DISCRETE INVERSE PROBLEMS FOR THE SCHRÖDINGER OPERATOR ON THE MULTI-DIMENSIONAL SQUARE LATTICE WITH PARTIAL CAUCHY DATA

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ABSTRACT. Motivated by the known results in the continuous case, in this paper we define the partial Cauchy data set for the discrete Schrödinger operator on a finite subset of the grid, and we prove reconstruction theorem for the potential. Our methods are applicable in some more general situation, we give a potential reconstruction theorem in the case of the hexagonal lattice as an example.

1. INTRODUCTION: CONTINUOUS AND DISCRETE INVERSE PROBLEMS

We consider the problem of reconstruction of a real valued potential from the partial Cauchy data set of a discrete version of the time independent Schrödinger equation. The motivation of the classical continuous problem is the inverse problem of electrical impedance tomography: we want to determine the electrical conductivity of a body by measurements of voltage and current on the boundary of the body. This is the classical Calderón problem [5]: let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with appropriate smooth boundary, and let ν be the unit outward normal vector to $\partial\Omega$. The electrical conductivity is a bounded and positive function γ on Ω, and we have for the electrical potential $f \in H^1(\Omega)$

$$\operatorname{div}(\gamma \operatorname{grad} f) = 0 \quad \text{on } \Omega \tag{1.1}$$

$$f|_{\partial\Omega} = f_0, \tag{1.2}$$

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where $f_0 \in H^{1/2}(\partial\Omega)$ is a boundary voltage potential. If this Dirichlet problem has a unique solution, then we can define the Dirichlet-to-Neumann map by the formula

$$\Lambda_\gamma(f) := \gamma \left.\frac{\partial f}{\partial \nu}\right|_{\partial\Omega}. \tag{1.3}$$

Then the inverse problem is to reconstruct the conductivity γ from Λ_γ. In the continuous setting we can transform this problem to an inverse Dirichlet-to-Neumann problem for the time independent Schrödinger equation:

$$-\Delta f + qf = 0 \quad \text{on } \Omega \tag{1.4}$$

$$f|_{\partial\Omega} = f_0, \tag{1.5}$$

$f_0 \in H^{1/2}(\partial\Omega)$. If the solution for this problem is unique, then the Dirichlet-to-Neumann map is

$$\Lambda_q(f) := \left.\frac{\partial f}{\partial \nu}\right|_{\partial\Omega}.$$

The question is the uniqueness of q by the knowledge of Λ_q. The connection between the two problems above is the transformation $q := -\frac{\triangle\sqrt{\gamma}}{\sqrt{\gamma}}$. At first Sylvester and Uhlmann proved in [15] that if $d \geq 3$, $q_1, q_2 \in C(\overline{\Omega})$ and $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 \equiv q_2$ (with suitable assumptions for the boundary of Ω). Nowadays there are a lot of generalizations of this result, e.g. the case of partial boundary measurements, when we measure f and $\frac{\partial f}{\partial \nu}$ only on an open subset Γ of $\partial\Omega$. This model is motivated e.g. by the medical applications of electrical impedance tomography. In this area there are a lot of open questions, and we know results only in some restricted cases. For example, in [9] Isakov proved uniqueness under the assumptions that $d = 3$ and $\partial\Omega \setminus \Gamma$ is a part of a plane or a sphere. In [7] Imanuvilov and Uhlmann proved uniqueness theorem for a cylindrical domain in 3 dimension in the partial measurement case.

Another possible generalization of the classical inverse D-to-N problem is related to the non unique solvability of problem (1.4)-(1.5).

In this more general situation following [8], let $\Omega \subset \mathbb{R}^d$, and let

$$C_q := \left\{ \left(f|_{\partial\Omega}, \left.\frac{\partial f}{\partial \nu}\right|_{\partial\Omega} \right) \mid f \in H^1(\Omega),\ (1.4) \text{ holds on } \Omega \right\}$$

which is the Cauchy data set for a bounded, real valued potential q. The question is the unique identification of q from C_q.

