILITY OF DIRECT AND INVERSE EIGENVALUE PROBLEMS: THE CASE OF COMPLEX POTENTIALS

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Abstract. We consider the inverse eigenvalue problem for Schrödinger operators on finite intervals. The potential is allowed to be complex. In [5] we obtained estimates between the L^p-norm of the perturbation of the potential and the $l^{p'}$-norm of the perturbation of the eigenvalues. In this paper we extend these results for complex-valued potentials.

1. Introduction

Consider the eigenvalue problem

$$-y" + q(x)y = \lambda y \quad \text{ on } [0,\pi], \tag{1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(\pi)\cos\beta + y'(\pi)\sin\beta = 0. \tag{1.2}$$

The set of eigenvalues is denoted by $\sigma(\alpha, \beta)$ or $\sigma(\alpha, \beta, q)$. The inverse Sturm-Liouville problem aims to identify the operator from a set of eigenvalues. We are interested in the stability of this problem, that is, estimates of the perturbation of the potential q by the perturbation of the eigenvalues. By a recent result of Marletta and Weikard [7], if the first N eigenvalues from $\sigma(0,0) \cup \sigma(0,\pi/2)$ of q and q^* are closer than ε then

$$\sup_{0\le x\le\pi} \left|\int_0^\pi (q-q^*)\right| \le c(\varepsilon \log N + N^{-1/2}).$$

If the eigenvalues are taken from more than two spectra, the analogous result needs more sophisticated considerations; this is one of the main objectives of the present paper. Concerning uniqueness problems, see [4] and the references therein. In a recent paper [5] we discussed the stability of the operator reconstruction using eigenvalues from infinitely many spectra. The interested reader can find there a detailed

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collection of known results. We considered real-valued potentials and quite routine extensions for those complex potentials where all eigenvalues are simple. However for general complex potentials finitely many multiple eigenvalues may appear, and handling with multiplicities requires many new ideas and additional effort; this is the content of the present paper. The main conditions under which stability holds are twofold. First the trigonometric system (1.13) built from the unperturbed eigenvalues has to satisfy some basis properties, and secondly, the eigenvalue perturbations $\lambda_n - \lambda_n^*$ should tend to zero. More detalis are given below.

We shall use the following notations throughout: $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, q and q^* are two real valued potentials, $q, q^* \in L^1(0,\pi)$. Sometimes we assume that $\|q\|_1, \|q^*\|_1 \le D$; $c(D)$ always means a positive constant, possibly different in different occurrences, depending only on D. The eigenvalues corresponding to the potential q and q^* are denoted by λ_n and λ_n^*, respectively. This means that λ_n^* can be continuously shifted to λ_n by the linear deformation of the potentials, that is, there exists a continuous function $\lambda(s)$ with $\lambda(0) = \lambda_n^*$, $\lambda(1) = \lambda_n$ and $\lambda(s) \in \sigma(\alpha_n, 0; q_s)$. We choose a correspondence $\lambda_n^* \mapsto \lambda_n$ such that every value λ_n occurs in the sequence no more times than its multiplicity.

1.1. **General results.** As in [5], we present two general theorems, which will essentially imply the subsequent more specialized results.

In what follows L_0^p denotes the subset of $L^p = L_p(0,\pi)$ in which $\int_0^\pi b = 0$, l^p denotes the infinite sequences beginning from the 0^{th} index with the usual p-norm, while l_0^p consists of sequences whose 0^{th} coordinate is zero. We shall write $z = \sqrt{\lambda}$. If the potential q^* is complex, it can have multiple eigenvalues. If λ^* is an eigenvalue with (algebraic) multiplicity ν, let us define

$$\omega(\lambda, q) = \prod_{j=0}^{\nu} (\lambda - \lambda_j) \tag{1.3}$$

where λ_j's are the corresponding eigenvalues of q. We shall prove that $\omega(\lambda, q)$ is analytic in a neighbourhood of (λ^*, q^*). If an eigenvalue λ_n^* appears more than once, i.e., $\lambda_{n-1}^* \ne \lambda_n^* = \ldots = \lambda_{n+\eta}^* \ne \lambda_{n+\eta+1}^*$ $(\eta \le \nu)$, then for $j = 0, \ldots, \eta$, let $\omega_{n+j} = \partial_\lambda^j \omega(\lambda_n^*, q)$ and $\varphi_{n+j} = \frac{\mathrm{d}^j}{\mathrm{d}z^j} \cos 2xz|_{z=\sqrt{\lambda_n^*}}$. Then, for a q^* fixed, let $\Delta\Lambda(q)$ denote the sequence $\left(\int_0^\pi (q - q^*), (\omega_n)_{n\ge1}\right)$. If λ_n^* is simple, then $\omega_n = \lambda_n^* - \lambda_n$, hence if all the λ_n^* are different, $\Delta\Lambda(q)$ is simply the sequence

$\left(\int_0^\pi (q-q^*), (\lambda_n^* - \lambda_n)_{n\geq 1}\right)$ and $\varphi_n = \cos 2\sqrt{\lambda_n^*}x$. Consider the system

$$C(\Lambda) = \{\varphi_n : n \geq 0\}, \ \varphi_0 = 1, \tag{1.4}$$

and the mapping

$$T: \quad b \mapsto (\langle b, \varphi_n \rangle) \qquad (n \geq 0) \qquad b \in L^1[0,\pi]. \tag{1.5}$$

In contrast with the real-valued case, Theorem 1.1 below holds only in a local sense, and the constants depend also on the potential q^*. Theorem 1.2 has the same form as in [5].

Theorem 1.1. *Let $q^* \in L^1[0,\pi]$ and the eigenvalues $\lambda_n^* \in \sigma(\alpha_n, 0; q^*)$, $|\lambda_n^*| \geq c(D) > 0$, $\Re\lambda_n^* \to \infty$ are given. Then the (nonlinear) mapping*

$$q - q^* \mapsto \Delta\Lambda(q) \tag{1.6}$$

is continuous from L_0^r to l^s $(1 \leq r, s \leq \infty)$ at $q = q^*$ if and only if the (linear) mapping (1.5) restricted to L_0^r is $L^r \to l^s$ continuous, that is,

$$\left(\sum |\langle h, \varphi_n \rangle|^s\right)^{1/s} \leq C\|h\|_r, \quad h \in L_0^r \tag{1.7}$$

(with the usual modification for $s = \infty$).

Moreover, if the mapping (1.5) from $L_0^r \to l^s$ $(1 \leq r, s \leq \infty)$ is bounded by a constant C, then there is an $\varepsilon > 0$ such that if $\|q - q^*\|_1 < \varepsilon$ then

$$\left(\sum_n |\omega_n|^s\right)^{\frac{1}{s}} \leq c(D, q^*)C\|q - q^*\|_r, \tag{1.8}$$

provided that $\|q\|_1, \|q^*\|_1 \leq D$ and $\Re\lambda_n^* \geq -D$.

The general answer to the question of the stability of the inverse problem depends on whether the inverse of the mapping defined in (1.5) is bounded.

Theorem 1.2. *Let $q^* \in L^1[0,\pi]$ and the eigenvalues $\lambda_n^* \in \sigma(\alpha_n, 0; q^*)$, $\lambda_n^* \neq 0$, $\Re\lambda_n^* \to \infty$ are given. The following statements are equivalent:*
A *The (possibly multivalued) inverse of the mapping (1.6)*

$$\Delta\Lambda(q) \mapsto q - q^* \tag{1.9}$$

with domain $\{(c_n) \in l_0^s | \exists q \in L^1 : q - q^* \in L_0^r \text{ and } c_n = \omega_n \ \forall n > 0\}$ is a (nonlinear) $l^s \to L^r$ continuous mapping at $\Delta\Lambda(q^*) = 0$, that is, $\Delta\Lambda(q) \to 0$ in l^s implies $q - q^* \to 0$ in L^r.
B *The inverse of (1.5) with domain $\{(c_n) \in l_0^s | \exists h \in L_0^r : \ \forall n \ c_n = \langle h, \varphi_n \rangle\}$ is a (linear) $l^s \to L^r$ continuous mapping $(1 \leq r, s \leq \infty)$, i.e.*

$$\|h\|_r \leq C\left(\sum |\langle h, \varphi_n \rangle|^s\right)^{1/s}, \quad h \in L_0^r \tag{1.10}$$

with obvious modification for $s=\infty$. The sum on the right hand side is allowed to be infinite.

Moreover, in this case

$$(1.11)\quad \|q-q^*\|_r \le c(D)C\left(\sum_n |\omega_n|^s\right)^{\frac{1}{s}} \le c(D)C\left(\sum_n |\lambda_n-\lambda_n^*|^s\right)^{\frac{1}{s}},$$

with the constant C in (1.10) provided that $\|q\|_1$, $\|q^*\|_1 \le D$, $q-q^* \in L_0^r$ and $\Re\lambda_n^* \ge -D$. The sum in the upper bound in (1.11) can again be infinite.

1.2. **Specialisations.** Theorems 1.3–1.10 in [5] require almost the same set of assumptions in the complex case. For the convenience of the reader we collect them in the condition (C1):

(C1) $\|q\|_1, \|q^*\|_1 \le D$, $-D \le \Re\lambda_n^* \to \infty$, $\lambda_n^* \ne 0$ and $\lambda_n \in \sigma(\alpha_n, 0, q)$ are the eigenvalues corresponding to λ_n^* in the above defined sense and $\lim_{n\to\infty} |\lambda_n^* - \lambda_n| = 0$.

