INVERSE PROBLEMS

FOR LINEAR DIFFERENTIAL OPERATORS

DSC DISSERTATION

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Contents

1. Introduction

1.1. **Basic notions and examples of inverse problems.**

The quiery "inverse problem" gives about 100.000 results in the browser Scholar Google, among them about 38.000 between 2000 and 2007. The journal Inverse Problems having a continuously growing scientific reputation publishes many papers. These facts illustrate the importance of this topic in science and technology and the interest of the scientific community in further development of the theory.

No precise definition of an inverse problem can be formulated. It assumes the existence of a direct problem. This means that we have a (physical, engineering, medical etc.) device with complete mathematical description (e.g. counting the eigenvalues of an operator, the gravitation generated by given mass distribution, the scattering of a sound wave by an obstacle in the space etc.). The inverse problem consists of recovering some (functional or data) characteristics of this device (reconstruction of the operator from its eigenvalues, of the mass distribution from part of the gravitational field, determination of the position and shape of the scatterer from measurements of the scattered waves). Following Hadamard [17] a problem is called well-posed if its solution exists, is unique and stable i.e. continuously depending on the input data. Now the majority of the important inverse problems proves to be ill-conditioned, while the corresponding direct problem is well-posed. Thus the instability of the inverse problems have to be remedied by some stabilization methods. Another common feature of the typical inverse problems is that the object or phenomenon to be investigated is not reachable for direct observation. It is too far away or not visible (radar, sonar technology), it is too small (quantummechanical inverse problems), it is in the inside of the human body (medical inverse problems, like tomography, magnetic resonance imaging etc.), or several hundred meters below the surface of the Earth (geological prospecting). Inverse problems occur e.g. in nondestructive evaluation of meterials and so on.

In this dissertation two kinds of inverse problems are considered in details: the inverse Sturm-Liouville problem in Chapter 2 and the quantummechanical inverse scattering with fixed energy in Chapter 3. However, just for illustration, some other inverse problems are briefly mentioned below (see Isakov [31] for more details).

Example 1. Gravimetry

Suppose we are given a mass distribution f on $\mathbf{R}^3$ vanishing outside a bounded domain $\Omega \subset \mathbf{R}^3$. Then the gravitational field u is the solution of

$$-\Delta u = f \quad \text{in} \;\; \mathbf{R}^3, \quad u \to 0 \;\; \text{if} \;\; |x| \to \infty.$$

The solution of this problem (the direct problem) can be expressed by

$$u(x) = \int_{\Omega} \frac{f(y)}{4\pi |x-y|} \, dy.$$

The gradient ∇u is the gravitational force. The inverse problem consists of recovering the mass density f given the gravitational force ∇u on the boundary $\partial\Omega$. It is used to recover the density of the Earth by measuring gravitation and to navigate aircrafts using high precision gravitational data.

Example 2. Conductivity

The conductivity equation has the form

$$\operatorname{div}(a \nabla u) = 0 \quad \text{in} \;\; \Omega, \qquad u = g \quad \text{in} \;\; \partial\Omega$$

where u is the electric potential within Ω and a is the conductivity. The direct problem is to find u given a and g ; using appropriate Soboleff spaces, this problem is well-defined and we can express

$$h = a \frac{\partial u}{\partial n} \quad \text{on} \;\; \Gamma$$

where Γ is a subset of the boundary $\partial\Omega$. The inverse conductivity problem is to reconstruct the conductivity a within Ω given h for one g (one boundary meaurement), or for many g (many boundary measurements). In the electrical impedance tomography this practically means that electric current sources are placed on the surface of the human body and voltage is measured for one or for many positions of these sources. Once the function a is reconstructed, this gives a portrait of the inner structure of the body, since the conductivity of different tissues (muscles, liver, lungs etc.) are different constants.