Related to the partial measurements problem, if we measure f and $\frac{\partial f}{\partial \nu}$ only on an open subset Γ of $\partial\Omega$, we give the partial Cauchy data problem. Let $\emptyset \neq \Gamma \subset \partial\Omega$ be open, and let $\Gamma_0 := \partial\Omega \setminus \overline{\Gamma}$. Let us define the partial Cauchy data set

$$C_q := \left\{ \left(f|_\Gamma , \left.\frac{\partial f}{\partial \nu}\right|_\Gamma \right) \mid f \in H^1(\Omega),\ \ f|_{\Gamma_0} = 0,\ (1.4) \text{ holds on } \Omega \right\}.$$

In [8] the authors proved that if $d = 2$, $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha > 0$ are complex valued potentials and $C_{q_1} = C_{q_2}$, then $q_1 \equiv q_2$. If $d \geq 3$, we do not know this type of results in the literature. A great survey in the topic is [12] written by Kenig and Salo.

The discrete version of these inverse problems are interesting themselves. The discrete version of the Calderón problem is related to resistor networks, where we have to determine the conductivities for an electrical network from the knowledge of the electrical potential and current in some nodes. The mathematical model for this is a weighted graph and a function defined on the vertices, where the weights of the edges are related to the conductivity of the edges, and the function is the electrical potential. The set of the nodes where we can measure the potential form the so called boundary nodes, the other vertices are the interior nodes. In the boundary nodes we can define the discrete version of normal derivative in some sense which is related to the current. In the interior nodes we know that Kirchoff's law is true which gives us a difference equation system, and we get the discrete version of (1.1)-(1.2) and (1.3).

The solvability of this problem depends heavily on the structure of the graph: in their magnificient article [6] Curtis, Ingerman and Morrow proved the reconstruction of the weights in the case of circular planar graphs.

In [4] the authors investigated the discretization of inverse conductivity problem to an inverse problem of resistor networks on circular planar graphs, in [3] the authors performed a similar procedure also in the case of partial measurements, but only in the case of circular network discretization. In [2] Chung and Berenstein proved uniqueness for the conductivities if the network satisfies some monotonicity condition.

In the discrete case there is no nice connection between the inverse conductivity and inverse Schrödinger problem, but the structure of a graph is very important in this latter case too. There are some results for inverse scattering problem for Schrödinger operators on $\mathbb{Z}^d$, $d \geq 2$ [11] and on hexagonal lattice [1]. In [13] the author investigated inverse Dirichlet-to-Neumann problems for the discrete Schrödinger operator on $\mathbb{Z}^d$.

In what follows we prove a reconstruction theorem for the discrete version of some partial Cauchy data problem in the case of the multi-dimensional square lattice.

2. Definition of the discrete problem and the main theorem

We formulate the discrete inverse D-to-N problem following the notations of [2]: consider a finite set $V \subset \mathbb{Z}^d$ in the d-dimensional grid $\mathbb{Z}^d$. We suppose $d \geq 2$. Denote S the set of points $y \in V$ for which all neighbors of y are in V and let $\partial S = V \setminus S$. Here, as usual, two points $x = (x_i)_{i=1}^d$ and $y = (y_i)_{i=1}^d$ of the grid $\mathbb{Z}^d$ are connected if $\sum_{i=1}^d |x_i - y_i| = 1$. We call points of S inner points and those of ∂S boundary points. In this paper we suppose that S is nonempty and that all boundary points are connected with some points of S. We delete the edges between points of ∂S. The remaining edges (x, y) are endowed with some positive weight $w(x, y)$, thus we get the graph $G(V, E)$. Note that we do not suppose that this graph is connected: in our proofs below we work with subgraphs of the original graph, which are not necessarily connected even though G is connected.

Denote by $x \sim y$ the fact that (x, y) is an edge. Let $\mathcal{N}(x) = \{y \in V | \, x \sim y\}$ be the set of neigbors of x, $\deg(x) = |\mathcal{N}(x)|$ be the degree of the node x and $d_w(x) = \sum_{y \in \mathcal{N}(x)} w(x, y)$ be the weight of x. Consider the weighted discrete Schrödinger equation

$$\Delta f(x) = q(x) f(x), \quad x \in S \tag{2.1}$$

where $f : V \to \mathbb{R}$, $q : S \to \mathbb{R}$ and the discrete Laplacian is

$$\Delta f(x) = \sum_{y \in \mathcal{N}(x)} \frac{w(x, y)}{d_w(x)} (f(y) - f(x)).$$