The statement of Theorem 1.3 will be the following: suppose (C1) and that $|\lambda_n^*| \ge c(D) > 0$. For $1 \le p \le 2$, there is an $\varepsilon > 0$ such that in $\|q-q^*\|_1 < \varepsilon$,

$$(1.12)\qquad \left(\sum_n |\omega_n|^{p'}\right)^{\frac{1}{p'}} \le c(D,q^*)M^{\frac{1}{p'}}\|q-q^*\|_p.$$

Theorems 1.4, 1.5, 1.6, 1.7, 1.8 and 1.9 in [5] remains valid for complex potentials under the Condition (C1). The proof is the same as in [5], but uses Theorems 1.1 and 1.2 of the present paper. To restate Theorem 1.10, we first redefine the system $e(\Lambda)$ for multiple eigenvalues.

Consider the system

$$(1.13)\qquad e(\Lambda) = \{1, e_n : 0 \ne n \in \mathbb{Z}\}.$$

where for $n>0$ $e_{\pm n} = e^{\pm 2iz_n^* x}$ if λ_n^* is a simple eigenvalue. If $\lambda_{n-1}^* \ne \lambda_n^* = \ldots = \lambda_{n+\nu}^* \ne \lambda_{n+\nu+1}^*$, let $e_{\pm(n+j)} = \frac{\mathrm{d}^j}{\mathrm{d}z^j} e^{\pm 2iz_n^* x}$. It is seen in Lemma 6.1 that if $e(\Lambda)$ is a frame or a Riesz basis in $L_2(-\pi,\pi)$, then the same is true for the system (1.4) in $L^2(0,\pi)$, with similar constants.

Theorem 1.3. *Assume (C1) and that the system (1.13) is a frame in $L_2(-\pi,\pi)$, with constants*

$$(1.14)\qquad m\|h\|^2 \le \sum |\langle h, \varphi_n\rangle|^2 \le M\|h\|^2 \quad h \in H.$$

Then

$$(1.15)\qquad c(D)m\|q-q^*\|_2^2 \le \sum |\omega_n|^2.$$

Moreover, if $|\lambda_n^*| \geq c(D) > 0$*, there is an* $\varepsilon > 0$ *such that in* $\|q - q^*\|_1 < \varepsilon$

$$\sum |\omega_n|^2 \leq c(D, q^*) M \|q - q^*\|_2^2. \tag{1.16}$$

1.3. **Finitely many known eigenvalues.** From a practical point of view, one can measure only finitely many eigenvalues, hence we need a theorem which gives an estimate tending to zero if an increasing number of eigenvalues are equal. The statements below generalize Theorems 1.13 and 1.14 of [5] to the case of multiple eigenvalues.

Theorem 1.4. *Suppose condition (C1). If the system (1.13) is a frame in* $L^2(-\pi, \pi)$ *and the* L^∞*-norm of the elements of the inverse frame is bounded by* C*, then*

$$\sup_{0 \leq x \leq \pi} \left| \int_0^x (q - q^*) \right| \leq C \sum_n \frac{c(D)}{\sqrt{|\lambda_n^*|}} |\omega_n|. \tag{1.17}$$

The previous theorem has an immediate consequence:

Theorem 1.5. *Assume (C1) and that* $\|q - q^*\|_2 \leq D$*. Suppose further that* $|\lambda_n - \lambda_n^*| < \varepsilon$ *if* $1 \leq n \leq N$*, for a given* $\varepsilon > 0$*, the system (1.13) is a frame in* $L^2(-\pi, \pi)$ *with frame operator* F *and the* L^∞*-norm of the elements of the biorthogonal system is bounded by* C*. Then*

$$\begin{aligned} \sup_{0 \leq x \leq \pi} \left| \int_0^x (q - q^*) \right| \leq & C c(D) \varepsilon \sum_{n=1}^N \frac{1}{\sqrt{|\lambda_n^*|}} + \\ & + C c(D) \left(C^{\frac{1}{2}} + \|F\|^{\frac{1}{2}} \right) \left(\sum_{n=N+1}^\infty \frac{1}{|\lambda_n^*|} \right)^{\frac{1}{2}}. \end{aligned} \tag{1.18}$$

If N *is large enough, or* λ_n^* *are simple for* $n > N$*, then the term* $C^{\frac{1}{2}}$ *can be omitted.*

If, for example, the first N Dirichlet eigenvalues and the first N Dirichlet-Neumann eigenvalues are given (i.e., there are $2N$ pairs of eigenvalues and $\sqrt{\lambda_n^*} = \frac{1}{2} n + \mathbf{o}(1)$), then $\|F\|$ and C depend only on D, and this estimate gives $c(D)(\varepsilon \log N + N^{-\frac{1}{2}})$, which is the main result of Marletta and Weikard [7]. Theorem 1.4 also contains the corresponding result in Marletta and Weikard [7]. To verify it we need that the system biorthogonal to (1.13) is uniformly bounded if the λ_n^* run over $\sigma(0,0) \cup \sigma(\pi/2, 0)$. We prove more:

Theorem 1.6. *Let* $0 \neq 4\lambda_n^* = n^2 + \mathbf{O}(1)$*,* $n \geq 1$ *be arbitrary real or complex numbers. Then the system (1.13) is a Riesz basis in* $L^2(-\pi, \pi)$ *and its biorthogonal system is uniformly bounded in* $L^\infty(-\pi, \pi)$*.*

2. Bounds for the eigenvalues and their multiplicity

As before, $c(D)$ denotes constants, depending only on D, possibly different in each occurrences. Let $\lambda \in \mathbb{C}$, $z = \sqrt{\lambda}$. Introduce the function $v(x, \lambda)$ as the solution of (1.1) with the initial conditions

$$v(\pi, \lambda) = 0, \quad v'(\pi, \lambda) = -1.$$

The lemmas in Section 3. of [5] works also for complex potentials, we shall cite them without replicating. However, for the sake of completeness, we repeat the following statements from [5]:

Lemma 2.1. [5] *Let $\|q\|_p$, $\|q^*\|_p \le D$ and $\lambda^* \in \sigma(0, \alpha; q^*)$ be an eigenvalue corresponding to $\lambda \in \sigma(0, \alpha; q)$ (that is, there is a continuous function $\lambda(s) \in \sigma(0, \alpha; q_s = sq^* + (1-s)q)$ with $\lambda(0) = \lambda$, $\lambda(1) = \lambda^*$). Then*

$$|\lambda - \lambda^*| \le c(D)$$

where $c(D)$ is independent of α, q, q^*, λ, λ^*.

Corollary 2.2. [5] *Let $\|q\|_p \le D$ and consider some eigenvalues $\lambda_n \in \sigma(\alpha_n, 0; q)$. If $\Re\lambda_n > -D$ then*

$$|\Im\sqrt{\lambda_n}| \le c(D).$$

Corollary 2.3. *If $\lambda^*_{n-1} \neq \lambda^*_n = \ldots = \lambda^*_{n+\nu} \neq \lambda^*_{n+\nu+1}$ then*

$$\left(\sum_{k=n}^{n+\nu} |\omega_k|^s\right)^{\frac{1}{s}} \le c(D) \left(\sum_{k=n}^{n+\nu} |\lambda_k - \lambda^*_k|^s\right)^{\frac{1}{s}}, \tag{2.1}$$

provided that $\|q\|_1$, $\|q^*\|_1 \le D$, $q - q^* \in L^r_0$ and $\Re\lambda^*_n \ge -D$.

Proof. If we denote the vector of $|\omega_k|$'s by $\mathbf{w}$ and the increasingly ordered vector of $|\lambda_k - \lambda^*_k|$'s by $\mathbf{l}$, obviously $\mathbf{w} \le c(D, \eta)\mathbf{l}$, where η is the multiplicity of λ^*_n. According to the next lemma, η also depends only on D. □

Lemma 2.4. [5] *If $\|q\|_p \le D$, $\lambda \in \sigma(\alpha, 0; q)$, $|\Im\lambda| \le D$ and $|\lambda| \ge c(D)$ with a sufficiently large constant independent of q, α then λ is a simple eigenvalue. The multiplicity of eigenvalues $|\lambda| \le c(D)$ is bounded by $c_1(D)$.*

Corollary 2.5. *Let $\lambda^* \in \sigma(\alpha, 0, q^*)$ be a possibly multiple eigenvalue and let $\lambda_0, \ldots, \lambda_n$ be the corresponding element(s) of $\sigma(\alpha, 0, q)$. If $\lambda^* \ge -D$, $||q||_1, ||q^*||_1 \le D$ then $\varepsilon(D) \le \frac{1+|\lambda_j|}{1+|\lambda^*|} \le c(D)$.*

Proof. Using Lemma 2.1,

$$\frac{1+|\lambda_j|}{1+|\lambda^*|} \le 1 + \frac{|\lambda_j - \lambda^*|}{1+|\lambda^*|} \le c(D), \tag{2.2}$$

and similarly $\frac{1+|\lambda^*|}{1+|\lambda_j|}$ is also bounded. □

3. Derivative with respect to the potential

Let y_1 and y_2 be two solutions of (1.1) such that $\begin{pmatrix} y_1(0,\lambda) \\ y_1'(0,\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} y_2(0,\lambda) \\ y_2'(0,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Statement 3.1. [6, 8, 10] *For* $q \in L^1(0,\pi)$