Example 3. Inverse eigenvalue problems

Many problems of this kind aim to reconstruct a mechanical system from its eigenfrequencies. The most famous question of this kind: "Can one hear the shape of a drum?" is due to M. Kac [33]. The problem is to determine the domain Ω from the eigenvalues of the Dirichlet Laplacian on Ω. A value λ is considered an eigenvalue if there exists a nontrivial solution u of the problem

$$-\Delta u = \lambda u \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

If we consider vibrational modes of a string, we are led to a similar problem. The well-known wave equation of the string has the form

$$\varrho U_{tt} - (TU_x)_x = \varrho F$$

where $U(x,t)$ is the transversal displacement of the point x at time t, $T(x)$ is the tension, $\varrho(x)$ is the density of the string and F represents the external (transversal) force at x. In case of constant tension and no external forces we get, after a separation of variables, the equation

$$-u" = \lambda \varrho u, \qquad u = u(x).$$

If the string is fixed at the endpoints, this means Dirichlet boundary conditions

$$u(0) = u(l) = 0.$$

The inverse problem given this way is to reconstruct ϱ from the eigenfrequencies λ_n; it can be paraphrased as "Can one hear the density of a string?".

In quantum mechanics the time-independent, one-dimensional Schrödinger equation has the form

$$-u" + q(x)u = \lambda u.$$

Here q is the potential of the operator. Since this operator is the Hamiltonian of the underlying physical system, the eigenvalues λ correspond to the energy levels of the system. Here

$$\int_x^{x+\Delta x} u^2$$

represents the probability of finding the particle between x and $x+\Delta x$. The inverse problem consists of recovering the system (the potential q) from the knowledge of the stable energy levels (i.e. the eigenvalues).

The most important category of inverse problems is the collection of inverse scattering problems; it is presented separately in the next point.

1.2. **Inverse scattering.**

Scattering theory has played a central role in the twentieth century mathematical physics. In a broad sense, scattering theory investigates the effect caused by an obstacle or by the inhomogenity of the medium on an incident wave. The total field u describing the wave phenomenon is considered as the sum

$$u = u_i + u_s,$$

u_i being the incident field and u_s the scattered field. The direct scattering problem is to find u_s from the knowledge of u_i and the differential equation satisfied by u. That is, the aim is to describe the wave scattered by an obstacle or by going through inhomogenities. Of much more interest is the inverse scattering problem of determining the properties of the scatterer from a knowledge of the asymptotic behaviour of u_s , the so-called far field pattern of the scattered wave. In other words, we aim to reconstruct the shape of the scatterer and the differential equation by measuring the scattered waves far away from the scatterer. This (simplified) presentation covers a huge range of problems and ideas; further infermation can be found e.g. in the monographs Chadan and Sabatier [9], Colton and Kress [13], Jones [32], Lax and Phillips [42], Newton [51], Reed and Simon [59], Isakov [31], Kirsch [36].

In what follows a short list of typical inverse scattering problems will be presented; the last one is the topic of Section 3.

Example 1 Scattering by an obstacle

We are given a bounded domain

$$D \subset \mathbf{R}^3.$$

The phenomenon of the scattering of acoustic plane waves by the obstacle D is described by the Helmholtz equation

$$-\Delta u = k^2 u \quad \text{in } \mathbf{R}^3 \setminus \overline{D}$$

where $k = \omega / c_0$ is the wave number, ω is the frequency, c_0 is the speed of sound in the homogeneous medium. We apply Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial D$$

for soft obstacle and Neumann-type boundary condition

$$\frac{\partial u}{\partial n} + i\lambda n = 0 \quad \text{on } \partial D$$

for hard obstacle. The solution u has the decomposition

$$u = u_i + u_s \tag{1.1}$$

where

$$u_i = e^{ik\xi\cdot x} \tag{1.2}$$

represents the planar wave coming from the direction ξ, $|\xi| = 1$ and the scattered field u_s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - iku_s\right) = 0. \tag{1.3}$$

Any solution u_s satisfying the above conditions admits the representation

$$u_s(x) = A(\hat{x}, \xi, k)\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right), \quad r = |x| \to \infty \tag{1.4}$$

where $\hat{x} = x/r$. Thus the scattered wave is asymptotically spherical with scattering amplitude A (called also the far field pattern) at the direction $\hat{x}$.

The inverse obstacle scattering problem is the recovery of the shape of the obstacle D from the scattering amplitude.