Thus, the Schrödinger equation (2.1) has the expanded form

$$(1 + q(x)) d_w(x) f(x) = \sum_{y \in \mathcal{N}(x)} w(x, y) f(y), \quad x \in S. \tag{2.2}$$

The discrete version of the outward normal derivative at the boundary points is

$$\partial f(z) = \sum_{y \in \mathcal{N}(z)} \frac{w(y, z)}{d_w(z)} (f(z) - f(y)).$$

If we consider the equation (2.2) with a given boundary condition $f|_{\partial S}$, and we assume that this equation has a unique solution, then $\Lambda_q : f|_{\partial S} \to \frac{\partial f}{\partial \nu}\big|_{\partial S}$ is the discrete D-to-N map. The inverse problem is the question of uniqueness of q from the knowledge of Λ_q. In [13] Morioka proved uniqueness for q if we know the D-to-N map to all energies in the case of $S \subset \mathbb{Z}^d$ is a finite set.

For rectangular domains in two dimensions Oberlin proved reconstruction algorithm in [14], Isozaki and Morioka generalized this result for $d \geq 2$ dimensions in [10]. Motivated by [8], in the following we prove a theorem in a more general situation.

If $f(z)$ is known for some $z \in \partial S$, then the same amount of information as in $\partial f(z)$ is contained in

$$\widetilde{\partial} f(z) = \sum_{y \in \mathcal{N}(z)} w(y,z) f(y).$$

We say that the node $x \in V$ is in the k-th level, $x \in V(k)$ if $\sum_{i=1}^d x_i = k$. We use analogously the notations $S(k)$ and $(\partial S)(k)$. To simplify notation we suppose that the nodes are between the (nonempty) levels -1 and $m+1$. For a boundary point $z \in (\partial S)(k)$ denote $\mathcal{N}_\downarrow(z)$ and $\mathcal{N}_\uparrow(z)$ the set of neighbors of z in $V(k-1)$ and $V(k+1)$ respectively. Consider the set

$$Z = \{z \in \partial S | \, \mathcal{N}_\uparrow(z) = \emptyset \text{ and } z - e_1 \notin S\}$$

where $e_1 = (1, 0, \dots, 0)$. That is, for all neighbors y of $z \in Z$ we have $y_1 = z_1$ and there are no neighbors one level up. Obviously, all points of Z are in levels ≥ 1 (an example for a graph in $\mathbb{Z}^2$ is on Figure 1).

In what follows we suppose that $f|_Z = 0$ and we do not measure $\widetilde{\partial} f$ on Z. Remark that Z is a nonempty subset of the boundary; e.g. for the points y of smallest first coordinate of $S(m)$ the points $y+e_2, \dots, y+e_d$ belong to Z. Typically for large dimensions Z can be a quite large part of the boundary. For example if $S = \{x = (x_i)_{i=1}^d | x_i = 1, \dots, n\}$ is the d-dimensional cube, then $|Z|/|\partial S| = (d-1)/(2d)$, which is close to $1/2$ if d is large.

We will consider the partial discrete Cauchy data

$$C(q) = \{(f|_{\partial S \setminus Z}, \widetilde{\partial} f|_{\partial S \setminus Z}) | \, f : V \to \mathbb{R}, \, f|_Z = 0, \text{ (2.2) holds in } S\}.$$

The main result of the present paper is that these partial Cauchy data are sufficient for the unique reconstruction of the operator:

Theorem 2.1. *Under the above assumptions, if $C(q_1) = C(q_2)$ then $q_1 = q_2$.*

Moreover, we provide an explicit method of reconstruction of the potential q from $C(q)$, inductively upward from $S(0)$ to $S(m)$, see formula (3.5) below.