(3.1)
$$\frac{\partial^{k+1} y_j}{\partial\lambda^k \partial q}(x) = \frac{\partial^k}{\partial\lambda^k}\big(y_j(t)[y_1(t)y_2(x) - y_2(t)y_1(x)]\chi_{(0,x)}(t)\big) \qquad j = 1,2,$$

(3.2)
$$\frac{\partial^{k+1} y_j'}{\partial\lambda^k \partial q}(x) = \frac{\partial^k}{\partial\lambda^k}\big(y_j(t)[y_1(t)y_2'(x) - y_2(t)y_1'(x)]\chi_{(0,x)}(t)\big) \qquad j = 1,2.$$

Lemma 3.2. *Let* $\Re\lambda = \Re z^2 \ge -D$, $\|q\|_1 \le D$. *Then*

$$|\partial_2^k y_1(x,\lambda)| \le \frac{c(D,k)}{1+|z|^k}, \tag{3.3}$$

$$|\partial_2^k y_2(x,\lambda)| \le \frac{c(D,k)}{1+|z|^{k+1}}. \tag{3.4}$$

Proof. By a well-known representation

$$y_1(x,\lambda) = \cos xz + \int_0^x K_1(x,t)\cos tz \, \mathrm{d}t, \tag{3.5}$$

$$y_2(x,\lambda) = \frac{\sin xz}{z} + \int_0^x K_2(x,t)\frac{\sin tz}{z} \, \mathrm{d}t, \tag{3.6}$$

where the kernel K_1, K_2 are continuous and $|K_i(x,t)| \le c(D)$, see [4], Lemma 5.1. Thus $|\partial_\lambda^k \cos xz| \le \frac{c(D,k)}{1+|z|^k}$ and $\partial_\lambda^k \frac{\sin xz}{z} \le \frac{c(D,k)}{1+|z|^{k+1}}$ gives the statement. □

Using this lemma, an elementary estimate of the supremum of the derivative yields that

Corollary 3.3.

$$|\partial_2^k y_2(x,\lambda,q) - \partial_2^k y_2(x,\lambda,q^*)| \le \frac{c(D,k)}{1+|z|^{k+2}}\|q-q^*\|_1. \tag{3.7}$$

Lemma 3.4. *Let y be an arbitrary solution of (1.1) with fixed initial conditions at $x = 0$ or at $x = \pi$. Then*

$$[W(\partial_\lambda^{n+1} y, y)]_0^\pi = (n+1) \int_0^\pi \partial_\lambda^n y(x) y(x)\ \mathrm{d}x, \tag{3.8}$$

where $W(f, g) = fg' - f'g$ is the Wronskian in the variable x.

Proof. Differentiating (1.1) k times, we get

$$-\partial_\lambda^k y'' + q\partial_\lambda^k y = \lambda \partial_\lambda^k y + k\partial_\lambda^{k-1} y.$$

Multiplying by $\partial_\lambda^l y$,

$$-\partial_\lambda^k y'' \partial_\lambda^l y + q\partial_\lambda^k y \partial_\lambda^l y = \lambda \partial_\lambda^k y \partial_\lambda^l y + k\partial_\lambda^{k-1} y \partial_\lambda^l y.$$

Similarly,

$$-\partial_\lambda^l y'' \partial_\lambda^k y + q\partial_\lambda^l y \partial_\lambda^k y = \lambda \partial_\lambda^l y \partial_\lambda^k y + l\partial_\lambda^{l-1} y \partial_\lambda^k y.$$

Subtracting,

$$\partial_\lambda^l y'' \partial_\lambda^k y - \partial_\lambda^k y'' \partial_\lambda^l y = k\partial_\lambda^{k-1} \partial_\lambda^l y - l\partial_\lambda^{l-1} y \partial_\lambda^k y.$$

Integrating from 0 to π:

$$[W(\partial_\lambda^k y, \partial_\lambda^l y)]_0^\pi = \int_0^\pi k\partial_\lambda^{k-1} y \partial_\lambda^l y - l\partial_\lambda^{l-1} y \partial_\lambda^k y. \tag{3.9}$$

The statement is the special case $k = n + 1$, $l = 0$. □

Lemma 3.5. *Let $q, q^* \in L_1(0, \pi)$, $\lambda^* \in \sigma(\alpha, \beta, q^*)$ be an eigenvalue of multiplicity $(n+1)$. If $\lambda_0, \ldots, \lambda_n \in \sigma(\alpha, \beta, q)$ denotes the corresponding eigenvalues of q, let us define*

$$\Omega(\lambda, q) = W(y(\pi, \lambda, q), y(\pi, \lambda^*, q^*)), \tag{3.10}$$

a shorthand notation for $W(y(x, \lambda, q), y(x, \lambda^*, q^*))|_{x=\pi}$, where $y(x, \lambda, q) = \sin \alpha y_1(x, \lambda, q) - \cos \alpha y_2(x, \lambda, q)$. Then

$$\partial_\lambda^{n+1} \Omega(\lambda^*, q^*) = (n+1) \int_0^\pi \partial_\lambda^n y(x) y(x)\ \mathrm{d}x, \tag{3.11}$$

$$\frac{\partial^{k+1}}{\partial \lambda^k \partial q} \Omega(\lambda^*, q^*) = -\partial_\lambda^k \big(y^2(t, \lambda^*, q^*)\big). \tag{3.12}$$

Proof. The first equation is implied by the previous lemma. The second equation follows from

$$\frac{\partial}{\partial q} \Omega(\lambda, q) = y(t, \lambda, q) \cdot [y_1(t, \lambda, q) W(y_2(\pi, \lambda, q), y(\pi, \lambda^*, q^*)) - \\ - y_2(t, \lambda, q^*) W(y_1(\pi, \lambda, q^*), y(\pi, \lambda, q^*))],$$

so

$$\frac{\partial}{\partial q}\Omega(\lambda^*, q^*) = -y^2(t, \lambda, q^*).$$

□

Lemma 3.6. *Let $q, q^* \in L_1(0,\pi)$, $\lambda^* \in \sigma(\alpha,\beta,q^*)$ is an eigenvalue of multiplicity $(n+1)$. If $\lambda_0,\dots,\lambda_n \in \sigma(\alpha,\beta,q)$ denote the corresponding eigenvalues of q, let us define $\omega(\lambda,q) = \prod_{j=0}^n(\lambda-\lambda_j)$. Then $\partial_\lambda^k\omega(\lambda^*,q)$ is an analytic function of q in a neighbourhood of q^*, and*

$$\|\frac{\partial^{k+1}\omega(\lambda^*,q^*)}{\partial\lambda^k\partial q}\|_\infty \le c(D,Q), \quad k=0,\dots,n \tag{3.13}$$

with $Q = \int_0^\pi \frac{\partial_\lambda^n y(x,\lambda^*,q^*)}{n!} y(x,\lambda^*,q^*)\ \mathrm{d}x$.

Proof. $\Omega(\lambda,q)$ is an analytic function of both of its variables. According to the Weierstrass Preparation Theorem,

$$\Omega(\lambda,q) = \tilde\omega(\lambda,q)H(\lambda,q),$$

where $\tilde\omega(\lambda,q) = (\lambda-\lambda^*)^{n+1} + \sum_{k=0}^n a_k(q)(\lambda-\lambda^*)^k$ is a polynomial of λ, $a_k(q^*) = 0$, if $k \le n$, and $H \ne 0$ in a neighbourhood of (λ^*, q^*). Theorem 2.4 of [3] gives explicit formulas for this neighbourhood by the terms of $a_k(q^*)$. As the eigenvalues depend continuously on the potential, we can choose another neighbourhood of U of q^* such that for a $q \in U$ the zeros of $\omega(\lambda,q)$ are inside the set $H \ne 0$. Then comparing the zeros and the principal coefficients for a $q \in U$ fixed, we can see that $\tilde\omega = \omega$.