Example 2 Inverse scattering by an inhomogeneous medium

In this case the medium where the sound propagates is homogeneous at large distances but has inhomogenities in a bounded domain of $\mathbf{R}^3$. This yields the Helmholtz equation

$$-\Delta u = k^2 n u \quad \text{on } \mathbf{R}^3$$

where $n(x)$, the index of refraction is inversely proportional to the square of the speed of sound at x. Since the medium is homogeneous far away, $n(x) = 1$ for large $|x|$. As above, we consider solutions of the form $u = u_i + u_s$, where u_i is of the form (1.2) and u_s satisfies the condition (1.3). We get again the representation (1.4) for u_s and we aim to recover the function $n(x)$ from the far field pattern.

Example 3 Inverse scattering in two-particle interactions

The Schrödinger operator

$$H = H_0 + V(x), \qquad H_0 = -\Delta$$

describes two interacting particles. As above, there are solutions u of

$$Hu = k^2 u$$

satisfying

$$u(x) = e^{ik\xi\cdot x} + A(\hat{x},\xi,k)\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right), \quad r = |x| \to \infty$$

and the inverse problem aims to reconstruct the interaction potential $V(x)$ from the scattering amplitude $A(\hat{x},\xi,k)$.
In a real experiment the plane wave $u_i = e^{ik\xi\cdot x}$ represents an incident beam of particles arriving from direction ξ against the target, and the outgoing spherical wave u_s corresponds to scattered particles. The ratio of the flux density of the scattered particles in the solid angle $d\hat{x}$ to that of the incident beam is $|A(\hat{x},\xi,k)|^2\, d\hat{x}$. The function $|A(\hat{x},\xi,k)|^2$ is called the cross section; it can easily be measured by a detector counting the number of outgoing particles in the solid angle $d\hat{x}$. However, the scattering amplitude A can be recovered (at least numerically) from its modulus through an integral equation for its phase, see Chadan and Sabatier [9], Chapter X. So we suppose A to be known in the inverse scattering problem and we are looking for the potential V through the knowledge of the scattering amplitude A.
In most cases of particle interactions the potential V proves to be spherically symmetrical:

$$V(x) = q(r) \quad r = |x|.$$

In this case the scattering is symmetrical around the axis of the incident beam, that is $A(\hat{x},\xi,k)$ depends only on $\hat{x}\cdot\xi$:

$$A(\hat{x},\xi,k) = F(\hat{x}\cdot\xi,k).$$

Expanding the scattering amplitude F by Legendre polynomials P_n gives

$$F(t,k) = \frac{1}{k}\sum_{n=0}^{\infty}(2n+1)e^{i\delta_n}\sin(\delta_n)P_n(t).$$

The constants $\delta_n = \delta_n(k)$ are called phase shifts.

There is an alternative way to get the phase shifts. Looking for solutions of $-\Delta u + Vu = k^2 u$ in the form

$$u(x) = \frac{\psi_n(r)}{\sqrt{r}}Y_m^n(\theta,\varphi)$$

with spherical harmonics Y_m^n gives in the radial variable r the equations

$$\psi_n" + \frac{1}{r}\psi_n' - \frac{(n+1/2)^2}{r^2}\psi_n + (k^2 - q(r))\psi_n = 0, \quad n = 0,1,\dots$$

Now the phase shifts $\delta_n(k)$ can be recovered from the asymptotical behaviour of ψ_n:

$$\psi_n(r) = c_n(k) r^{n+1/2}(1 + \mathbf{o}(1)) \quad r \to 0+$$
$$\sqrt{r}\psi_n(r) = c_n(k) \sin[kr - \pi n/2 + \delta_n(k)] + o(1) \quad r \to \infty.$$

Since the phase shifts and the scattering amplitude determine each other, the inverse problem can be reformulated as to recover the potential from a set of phase shifts. Two special cases are of interest:

a) inversion with fixed impulse moment $n = n_0$ where we know $\delta_{n_0}(k)$ for all real $k > 0$,

b) inversion with fixed energy $k = k_0$ where the phase shifts $\delta_n(k_0)$, $n \geq 0$ are known.

Both problems are studied e.g. in Levitan [45] and in Chadan and Sabatier [9]. The second problem is the subject of Chapter 3. of this dissertation.