The tools of the proof of theorem 2.1 are applicable for some special subgraphs of the square lattice, specially the same proof will be good

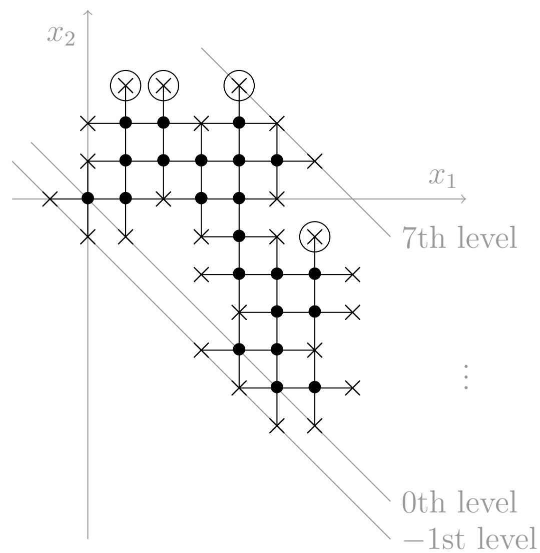

FIGURE 1. Az example for a graph embedded in $\mathbb{Z}^2$ with its levels: the boundary nodes are labeled with $\times$, the interior nodes with $\bullet$. For this graph $m = 6$, and the nodes of set Z are in circles.

for the hexagonal lattice too (this graph can be embedded in $\mathbb{Z}^2$ as described in [1]).

Theorem 2.2. *Let us consider the hexagonal lattice graph H instead of $\mathbb{Z}^2$, and the definitions above with $V \subset H$. Then under the above assumptions, if $C(q_1) = C(q_2)$ then $q_1 = q_2$.*

3. PROOFS

The following Lemma is the key tool in all of the later statements. We exploit the sorting of the points of the $k-1$-th level in diminishing order of the first coordinate.

Lemma 3.1. *Let $0 \le k \le m+1$ and $\widetilde{V}_{k-1} = \{y \in V(k-1) | \, y \sim y+e_1\}$, $\widetilde{\widetilde{V}}_{k-1} = V(k-1) \setminus \widetilde{V}_{k-1}$. Then there is a disjoint decomposition*

$$\widetilde{V}_{k-1} = Y_1 \cup Y_2 \cup \dots, \tag{3.1}$$

such that

$$\mathcal{N}_\downarrow(y+e_1) \subset \{y\} \cup Y_{i-1} \cup \widetilde{\widetilde{V}}_{k-1}, \quad y \in Y_i, \quad i \ge 1 \tag{3.2}$$

and for $i = 1$ we set $Y_0 = \emptyset$.

Proof. Let $s = \max\{y_1 | \, y \in \widetilde{V}_{k-1}\}$ and $Y_i = \{y \in \widetilde{V}_{k-1} | \, y_1 = s - i + 1\}$. This is a decomposition satisfying (3.2). Indeed, if $y \in Y_1$ then any neighbor $x \in \mathcal{N}_\downarrow(y+e_1)$ other than y satisfies $x_1 = s+1$, hence $\mathcal{N}_\downarrow(y+e_1) \subset \{y\} \cup \widetilde{\widetilde{V}}_{k-1}$. Similarly, for $y \in Y_i$ any neighbor $x \in \mathcal{N}_\downarrow(y+e_1)$ other than y satisfies $x_1 = s - i + 2$, consequently $x \in Y_{i-1} \cup \widetilde{\widetilde{V}}_{k-1}$. The lemma is proved. □

Lemma 3.2. *Let $-1 \le k \le m+1$ be fixed and suppose that*
a) $f = 0$ in all boundary points z of level $\ge k$ for which $z + e_1 \notin S$,
b) $\widetilde{\partial} f = 0$ in all boundary points z of level $\ge k+1$ for which $z - e_1 \in S$,
c) the differential equation (2.2) holds in all inner points of level $\ge k+1$.
Then $f = 0$ in all points of level $\ge k$.