From $j!a_j(q^*) = \partial_\lambda^j\omega(\lambda^*,q^*) = 0$ $(j \ne n+1)$ it follows that

$$\partial_\lambda^{n+1+k}\Omega(\lambda^*,q^*) = \frac{(n+1+k)!}{k!}\partial_\lambda^k H(\lambda^*,q^*). \tag{3.14}$$

$$\begin{aligned}\partial_\lambda^k\partial_q\Omega(\lambda^*,q^*) &= \sum_{j=0}^k \binom{k}{j} j!a_j'(q^*)\partial_\lambda^{k-j}H(\lambda^*,q^*)\\ &= k!\sum_{j=0}^k \frac{\partial_\lambda^{(n+1+k-j)}\Omega(\lambda^*,q^*)}{(n+1+k-j)!}a_j'(q^*).\end{aligned} \tag{3.15}$$

Hence for $k=0$

$$\partial_q\omega(\lambda^*,q^*) = a_0'(q^*) = \frac{\partial_q\Omega(\lambda^*,q^*)}{\partial_\lambda^{n+1}\Omega(\lambda^*,q^*)}(n+1)! = -\frac{y^2(t,\lambda^*,q^*)}{Q}, \tag{3.16}$$

and the statement follows by Lemma 3.2. We continue by induction. Assume that the statement is true for all $j < k$, then, from (3.15),

$$Qa_k'(q^*) = \partial_\lambda^k \partial_q \Omega(\lambda^*, q^*) - k! \sum_{j=0}^{k-1} \frac{\partial_\lambda^{(n+1+k-j)} \Omega(\lambda^*, q^*)}{(n+1+k-j)!} a_j'(q^*). \tag{3.17}$$

By (3.12), the first summand is at most $c(D)$. By (3.11) the coefficients of $a_j'(q^*)$ can be estimated by $c(D)$, thus the right-hand side is at most $c(D, Q)$, using the assumption of induction. Hence the statement is true for all $k \le n$. □

The statement of the lemma implies $|\frac{\partial^k \omega(\lambda^*, q)}{\partial \lambda^k}| < c(D, Q, q^*)||q - q^*||_1$, if q is in a neighbourhood of q^*. By Lemma 2.1, $|\frac{\partial^k \omega(\lambda^*, q)}{\partial \lambda^k}| < c(D)$, hence

Theorem 3.7. *Let $q, q^* \in L_1(0, \pi)$, $\|q\|_1 \le D$, $\|q^*\|_1 \le D$, $\lambda^* \in \sigma(\alpha, \beta, q^*)$ is an eigenvalue of multiplicity $(n + 1)$. If $\lambda_0, \dots, \lambda_n \in \sigma(\alpha, \beta, q)$ denotes the corresponding eigenvalues of q, then there is a constant $c(D, Q, q^*)$ such that*

$$|\frac{\partial^k \omega(\lambda^*, q)}{\partial \lambda^k}| < c(D, Q, q^*)||q - q^*||_1, \quad k = 0, \dots, n. \tag{3.18}$$

Lemma 3.8. *Let $p(x, \cdot)$ be a polynomial of degree at most ν, interpolating $v(x, \cdot)$ in the points λ_0, λ_1, ..., λ_ν. Then for $l \le \nu$,*

(3.19)

$$\partial_2^l \big(v(x, \lambda^*) - p(x, \lambda^*)\big) = \sum_{j=0}^{l} \binom{l}{j} v(x, \cdot)[\lambda_0, \dots, \lambda_\nu, \lambda^*, \dots, \lambda^*] j! \omega^{(l-j)}(\lambda^*),$$

where $\omega(\lambda)$ denotes $\prod_{j=0}^{\nu} (\lambda - \lambda_j)$ and λ^* occurs $j + 1$ times in the divided differences.

Proof. Let $P(x, \cdot)$ be the interpolation of $v(x, \cdot)$ on the base points $\lambda_0, \dots, \lambda_\nu, \lambda^*, \dots, \lambda^*$, where the multiplicity of λ^* is $\nu + 1$. From the Newton interpolation scheme we get

(3.20)

$$P(x, \lambda) = p(x, \lambda) + \sum_{j=0}^{\nu} v(x, \cdot)[\lambda_0, \dots, \lambda_\nu, \lambda^*, \dots, \lambda^*](\lambda - \lambda^*)^j \omega(\lambda),$$

where the point λ^* occurs $j+1$ times in the jth divided difference. The statement follows by differentiation. □

Lemma 3.9. *If $|\lambda_j - \lambda^*| \le c(D)$,*

$$v(x, \cdot)[\lambda_0, \dots, \lambda_\nu, \lambda^*, \dots, \lambda^*] \le \frac{c(D)}{1 + |z^*|} \tag{3.21}$$

Proof. The divided difference can be represented by

(3.22)
$$v(x,\cdot)[\lambda_0,\dots,\lambda_\nu,\lambda^*,\dots,\lambda^*] = \frac{1}{2\pi i}\int_\gamma \frac{v(x,\lambda)}{(\lambda-\lambda^*)^j \prod_{j=0}^{\nu}(\lambda-\lambda_j)}\,\mathrm{d}\lambda,$$

where the path of integration is the boundary of a domain containing the base points. Let us choose a circle around λ^* with radius $c(D)$ such that $|\lambda-\lambda_j|\geq 1$, if λ is on the boundary, then the statement follows. ☐

Lemma 3.10. *If $|\lambda_j-\lambda^*|\leq c(D)$ and $l\leq\nu$,*

$$\big|\partial_2^l\big(v(x,\lambda^*)-p(x,\lambda^*)\big)\big| \leq \frac{c(D)}{1+|z^*|}\max_{j\leq l}|\partial_\lambda^j\omega(\lambda^*)|. \tag{3.23}$$

Proof. All the divided differences can be estimated by $\frac{c(D)}{1+|z^*|}$. Hence the statement follows from Lemma 3.8. ☐

4. The main tool

Lemma 4.1. *Let $\lambda\in\sigma(\alpha,0,q)$ and $\lambda^*\in\sigma(\alpha,0,q^*)$ are possible multiple eigenvalues with multiplicity $m+1$ and $n+1$, respectively. Then, for $k\leq m$ and $l\leq n$,*

$$\int_0^\pi (q(x)-q^*(x))\partial_2^k v(x,\lambda)\partial_2^l v^*(x,\lambda^*)\,\mathrm{d}x =$$
$$= \int_0^\pi \big[(\lambda-\lambda^*)\partial_2^k v(x,\lambda)\partial_2^l v^*(x,\lambda^*) +$$
$$+ k\partial_2^{k-1}v(x,\lambda)\partial_2^l v^*(x,\lambda^*) - l\partial_2^k v(x,\lambda)\partial_2^{l-1}v^*(x,\lambda^*)\big]\,\mathrm{d}x.$$

Proof.
$$\begin{aligned}
0 &= \big[W(\partial_2^k v(x,\lambda),\partial_2^l v^*(x,\lambda^*))\big]_0^\pi\\
&= \int_0^\pi \big[\partial_2^k v(x,\lambda)\partial_2^l v^{*\prime\prime}(x,\lambda^*) - \partial_2^k v''(x,\lambda)\partial_2^l v^*(x,\lambda^*)\big]\,\mathrm{d}x =\\
&= \int_0^\pi \big[\partial_2^k v(x,\lambda)\partial_2^l v^*(x,\lambda^*)(q^*(x)-\lambda^*-q(x)+\lambda) +\\
&+ k\partial_2^{k-1}v(x,\lambda)\partial_2^l v^*(x,\lambda^*) - l\partial_2^k v(x,\lambda)\partial_2^{l-1}v^*(x,\lambda^*)\big]\,\mathrm{d}x.
\end{aligned}$$

☐

Lemma 4.2. *Assume that the multiplicity of λ^* in $\sigma(\alpha,0,q^*)$ is ν. Let $p(x,\cdot)$ be a polynomial of degree at most ν, interpolating $v(x,\cdot)$ in the* *eigenvalues* λ_0*,* λ_1*, …,* λ_ν *corresponding to* λ^**. Then for* $l \le \nu$*,*

$$\int_0^\pi (q(x) - q^*(x))p(x,\lambda)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x =$$

$$= -\omega(\lambda, q)\int_0^\pi v(x,\cdot)[\lambda_0,\dots,\lambda_\nu]\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x +$$

$$+(\lambda-\lambda^*)\int_0^\pi p(x,\lambda)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x - l\int_0^\pi p(x,\lambda)\partial_2^{l-1} v^*(x,\lambda^*)\ \mathrm{d}x,$$

where $\omega(\lambda, q)$ *is defined in (1.3), and* $v(x,\cdot)[\lambda_0,\dots,\lambda_\nu]$ *denotes divided differences.*

Proof. The divided differences are continuous functions of the base points, thus also is the interpolating polynomial, as it is seen from the Newton interpolation. Hence we can assume that the points λ_0, λ_1, …, λ_ν are all different. Then, according to the Lagrange method, $p(x,\lambda) = \sum_{j=0}^{\nu} p_j(\lambda)v(x,\lambda_j)$, where the p_j's are fixed polynomials. Hence

$$\int_0^\pi (q(x) - q^*(x))p(x,\lambda)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x =$$

$$= \sum_{j=0}^{\nu} p_j(\lambda)\int_0^\pi (q(x) - q^*(x))v(x,\lambda_j)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x =$$

$$= \sum_{j=0}^{\nu} p_j(\lambda)\int_0^\pi (\lambda_j - \lambda^*)v(x,\lambda_j)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x -$$

$$-l\sum_{j=0}^{\nu} p_j(\lambda)\int_0^\pi v(x,\lambda_j)\partial_2^{l-1} v^*(x,\lambda^*)\ \mathrm{d}x =$$

$$= \int_0^\pi \sum_{j=0}^{\nu} p_j(\lambda)(\lambda_j - \lambda)v(x,\lambda_j)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x +$$

$$+(\lambda-\lambda^*)\int_0^\pi p(x,\lambda)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x - l\int_0^\pi p(x,\lambda)\partial_2^{l-1} v^*(x,\lambda^*)\ \mathrm{d}x.$$

Here the first term is a polynomial of degree (at most) $\nu + 1$. It is zero at λ_j ($j = 0,\dots,\nu$), and its leading coefficient equals to that of $-p(x,\lambda)$. Thus it coincides with the first term in the right-hand side of the statement. □