Proof. We prove that $f = 0$ in the levels $\ge k$ by backward induction on k. For $k = m+1$ this follows from a). If we know for some $k \le k_1 \le m$ that f is zero in the levels $\ge k_1 + 1$, apply the decomposition (3.1) with $k_1 + 1$ instead of k. In the points $z \in \widetilde{\widetilde{V}}_{k_1}$ we have $z + e_1 \notin S$, hence by a) $f|_{\widetilde{\widetilde{V}}_{k_1}} = 0$. Now let $y \in Y_i$. If $y + e_1 \in S$ then the equation (2.2) holds in the point $y + e_1$, thus, by the induction hypothesis and by (3.2)

$$
\begin{aligned}
(3.3) \qquad 0 = & \sum_{x \in \mathcal{N}_\downarrow(y+e_1)} f(x) w(x, y+e_1) \\
= & f(y) w(y, y+e_1) + \sum_{x \in Y_{i-1},\, x \sim y+e_1} f(x) w(x, y+e_1).
\end{aligned}
$$

If $y + e_1 \notin S$ then y must be in S, hence the same conclusion (3.3) follows from b). Now by an inner induction on $i \ge 1$ (3.3) gives that $f|_{Y_i} = 0$. That is, $f = 0$ in the levels $\ge k_1$ for every $k \le k_1 \le m$ hence in levels $\ge k$. □

The next lemma clarifies how to reconstruct $q|_{S_0}$ from the partial Cauchy data set $C(q)$. For this goal we show that we can prescribe a function f from the values f and $\widetilde{\partial} f$ on $\partial S \setminus Z$ which fulfills the differential equation (2.2), and from the values of f we can compute $q|_{S_0}$.

Lemma 3.3. *Suppose that $f : V \to \mathbb{R}$ satisfies the following assumptions:*
a) $f = 0$ in all boundary points z of level ≥ 0; in particular $f|_Z = 0$,
b) $\widetilde{\partial} f = 0$ in all boundary points z of level ≥ 2 for which $z - e_1 \in S$,
c) $\widetilde{\partial} f(y - e_1) = (-1)^{y_1}$ for all $y \in S(0)$,
d) $f = 0$ in the points $z \in V(-1)$ for which $z + e_1 \notin S$,

e) the differential equation (2.2) holds in all of S*.*
Then
i) there exists exactly one function f *with the above assumptions,*
ii) f *vanishes nowhere in* $S(0)$*,*
iii) $q|_{S_0}$ *can be uniquely reconstructed from the data* $(f|_{\partial S\setminus Z}, \widetilde{\partial} f|_{\partial S\setminus Z})$*.*

Note that in the assumptions a)-d) we prescribe f only in some boundary nodes of $\partial S \setminus Z$ and $\widetilde{\partial} f$ in some nodes of $\partial S \setminus Z$, since

$$\{z \mid z \in (\partial S)(k), k \geq 2, z - e_1 \in S\} \subset \partial S \setminus Z$$

and also $S(0) - e_1 \subset \partial S \setminus Z$. The major sets of lemma 3.3 for our graph example is on Figure 2.

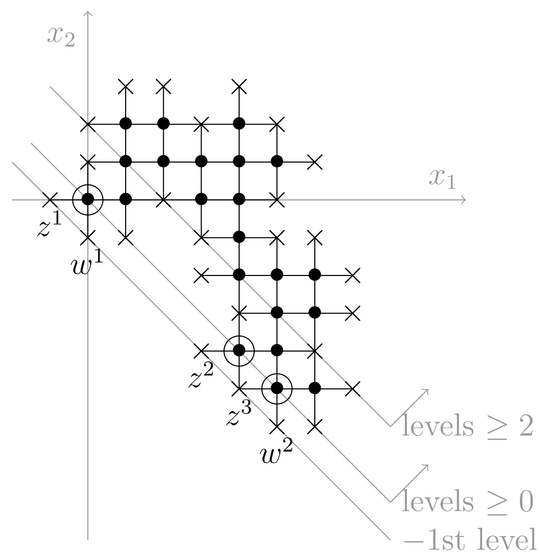

FIGURE 2. The major sets from lemma 3.3 for the graph example above. The nodes of $S(0)$ are in circles, $\widetilde{V}_{-1} = \{z^1, z^2, z^3\}$, $\widetilde{\widetilde{V}}_{-1} = \{w^1, w^2\}$.