Corollary 4.3. *Assume that the multiplicity of* λ^* *in* $\sigma(\alpha, 0, q^*)$ *is* ν*. Let* $p(x,\cdot)$ *be a polynomial of degree at most* ν*, interpolating* $v(x,\cdot)$ *in* *the eigenvalues* λ_0, λ_1, …, λ_ν *corresponding to* λ^*. *Then for* $k, l \leq \nu$,

$$\int_0^\pi (q(x) - q^*(x))\partial_2^k p(x,\lambda^*)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x =$$

$$= -\partial_\lambda^k \omega(\lambda^*, q) \int_0^\pi v(x,\cdot)[\lambda_0,\dots,\lambda_\nu]\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x +$$

$$+k \int_0^\pi \partial_2^{k-1} p(x,\lambda^*)\partial_2^l v^*(x,\lambda^*)\ \mathrm{d}x - l \int_0^\pi \partial_2^k p(x,\lambda^*)\partial_2^{l-1} v^*(x,\lambda^*)\ \mathrm{d}x.$$

Lemma 4.4. *Assume that the multiplicity of* λ^* *in* $\sigma(\alpha, 0, q^*)$ *is* ν. *Let* $p(x,\cdot)$ *be a polynomial of degree at most* ν, *interpolating* $v(x,\cdot)$ *in the eigenvalues* λ_0, λ_1, …, λ_ν *corresponding to* λ^*. *Then for* $m \leq \nu$,

$$\int_0^\pi (q(x) - q^*(x))\partial_\lambda^m [p(x,\lambda^*)v^*(x,\lambda^*)]\ \mathrm{d}x =$$

$$= -\int_0^\pi v(x,\cdot)[\lambda_0,\dots,\lambda_\nu]\partial_\lambda^m[\omega(\lambda^*,q)v^*(x,\lambda^*)]\ \mathrm{d}x.$$

Proof.

$$\int_0^\pi (q(x) - q^*(x))\partial_\lambda^m [p(x,\lambda^*)v^*(x,\lambda^*)]\ \mathrm{d}x =$$

$$\sum_{k=0}^m \binom{m}{k} \int_0^\pi (q(x) - q^*(x))\partial_\lambda^k p(x,\lambda^*)\partial_\lambda^{m-k} v^*(x,\lambda^*)\ \mathrm{d}x =$$

$$= -\sum_{k=0}^m \binom{m}{k} \partial_\lambda^k \omega(\lambda^*, q) \int_0^\pi v(x,\cdot)[\lambda_0,\dots,\lambda_\nu]\partial_\lambda^{m-k} v^*(x,\lambda^*)\ \mathrm{d}x +$$

$$+ \sum_{k=1}^m \binom{m}{k} k \int_0^\pi \partial_\lambda^{k-1} p(x,\lambda^*)\partial_\lambda^{m-k} v^*(x,\lambda^*)\ \mathrm{d}x -$$

$$- \sum_{k=0}^{m-1} \binom{m}{k} (m-k) \int_0^\pi \partial_\lambda^k p(x,\lambda^*)\partial_2^{m-k-1} v^*(x,\lambda^*)\ \mathrm{d}x.$$

The last two term both equals to $m \int_0^\pi \partial_\lambda^{m-1}[p(x,\lambda^*)v^*(x,\lambda^*)]\ \mathrm{d}x$, thus cancel each other, while the first term is exactly what we have stated. □

Lemma 4.5. *Let* $\|q\|_1, \|q^*\|_1 \leq D$, *and let the multiplicity of* λ^* *in* $\sigma(\alpha, 0, q^*)$ *is* ν. *If* $\Re\lambda_n^* \geq -D$, *then for* $m \leq \nu$,

(4.1)
$$|\int_0^\pi (q(x) - q^*(x))\partial_2^m [v(x,\lambda^*)v^*(x,\lambda^*)]\ \mathrm{d}x| \leq \frac{c(D)}{1 + |z^*|^2} \max_{0 \leq j \leq m} |\partial_\lambda^j \omega(\lambda^*, q)|.$$

Proof. As before, we denote by $p(x,\cdot)$ the polynomial of degree at most ν, interpolating $v(x,\cdot)$ in the eigenvalues λ_0, λ_1, …, λ_ν corresponding to λ^*.

$$\int_0^\pi (q(x)-q^*(x))\partial_\lambda^m[v(x,\lambda^*)v^*(x,\lambda^*)]\ \mathrm{d}x =$$
$$\int_0^\pi (q(x)-q^*(x))\partial_\lambda^m[(v(x,\lambda^*)-p(x,\lambda^*))v^*(x,\lambda^*)]\ \mathrm{d}x -$$
$$-\int_0^\pi v(x,\cdot)[\lambda_0,\dots,\lambda_\nu]\partial_\lambda^m[\omega(\lambda^*)v^*(x,\lambda^*)]\ \mathrm{d}x = I_1 - I_2.$$

By Lemma 3.2 and Lemma 3.10

$$|I_1| \le \frac{c(D)}{1+|z^*|^2}\max_{0\le j\le m}|\partial_\lambda^j\omega(\lambda^*,q)|\|q(x)-q^*(x)\|_1 \le \frac{c(D)}{1+|z^*|^2}\max_{0\le j\le m}|\partial_\lambda^j\omega(\lambda^*,q)|.$$

By Lemma 3.2 and Lemma 3.9

$$|I_2| \le \frac{c(D)}{1+|z_n^*|^2}\max_{0\le j\le m}|\partial_\lambda^j\omega(\lambda^*,q)|, \tag{4.2}$$

which implies the formulated estimate. ☐

Lemma 4.6. *(Lemma 5.2 of Horváth* [4]*) Let* $\|q\|_1, \|q^*\|_1 \le D$. *Then there exists a continuous kernel function* M_1 *such that*

$$1 - 2z^2 v(\pi-x,\lambda)v^*(\pi-x,\lambda) = = \cos 2xz + \int_0^x \cos 2tz M_1(x,t,q,q^*)\ \mathrm{d}t, \tag{4.3}$$

and

$$|M_1(x,t,q,q^*)| \le c(D), \tag{4.4}$$
$$|M_1(x,t,q_1,q^*) - M_1(x,t,q_2,q^*)| \le c(D)\|q_1-q_2\|_1. \tag{4.5}$$

Corollary 4.7. [4] *Let* $h \in L^p(0,\pi)$. *Then*

$$\int_0^\pi h - 2z^2\int_0^\pi h(x)v(x,\lambda)v^*(x,\lambda)\ \mathrm{d}x = \int_0^\pi A_q\left(h(\pi-x)\right)\cos 2xz\ \mathrm{d}x, \tag{4.6}$$

where

$$(A_q h)(x) = h(x) + \int_x^\pi M(x,t)h(t)\,dt \qquad h \in L_1(0,\pi). \tag{4.7}$$

with $M(x,t) = M_1(t,x,q,q^*)$.

Proof. Multiplying (4.3) by $h(\pi - x)$, integrating from 0 to π and changing the order of integrations gives the formula (4.6). □

Lemmas 4.6-4.9 in [5] are also true for the operators A_q defined in (4.7).

Corollary 4.8. *Let $\|q\|_p, \|q^*\|_p \le D$, $\Re\lambda^* \ge -D$, and $\lambda_0, \dots, \lambda_\nu$ are the corresponding elements of $\sigma(\alpha, 0, q)$. Assume further that $m \le \nu$, $\int_0^\pi (q - q^*) = 0$. Then*

$$\max_{0 \le j \le m} \left|\partial_z^j \int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x)\right) \cos 2xz^* \, \mathrm{d}x\right| \le c(D) \max_{0 \le j \le m} |\partial_\lambda^j \omega(\lambda^*, q)|.$$

Proof. The formulated estimate follows from (4.6) and the previous lemma with derivatives with respect to λ instead of z. As for multiple eigenvalues $|z^*| \le c(D)$ (Lemma 2.4), we can change the derivatives on the left-hand side. □

Lemma 4.9. *Let $\|q\|_1, \|q^*\|_1 \le D$, $\int_0^\pi (q - q^*) = 0$. Then there are further constants depending on D such that if $|z^*| \ge c(D)$,*

$$|\lambda - \lambda^*| \le c(D) \left| \int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x)\right) \cos 2z^* x \, \mathrm{d}x\right|. \tag{4.8}$$

Proof. By Lemma 2.4, if $|z^*| \ge c(D)$, λ^* is a simple eigenvalue. Then according to (6.2) in [5], if $|z^*| \ge c(D)$,

$$|\lambda - \lambda^*| \le c(D)(1 + |\lambda^*|) \left| \int_0^\pi (q(x) - q^*(x)) v(x, \lambda^*) v^*(x, \lambda^*) \, \mathrm{d}x\right|, \tag{4.9}$$

which, through (4.6), implies the statement. □

Lemma 4.10. *Let $\|q\|_1, \|q^*\|_1 \le D$. Assume further that $\Re\lambda^* \ge -D$, $\lambda_0, \dots, \lambda_\nu$ are the corresponding elements of $\sigma(\alpha, 0, q)$, and $m \le \nu$. In this lemma we locally redefine Q in terms of v, now let $Q = \int_0^\pi \frac{\partial_\lambda^\nu v^*(x,\lambda^*)}{\nu!} v^*(x, \lambda^*) \, \mathrm{d}x$. Then there are further constants depending on D and q^* such that if $\|q - q^*\|_1 \le c(D, Q, q^*)$,*

$$\lambda^{*m+1} \partial_\lambda^m \omega(\lambda^*, q) = \sum_{0 \le j \le m} O_Q(1) \partial_z^j \int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x)\right) \cos 2xz^* \, \mathrm{d}x \tag{4.10}$$

where $O_Q(1)$ denotes different quantities with absolute value at most $c(D, Q)$.