Proof. By Lemma 3.2 $f = 0$ in all levels ≥ 1. Let $s = \min\{y_1 | \, y \in S(0)\}$, $Y_{i,0} = \{y \in S(0) | \, y_1 = s + i - 1\}$, $i \geq 1$ and $Y_{0,0} = \emptyset$. If $y \in S(0)$, then $y - e_1 \in (\partial S)(-1)$, and for $x \sim y - e_1$, $x \neq y$ we have $x \in S(0)$ and $x_1 = y_1 - 1$, hence for $y \in Y_{i,0}$ we get $\mathcal{N}_\uparrow(y - e_1) \subset \{y\} \cup Y_{i-1,0}$. Consequently, using c) we obtain for $y \in Y_{i,0}$ that

$$(-1)^{s+i-1} = \widetilde{\partial} f(y - e_1) = w(y, y - e_1) f(y) + \sum_{x \in Y_{i-1,0},\, x \sim y - e_1} w(x, y - e_1) f(x). \tag{3.4}$$

This formula shows by induction on i that f is uniquely defined in $y \in Y_{i,0}$ and, moreover, that $\operatorname{sign} f(y) = (-1)^{s+i-1}$. Thus, f is nowhere zero in $S(0)$. From d) we know that $f|_{\widetilde{\widetilde{V}}_{-1}} = 0$, we show that f is uniquely determined also on $\widetilde{V}_{-1}$. Consider the decomposition of $\widetilde{V}_{-1}$ given in Lemma 3.1 for $k = 0$. From Lemma 3.1 and from e) we obtain that if $z \in Y_i$, then

$$(3.5)\quad (1 + q(z + e_1))d_w(z + e_1)f(z + e_1) = \sum_{x \in \mathcal{N}(z+e_1)} w(x, z + e_1)f(x)$$
$$= w(z, z + e_1)f(z) + \sum_{x \in Y_{i-1},\, x \sim z+e_1} w(x, z + e_1)f(x)$$

since $f \equiv 0$ in all levels ≥ 1 and on $\widetilde{\widetilde{V}}_{-1}$. This shows by induction on i that f is uniquely defined in Y_i (if $i = 1$, then $Y_0 = \emptyset$ and the sum for Y_0 is empty). We get that the assumptions a)-e) determine a unique $f : V \to \mathbb{R}$ with the prescribed properties, that is, statements i) and ii) are proved. Finally suppose that we know the vector $(f|_{\partial S \setminus Z}, \widetilde{\partial} f|_{\partial S \setminus Z}) \in C(q)$ which corresponds to this f. We have seen above that from these data f is uniquely determined in levels ≥ 0 (on the -1st level we know f from the data), and we can compute f on $S(0)$ without the knowledge of q by induction using (3.4). Moreover, f is nowhere zero in $S(0)$. Then we can express $q(z + e_1)$ from (3.5) for every $z \in Y_i$, since every $y \in S(0)$ has the form $y = z + e_1$ for some $z \in Y_i$. Thus, the property iii) is also verified. □

Since q is known in $S(0)$, we can reduce the "width" of the graph, i.e. the value of m by deleting the level -1. More precisely we delete the following sets A, B and C:

$$A = V(-1),\ B = \{z \in (\partial S)(1) | \mathcal{N}_{\uparrow}(z) = \emptyset\},$$
$$C = \{y \in S(0) | \mathcal{N}(y) \cap S(1) = \emptyset\}.$$

The points $S(0) \setminus C$ will be boundary points in the reduced graph. Thus the reduced node sets are $V' = V \setminus (A \cup B \cup C)$, $S' = S \setminus S(0)$, $\partial(S') = (\partial S \cup S(0)) \setminus (A \cup B \cup C)$ and $Z' = Z \setminus B$. Let $q' = q|_{S'}$. In what follows we show how to obtain the Cauchy data $C'(q')$ of the reduced graph from $C(q)$.

Lemma 3.4. *a) If $f : V \to \mathbb{R}$ satisfies (2.2) on S and vanishes on Z then $f|_{V'}$ satisfies (2.2) on S' and vanishes on Z'.*
b) Conversely, if $f : V' \to \mathbb{R}$ satisfies (2.2) on S' and vanishes on Z' then f has an extension to V with the properties listed in a).
c) To every element $X(f) = (f|_{\partial S \setminus Z}, \widetilde{\partial} f|_{\partial S \setminus Z})$ of $C(q)$ we can construct

an element $X'(f') \in C'(q')$ such that $f|_{V'} = f'$. In this way every element of $C'(q')$ can be constructed.