Proof. We have $Q \neq 0$, otherwise the multiplicity of the eigenvalue λ^* would be greater than ν.

$$\int_0^\pi (q(x) - q^*(x))\partial_\lambda^m [v(x,\lambda^*)v^*(x,\lambda^*)]\ \mathrm{d}x =$$
$$\int_0^\pi (q(x) - q^*(x))\partial_\lambda^m [(v(x,\lambda^*) - p(x,\lambda^*))v^*(x,\lambda^*)]\ \mathrm{d}x -$$
$$-\int_0^\pi \big(v(x,\cdot)[\lambda_n,\ldots,\lambda_{n+\nu}] - \frac{\partial_\lambda^\nu v(x,\lambda^*)}{\nu!}\big)\partial_\lambda^m [\omega(\lambda^*,q)v^*(x,\lambda^*)]\ \mathrm{d}x -$$
$$-\int_0^\pi \big(\frac{\partial_\lambda^\nu v(x,\lambda^*)}{\nu!} - \frac{\partial_\lambda^\nu v^*(x,\lambda^*)}{\nu!}\big)\partial_\lambda^m [\omega(\lambda^*,q)v^*(x,\lambda^*)]\ \mathrm{d}x -$$
$$-\int_0^\pi \frac{\partial_\lambda^\nu v^*(x,\lambda^*)}{\nu!}\partial_\lambda^m [\omega(\lambda^*,q)v^*(x,\lambda^*)]\ \mathrm{d}x = I_1^m - I_2^m - I_3^m - I_4^m.$$

By Lemma 3.2 and Lemma 3.10

$$|I_1^m| \leq c(D)\|q(x) - q^*(x)\|_1 \max_{0\leq j\leq m} |\partial_\lambda^j \omega(\lambda^*,q)|. \tag{4.11}$$

Since $v(x,\cdot)[\lambda_n,\ldots,\lambda_{n+\nu}] = \frac{\partial_\lambda^\nu p}{\nu!}$, we get similarly

$$|I_2^m| \leq c(D) \max_{0\leq j\leq \nu} |\partial_\lambda^j \omega(\lambda^*,q)| \max_{0\leq j\leq m} |\partial_\lambda^j \omega(\lambda^*,q)|. \tag{4.12}$$

By Lemma 3.2 and Corollary 3.3

$$|I_3^m| \leq c(D)\|q(x) - q^*(x)\|_1 \max_{0\leq j\leq m} |\partial_\lambda^j \omega(\lambda^*,q)|. \tag{4.13}$$

Finally,

$$|I_4^m - Q\partial_\lambda^m \omega(\lambda^*,q)| = \sum_{j=0}^{m-1} O(1)\partial_\lambda^j \omega(\lambda^*,q), \tag{4.14}$$

where $O(\alpha)$ denotes different quantities with absolute value at most $c(D)|\alpha|$. Hence if $q \to q^*$,

$$\int_0^\pi (q(x) - q^*(x))\partial_\lambda^m [v(x,\lambda^*)v^*(x,\lambda^*)]\ \mathrm{d}x =$$
$$= (-Q + o(1))\partial_\lambda^m \omega(\lambda^*,q) + \sum_{j=0}^{m-1} O(1)\partial_\lambda^j \omega(\lambda^*,q),$$

i.e.,

$$\partial_\lambda^m \int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2xz^*\ \mathrm{d}x =$$
$$= 2\lambda^*(Q+o(1))\partial_\lambda^m\omega(\lambda^*,q)+\sum_{j=0}^{m-1}O(1)\partial_\lambda^j\omega(\lambda^*,q).$$

This proves the statement for $m=0$. If $m>0$ and then $|z^*|\le c(D)$, then

$$\lambda^*\partial_\lambda^m\omega(\lambda^*,q) =(\frac{1}{2Q}+o(1))\partial_\lambda^m\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2xz^*\ \mathrm{d}x$$
$$+\sum_{j=0}^{m-1}O_Q(1)\partial_\lambda^j\omega(\lambda^*,q).$$

Applying this for all $j\le m$,

$$\lambda^{*m+1}\partial_\lambda^m\omega(\lambda^*,q) =$$
$$(\frac{1}{2Q}+o(1))\lambda^{*m}\partial_\lambda^m\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2xz^*\ \mathrm{d}x$$
$$+\sum_{j=0}^{m-1}O_Q(1)\lambda^{*j}\partial_\lambda^j\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2xz^*\ \mathrm{d}x =$$
$$2^{-m}z^{*m}(\frac{1}{2Q}+o(1))\partial_z^m\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2xz^*\ \mathrm{d}x$$
$$+\sum_{j=0}^{m-1}O_Q(1)z^{*j}\partial_z^j\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2xz^*\ \mathrm{d}x,$$

and the statement follows. In the last step we used that

$$\lambda^k\partial_\lambda^k f(\sqrt{\lambda})=2^{-k}\lambda^{\frac{k}{2}}f^{(k)}(\sqrt{\lambda})+\sum_{j=0}^{k-1}c_{kj}\lambda^{\frac{k-j}{2}}f^{(k-j)}(\sqrt{\lambda}).$$

□

Corollary 4.11. *Let* $\|q\|_p,\|q^*\|_p\le D$, $\lambda_n^*\in\sigma(\alpha,0,q^*)$, $\Re\lambda_n^*\ge -D$, *and* $\lambda_n,\dots,\lambda_{n+\nu}$ *be the corresponding elements of* $\sigma(\alpha_n,0,q)$. *Assume* *further that* $m \le \nu$, $\int_0^\pi (q - q^*) = 0$. *Then*

$$\Big(\sum_{j=0}^{m} \Big| \int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x)\right)\varphi_{n+j}(x)\ \mathrm{d}x\Big|^s\Big)^{\frac{1}{s}} \le$$

$$c(D)\Big(\sum_{j=0}^{m} |\partial_\lambda^j \omega(\lambda_n^*, q)|^r\Big)^{\frac{1}{r}}. \tag{4.15}$$

There are further constants depending on D, q^* *and the set of eigenvalues such that if* $\lambda_n^* \ne 0$ *and* $\|q - q^*\|_1 \le c(D, q^*, \min_n |\lambda_n^*|)$, *then*

$$\Big(\sum_{j=0}^{m} |\partial_\lambda^j \omega(\lambda_n^*, q)|^r\Big)^{\frac{1}{r}} \le$$

$$c(D, q^*, \min_n |\lambda_n^*|)\Big(\sum_{j=0}^{m} \Big| \int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x)\right)\varphi_{n+j}(x)\ \mathrm{d}x\Big|^s\Big)^{\frac{1}{s}}. \tag{4.16}$$

Proof. The first inequality follows from Corollary 4.8. The constants in (4.10) depend continuously on Q and then on λ. Hence Lemma 4.10 proves the second inequality if $|\lambda_n^*| < c(D)$, while for $\lambda_n^* > c(D)$ it follows from Lemma 4.9. □

Lemma 4.12. *Let* q, $q^* \in L_1(0, \pi)$, $\lambda_n^* \to \infty$, *where* $\lambda_n^* \in \sigma(\alpha_n, 0, q^*)$ *and let* λ_n *be the corresponding element of* $\sigma(\alpha_n, 0, q)$. *Then*

$$\lambda_n - \lambda_n^* \to 0 \ \ (n \to \infty) \quad \Leftrightarrow \quad \int\limits_0^\pi (q - q^*) = 0. \tag{4.17}$$

Proof. By (4.8) in [5]

$$\begin{aligned}\int_0^\pi (q - q^*) - 2\lambda_n^* \int_0^\pi (q(x) - q^*(x)) v(x, \lambda_n^*) v^*(x, \lambda_n^*)\ \mathrm{d}x \\ = \int_0^\pi A_q \left(q(\pi - x) - q^*(\pi - x)\right)\cos 2\sqrt{\lambda_n^*}x\ \mathrm{d}x.\end{aligned} \tag{4.18}$$

Here the right hand side tends to zero by the Riemann lemma and the second summand on the left has the exact order $\lambda_n - \lambda_n^*$ for large n, see (4.1) and (4.9). This proves Lemma 4.12. □

5. Proof of the general results

Proof of Theorem 1.2.