Proof. The statement a) is straightforward since all neighbors of S' are in V' and $Z' \subset Z$. To verify b) define $f = 0$ on B; this implies $f = 0$ on Z. Let f be arbitrary on C, it will not influence the fulfillment of (2.2) in S'. In $A = V(-1)$ we apply again the decomposition of Lemma 3.1 with $k = 0$. Define $f = 0$ on $\widetilde{\widetilde{V}}_{-1}$. By induction on i we see that for any $z \in Y_i$ the value $f(z)$ can be uniquely obtained from

$$(1 + q(z + e_1))d_w(z + e_1)f(z + e_1) = w(z, z + e_1)f(z)$$
$$+ \sum_{x\in Y_{i-1},\, x\sim z+e_1} w(x, z + e_1)f(x) + \sum_{x\in \mathcal{N}_\uparrow(z+e_1)} w(x, z + e_1)f(x), \quad z \in Y_i.$$

In this way the equation (2.2) holds in the points $z + e_1$, $z \in Y_i$. Since every points of $S(0)$ have such a representation, the statement b) is verified. To see c) we determine first the values of f in $S(0)$ as in the proof of Lemma 3.3. Indeed, let again $s = \min\{y_1 | \, y \in S(0)\}$, $Y_{i,0} = \{y \in S(0) | \, y_1 = s+i-1\}$, $i \geq 1$ and $Y_{0,0} = \emptyset$. Then by induction on i from the formula

$$\widetilde{\partial} f(y-e_1) = w(y, y-e_1)f(y)+ \sum_{x\in Y_{i-1,0},\, x\sim y-e_1} w(x, y-e_1)f(x), \quad y \in Y_{i,0}$$

we can express $f(y)$ for any $y \in Y_{i,0}$ and finally for any $y \in S(0)$. In the new boundary points $y \in S(0) \setminus C$ we can compute

$$\widetilde{\partial}_{\text{new}} f(y) = (1+q(y))d_w(y)f(y)- \sum_{z\in B,\, y\sim z} w(z, y)f(z)- \sum_{z\in \mathcal{N}_\downarrow(y)} w(z, y)f(z)$$

since the equation (2.2) holds in y and $q(y)$ is reconstructed from $C(q)$ by Lemma 3.3. The boundary points $\partial S \cap \partial(S')$ have neighbors only in S and no inner points remained in $V(0)$. Thus we have to modify $\widetilde{\partial} f$ only in those points $z \in (\partial S)(1)$ which have neighbors in $S(0)$ and $S(2)$, too, namely

$$\widetilde{\partial}_{\text{new}} f(z) = \widetilde{\partial}_{\text{old}} f(z) - \sum_{z'\in \mathcal{N}_\downarrow(z)} w(z, z')f(z').$$

In those points of $\partial(S')$ which were not considered above the values of f and ∂f are the same as in X_f. Thus from the knowledge of $C(q)$ we constructed an element $X'(f') \in C'(q')$ such that $f' = f|_{V'}$. As we have seen in b), in this way we obtain all elements of $C'(q')$. □

Proof of Theorem 2.1 We use induction on m, the width of the graph. If $m = 0$, Lemma 3.3 gives q on S. For larger m we apply the reduction of the width described above, the construction of the

Cauchy data for the reduced graph is given in Lemma 3.4. By induction hypothesis, q can be obtained in $S' = S \setminus S(0)$ and by Lemma 3.3 q is determined in $S(0)$. The theorem is proved. $\square$

Proof of Theorem 2.2 In the proofs above we use only the properties of the multi-dimensional square lattice that the node set of the graph can be decomposed into disjoint subsets $V(k)$ such that edges are only between $V(k)$ and $V(k \pm 1)$, Lemma 3.1 is true for an arbitrary finite subgraph of the lattice, and if x is some node of $\mathbb{Z}^d$, then $x \pm e_1 \in \mathbb{Z}^d$ too. In particular every subgraph of $\mathbb{Z}^d$ given by deleting arbitrary edges not parallel with e_1 is convenient and the proofs above remain true. We can see that the hexagonal lattice forms such a graph embedded in $\mathbb{Z}^2$, see the figures 1 and 2 in [1], so with the lemmas above prove Theorem 2.2 as Theorem 2.1 too. $\square$

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