According to Lemma 4.6 in [5], $A_q\left(q(\pi-x)-q^*(\pi-x)\right)\in L_0^r$. If the inverse of (1.5) is continuous, then

$$\begin{aligned}\|q-q^*\|_r &\le c(D)\|A_q\left(q(\pi-x)-q^*(\pi-x)\right)\|_r \le \\ \le c(D)C&\left(\sum_n |\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\varphi_n(x)\,\mathrm{d}x|^s\right)^{\frac{1}{s}} \\ \le& c(D)C\left(\sum_n |\omega_n|^s\right)^{\frac{1}{s}} \le c(D)C\left(\sum_n |\lambda_n-\lambda_n^*|^s\right)^{\frac{1}{s}},\end{aligned} \tag{5.1}$$

which implies the continuity of (1.9) at $q=q^*$. In contrast, if the inverse mapping is not bounded, we can choose a sequence $h_k\in L_0^r$ such that $\|h_k\|_r=1$ but

$$\lim_{k\to+\infty}\left(\sum_n |\int_0^\pi h_k(x)\varphi_n(x)\,\mathrm{d}x|^s\right)^{\frac{1}{s}}=0. \tag{5.2}$$

Now Corollary 4.9 in [5] implies that for appropriately small $\gamma=\gamma(D)>0$ there exist potentials $q_k\in L^1$, $q_k-q^*\in L_0^r$ such that

$$A_{q_k}\left(q_k(\pi-x)-q^*(\pi-x)\right)=\gamma h_k(x). \tag{5.3}$$

We can choose $\gamma(D,q^*)>0$ so small that (4.16) holds for all q_k. Then

$$\begin{aligned}&\lim_{k\to+\infty}\left(\sum_n |\omega_n(q_k)|^s\right)^{\frac{1}{s}} \le \\ \le c(D,q^*,\min_n|\lambda_n^*|)&\lim_{k\to+\infty}\left(\sum_n |\int_0^\pi \gamma h_k(x)\varphi_n(x)\,\mathrm{d}x|^s\right)^{\frac{1}{s}}=0,\end{aligned} \tag{5.4}$$

but

$$\|q_k-q^*\|_r\ge c(D)\|A_q\left(q_k(\pi-x)-q^*(\pi-x)\right)\|_r=\gamma c(D,q^*)>0, \tag{5.5}$$

thus (1.9) is not continuous. □

Proof of Theorem 1.1.

For $s=\infty$ the statement follows from Theorem 3.7. Assume now $s<\infty$. As $|\lambda_n^*|\ge c(D)$, $c(D,q^*,\min_n|\lambda_n^*|)$ can be substituted by $c(D,q^*)$. If the mapping (1.5) is continuous, then we can choose $c(D)$

by Lemma 2.4 such that

$$\left(\sum_n |\omega_n|^s\right)^{\frac{1}{s}} \le \left(\sum_{|z_n^*|\le c(D)} |\omega_n|^s\right)^{\frac{1}{s}} + \left(\sum_{|z_n^*|\ge c(D)} |\lambda_n - \lambda_n^*|^s\right)^{\frac{1}{s}}$$

$$\le c(D,q^*)\left(\#\{|z_n^*| \le c(D)\}\right)^{\frac{1}{s}} \|q-q^*\|_1 +$$

(5.6)

$$+c(D)\Big(\sum_{|z_n^*|\ge c(D)} |\int_0^\pi A_q\left(q(\pi-x)-q^*(\pi-x)\right)\cos 2\sqrt{\lambda_n^*}x \; \mathrm{d}x|^s\Big)^{\frac{1}{s}}$$

$$\le c(D,q^*)\left(\#\{|z_n^*| \le c(D)\}\right)^{\frac{1}{s}} \|q-q^*\|_1 +$$
$$+ c(D)C\|A_q\left(q(\pi-x)-q^*(\pi-x)\right)\|_r$$
$$\le c(D,q^*)\left(\#\{|z_n^*| \le c(D)\}\right)^{\frac{1}{s}} \|q-q^*\|_r +$$
$$+ c(D)C\|q-q^*\|_r.$$

In the second sum we applied Lemma 4.9 and Lemma 4.12.

To complete the first part, we need only the following

Statement 5.1.

$$\left(\#\{|z_n^*| \le c(D)\}\right)^{\frac{1}{s}} \le c(D)C, \tag{5.7}$$

where C is the norm of the mapping (1.5).

Proof. $|\cos z| \ge |\cos \Re z|$, thus for $|z| \le c(D)$, $\int_0^\pi |\cos xz|^2 \; \mathrm{d}x \ge c(D)$. Then if $|z|, |w| \le c(D)$ and $|z-w| \le \delta(D)$,

$$|\int_0^\pi \cos xw\overline{\cos xz} \; \mathrm{d}x| \ge c(D). \tag{5.8}$$

For a w fixed

$$\left(\#\{|2z_n^* - w| \le \delta(D)\}\right)^{\frac{1}{s}} \le c(D)\left(\sum_{|2z_n^*-w|\le\delta(D)} |\langle \cos xw, \varphi_n\rangle|^s\right)^{\frac{1}{s}} \le$$

$$\le c(D)\left(\sum_n |\langle \cos xw, \varphi_n\rangle|^s\right)^{\frac{1}{s}} \le c(D)C\|\cos xw\|_r \le c(D)C,$$

hence

$$\#\{|z_n^*| \le c(D)\} \le \sum_w \#\{|2z_n^* - w| \le \delta(D)\}$$
$$\le c(D)(c(D)C)^s \le (c(D)C)^s.$$

∎

To prove the second part, if the mapping (1.5) is not bounded, there is a sequence $h_k \in L_0^r$ such that $\lim_{k\to\infty} \|h_k\|_r = 0$ but

$$\left(\sum_n \Big| \int_0^\pi h_k(x)\varphi_n(x)\,\mathrm{d}x\Big|^s\right)^{\frac{1}{s}} \geq 1. \tag{5.9}$$

holds. If $\|h_k\|_r$ (and then $\|h_k\|_1$) is small enough, by Corollary 4.9 in [5] there are potentials $q_k \in L^1$, $q_k - q^* \in L_0^r$ such that

$$A_{q_k}\left(q_k(\pi - x) - q^*(\pi - x)\right) = h_k(x). \tag{5.10}$$

Then by Corollary 4.11 and Lemma 4.12,

(5.11)

$$\left(\sum_n |\omega_n(q_k)|^s\right)^{\frac{1}{s}} \geq$$
$$\geq \frac{1}{c(D)}\Big(\sum_n \Big|\int_0^\pi A_{q_k}\left(q_k(\pi - x) - q^*(\pi - x)\right)\varphi_n(x)\,\mathrm{d}x\Big|^s\Big)^{\frac{1}{s}} \geq \frac{1}{c(D)}$$

but

$$\lim_{k\to+\infty} \|q_k - q^*\|_r \geq c(D)\|A_{q_k}\left(q_k(\pi - x) - q^*(\pi - x)\right)\|_r = 0, \tag{5.12}$$

thus (1.6) is not continuous. □

6. Finitely many known eigenvalues

If $\lambda^*_{n-1} \neq \lambda^*_n = \lambda^*_{n+1} = \ldots = \lambda^*_{n+\nu} \neq \lambda^*_{n+\nu+1}$, let us define

$$s_{n+j}(x) = \frac{\mathrm{d}^j}{\mathrm{d}z^j} \sin 2xz^*_n \quad j = 0, \ldots, \nu. \tag{6.1}$$

Lemma 6.1. *Assume that the system (1.13) is a frame (resp., a Riesz basis) in $L^2[-\pi, \pi]$. Then both (1.4) and the system*

$$S(\Lambda) = \{s_n(x) : n \geq 1\} \tag{6.2}$$

are frames (resp., Riesz bases) in $L^2[0, \pi]$. If the elements of the inverse frame of (1.13) are bounded by C in a p-norm, then the elements of the inverse frame of (1.4) and of (6.2) are both bounded by $2C$ in the same norm.

Proof. The proof is the same as of Lemma 8.3 in [5]. □

Proof of Theorem 1.4. From $\int_0^\pi A_q(q-q^*)=0$ we obtain

$$\Big|\int_0^\pi \left[\int_0^x A_q(q(\pi-t)-q^*(\pi-t))\,\mathrm{d}t\right] s_{n+j}(x)\,\mathrm{d}x\Big| =$$
$$=\Big|\int_0^\pi A_q(q(\pi-x)-q^*(\pi-x))\frac{\varphi_n(x)}{2z_n^*}\,\mathrm{d}x\Big|$$

As (6.2) is a frame and its inverse frame is bounded by C in L^∞, we can estimate by Corollary 4.11 as follows:

$$\Big|\int_0^x A_q(q-q^*)\Big| \le 2C\sum_n \Big|\int_0^\pi\int_0^x A_q(q-q^*)\cdot s_n(x)\,\mathrm{d}x\Big| \le$$
$$\le C\sum_n \frac{c(D)}{\sqrt{|\lambda_n^*|}}|\omega_n|.$$

Then the statement follows from Lemma 9.1 in [5]. □

Consider the case when we know the eigenvalues $\lambda_n^* \in \sigma(\alpha_n, 0, q^*)$, of which the first N may contain an error ε, while the others can contain unknown errors tending to zero.

Proof of Theorem 1.5.

By Theorem 1.4 and a Cauchy-Schwartz inequality,

$$\sup_{0\le x\le\pi}\Big|\int_0^x (q-q^*)\Big| \le C\sum_n \frac{c(D)}{\sqrt{|\lambda_n^*|}}|\omega_n|$$
$$\le Cc(D)\varepsilon\sum_{n=1}^N \frac{1}{\sqrt{|\lambda_n^*|}}+$$
$$+\,Cc(D)\left(\sum_{n=N+1}^\infty \frac{1}{|\lambda_n^*|}\right)^{\frac12}\left(\sum_{n=N+1}^\infty |\omega_n|^2\right)^{\frac12}.$$

Similarly to (5.6), by Lemma 2.1 it follows that $\sum_n |\omega_n|^2 \le c(D)C + c(D)\|F\|$, and if N is large enough in order to λ_n^* to be simple for $n > N$, then $\sum_{n=N+1}^\infty |\omega_n|^2 \le c(D)\|F\|$. □

Proof of Theorem 1.6 Let $\mu_n = 2\sqrt{\lambda_n^*}$ then $|\mu_n - n| = \mathbf{O}(\frac1n) \le d < \frac14$ for n large. This means that after shifting every μ_n to n the average shift over large intervals will be $\le d < \frac14$. Since the multiplicity of μ_n is bounded, this implies that (1.13) is a Riesz basis in $L_2(-\pi,\pi)$, see [1] and [2] for multiple values μ_n. According to Step 3. of Theorem 1.15 in [5], if all μ_n were different, there would be a function of bounded variation $\beta \in \mathrm{BV}[-\pi, \pi]$ such that the entire function

$$0 \neq G(z) = \int_{-\pi}^{\pi} e^{izx} d\beta(x)$$

satisfies $0 = G(0) = G(\pm\mu_n)$.

Statement 6.2. *If we change a* μ_n *to* $\hat{\mu}_n$*,*

$$\hat{G}(z) = \frac{z - \hat{\mu}_n}{z - \mu_n} G(z) = \int_{-\pi}^{\pi} e^{izx} d\hat{\beta}(x),$$

where $\hat{\beta} \in \mathrm{BV}[-\pi, \pi]$*, namely*

$$d\hat{\beta}(t) = d\beta(t) + i(\hat{\mu}_n - \mu_n) e^{-i\mu_n t} \int_{-\pi}^{t} e^{i\mu_n s} d\beta(s).$$

Proof. This is a standard calculation, using an integration by parts, see e.g. in Young [9], p 128-129. □

Consequently if finitely many μ_n are equal, we shift them to different values and reshift one by one to μ_n and finally get a function (which we shall denote again by β) $\beta \in \mathrm{BV}[-\pi, \pi]$ such that the entire function

$$0 \neq G(z) = \int_{-\pi}^{\pi} e^{izx} d\beta(x)$$

satisfies $0 = G(0) = G(\pm\mu_n)$, and the multiplicity of $\pm\mu_n$ as a root of G equals to its multiplicity in the sequence.

Statement 6.3. *If the multiplicity of* μ *is at least* k*,*

$$\frac{G(z)}{(z-\mu)^k} = (-i)^k \int_{-\pi}^{\pi} e^{i(z-\mu)x} \int_{-\pi}^{x} \frac{(x-t)^{k-1}}{(k-1)!} e^{i\mu t} \, \mathrm{d}\beta(t) \, \mathrm{d}x,$$

where $\beta \in \mathrm{BV}[-\pi, \pi]$.

Proof. For μ of multiplicity k we have

$$0 = G^{(j)}(\mu) = \int_{-\pi}^{\pi} (ix)^j e^{i\mu x} d\beta(x), \quad 0 \leq j < k, \tag{6.3}$$

$$0 \neq G^{(k)}(\mu) = \int_{-\pi}^{\pi} (ix)^k e^{i\mu x} d\beta(x). \tag{6.4}$$

Thus integrating by parts gives that

$$\begin{aligned}&\int_{-\pi}^{\pi} e^{i(z-\mu)x} \int_{-\pi}^{x} \frac{(x-t)^{k-1}}{(k-1)!} e^{i\mu t} \, \mathrm{d}\beta(t) \, \mathrm{d}x \\ = -&\int_{-\pi}^{\pi} \frac{e^{i(z-\mu)x}}{i(z-\mu)} \int_{-\pi}^{x} \frac{(x-t)^{k-2}}{(k-2)!} e^{i\mu t} \, \mathrm{d}\beta(t) \, \mathrm{d}x\end{aligned}$$

Repeating this process we arrive at

$$\begin{aligned}&\int_{-\pi}^{\pi} e^{i(z-\mu)x} \int_{-\pi}^{x} \frac{(x-t)^{k-1}}{(k-1)!} e^{i\mu t} \, \mathrm{d}\beta(t) \, \mathrm{d}x \\ =&(-1)^{k-1} \int_{-\pi}^{\pi} \frac{e^{i(z-\mu)x}}{[i(z-\mu)]^{k-1}} \int_{-\pi}^{x} e^{i\mu t} \, \mathrm{d}\beta(t) \, \mathrm{d}x \\ =&\int_{-\pi}^{\pi} \frac{e^{i(z-\mu)x}}{[-i(z-\mu)]^{k}} e^{i\mu x} \, \mathrm{d}\beta(x) = \frac{G(z)}{[-i(z-\mu)]^{k}}\end{aligned}$$

as asserted. □

Statement 6.4. *The system biorthogonal to (1.13) consists of the functions*

$$\psi(x) = \frac{1}{iG'(\mu)} e^{-i\mu x} \int_{-\pi}^{x} e^{i\mu t} \, \mathrm{d}\beta(t) \tag{6.5}$$

for simple μ and there are coefficients $\alpha_{r,j}$ such that

$$\psi_j(x) = e^{-i\mu x} \int_{-\pi}^{x} (\alpha_{k-j-1,j} t^{k-j-1} + \cdots + \alpha_{0,j}) e^{i\mu t} \, \mathrm{d}\beta(t) \tag{6.6}$$

is the function corresponding to $(ix)^j e^{i\mu x}$ in case of μ with multiplicity k, $0 \le j < k$.

Proof. If the multiplicity of μ and μ^* is k and k^* respectively then for $0 \le j \le k^* - 1$ and $1 \le r \le k$ we have

$$\begin{aligned}0 &= \frac{d^j}{dz^j} \frac{G(z)}{(z-\mu)^r}\Big|_{z=\mu^*} \\ &= (-i)^r \int_{-\pi}^{\pi} (ix)^j e^{i\mu^* x} \cdot e^{-i\mu x} \int_{-\pi}^{x} \frac{(x-t)^{r-1}}{(r-1)!} e^{i\mu t} \, \mathrm{d}\beta(t) \, \mathrm{d}x\end{aligned}$$

which shows the biorthogonality of the ψ-system for different μ and μ^*. If μ is simple, the constant factor in (6.5) has been verified in [5]. For

μ of multiplicity k an exchange of order of integrations gives

$$\begin{aligned}
&\int_{-\pi}^{\pi} x^r e^{i\mu x} \cdot \psi_j(x)\ \mathrm{d}x \\
=&\int_{-\pi}^{\pi} \frac{\pi^{r+1} - t^{r+1}}{r+1} (\alpha_{k-j-1,j} t^{k-j-1} + \cdots + \alpha_{0,j}) e^{i\mu t}\ \mathrm{d}\beta(t) \\
=&\frac{-1}{r+1} \int_{-\pi}^{\pi} (\alpha_{k-j-1,j} t^{k-j+r} + \cdots + \alpha_{0,j} t^{r+1}) e^{i\mu t}\ \mathrm{d}\beta(t).
\end{aligned}$$

The right hand side is clearly zero if $r < j$. For $r = j$ we have

$$\begin{aligned}
1 =& \int_{-\pi}^{\pi} (ix)^j e^{i\mu x} \cdot \psi_j(x)\ \mathrm{d}x \\
=& \frac{-i^j}{j+1} \int_{-\pi}^{\pi} \alpha_{k-j-1,j} t^k e^{i\mu t}\ \mathrm{d}\beta(t) = \frac{-(-i)^{k-j}}{j+1} \alpha_{k-j-1,j} G^{(k)}(\mu)
\end{aligned}$$

whence $\alpha_{k-j-1,j}$ can be expressed since $G^{(k)}(\mu) \neq 0$. For $r > j$ we inductively suppose that $\alpha_{k-j-1,j}, \ldots \alpha_{k-r,j}$ are already determined and we express $\alpha_{k-r-1,j}$ from

$$\begin{aligned}
0 =& \int_{-\pi}^{\pi} x^r e^{i\mu x} \cdot \psi_j(x)\ \mathrm{d}x \\
=& \frac{-1}{r+1} \int_{-\pi}^{\pi} (\alpha_{k-j-1,j} t^{k-j+r} + \cdots + \alpha_{k-r-1,j} t^k) e^{i\mu t}\ \mathrm{d}\beta(t).
\end{aligned}$$

Finally with $r = k-1$ we get $\alpha_{0,j}$. Thus the ψ-system is defined and the biorthogonality equations are all satisfied. □

By the last statement the uniform boundedness of the ψ-system holds if and only if $|G'(\mu_n)| \geq c > 0$ for all simple μ_n. Shift all μ_j with multiplicity $k > 1$ into k different values μ_j^* and denote by G^* the corresponding G-function. In [5] we proved that $|{G^*}'(\mu_n)| \geq c > 0$ for all simple μ_n. From

$$G(z) = G^*(z) \prod_j \frac{z - \mu_j}{z - \mu_j^*}$$

(finite product) it follows that for simple μ_n

$$G'(\mu_n) = {G^*}'(\mu_n) \prod_j \frac{\mu_n - \mu_j}{\mu_n - \mu_j^*}.$$

The product on the right is clearly nonzero and is close to 1 for large indices n, thus $|G'(\mu_n)| \geq c_1 > 0$ for all simple μ_n. This finishes the proof. □

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