2. Inverse Sturm- Liouville problems

The general Sturm-Liouville eigenvalue problem is defined as follows. We are given the equation

$$-(pw')' + lw = \lambda r w \tag{2.1}$$

on a finite or infinite interval, p, l, r are coefficient functions, $p > 0$, $r > 0$. We give some boundary conditions at the endpoints. It turns out that (under some regularity conditions on p, l, r) a nontrivial solution $w \neq 0$ exists only for some special values λ; these are called eigenvalues.
If p and r are sufficiently smooth, we can apply Liouville transform to obtain a reduced form of (2.1):

$$-y'' + q(x)y = \lambda y. \tag{2.2}$$

In what follows we prefer this simplified form of the problem, closely related to the quantummechanical problem of a particle moving along a line segment. Suppose that (2.2) holds on a finite segment, say, on $[0, \pi]$. We will restrict ourselves to separated boundary conditions of the form

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{2.3}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0. \tag{2.4}$$

We will assume throughout that the operator is regular in the sense that

$$q \in L_1(0,\pi). \tag{2.5}$$

It is known that in this case there exists a real sequence of values λ_n tending to $+\infty$ such that the system (2.2)-(2.4) has a nontrivial solution only for $\lambda = \lambda_n$ and the corresponding eigenfunctions y_n form an orthonormal basis in $L_2(0,\pi)$. Consequently the spectrum of the operators consists only of eigenvalues. To indicate the dependence on the boundary conditions we will use the notation

$$\sigma(\alpha,\beta) = \{\lambda_n :\ n \geq 1\} \tag{2.6}$$

for the set of eigenvalues. This chapter is devoted to the problem of unique recovery of the potential q from a set of eigenvalues.

2.1. **Ambarzumian-type theorems.**

The spectrum $\sigma(\pi/2,\pi/2)$ in (2.2)-(2.4) gives the sequence of eigenvalues under Neumann boundary conditions. In case $q = 0$ the Neumann spectrum has the form

$$\lambda_n = n^2 \quad n \geq 0 \tag{2.7}$$

and the corresponding eigenfunctions are $\cos(nx)$. The following statement, considered as the starting point of inverse spectral theory, says that the converse is also true:

Theorem *Ambarzumian* [2]
If the Neumann eigenvalues of a potential $q \in L_1$ are $\{n^2 :\ n \geq 0\}$ then $q = 0$ a.e.

Later it turned out that a sharper version also holds (see e.g. Levitan and Gasymov [46]): if the first Neumann eigenvalue is $\lambda_0 = 0$ and if $\lambda_n = n^2$ for infinitely many n then $q = 0$ a.e. Indeed, from the asymptotical distribution

$$\lambda_n = n^2 + \frac{1}{\pi}\int_0^\pi q + o(1) \qquad n \to \infty \tag{2.8}$$

of the Neumann eigenvalues it follows that

$$\int_0^\pi q = 0. \tag{2.9}$$

The first eigenvalue $\lambda_0 = 0$ is the minimum of the quadratic form

$$(Hy, y) = \int_0^\pi (y'^2 + q(x)y^2)\,dx, \tag{2.10}$$

the minimizer being const$\cdot y_0$. Since for $y = 1$ $(Hy, y) = 0$, hence y is parallel to y_0 , hence $0 = -y_0" + qy_0 = q$ a.e.

Several generalizations of the Ambarzumian theorem have been found. Chakravarty and Acharyya [10] considered the case of 2x2 real symmetrical matrix potentials, Chern and Shen [9] then gave the extension for $n \times n$ potentials. More precisely let $y(x) \in \mathbf{R}^n$, $q(x) \in \mathbf{R}^{n\times n}$ be symmetrical and $y'(0) = y'(\pi) = 0$. Now if $\lambda_0 = 0$ and there are infinitely many eigenvalues of the form k^2 then $q = 0$ a.e.

We do not know whether there is a nonzero potential that can be uniquely recovered from its Neumann spectrum or any potential uniquely defined by one (non-Neumann) spectrum. To find such an example or to disprove its existence is a challenging open problem. However, if we allow an additional integral condition for q, uniqueness can be established from the knowledge of a non-Neumann spectrum. This is given by Chern, Law and Wang in 2001, [12]. They proved that if $\sigma(\alpha, \alpha) = \{n^2 : n \geq 1\}$ and if

$$\int_0^\pi q(x) \cos 2(x - \alpha)\,dx = 0 \tag{2.11}$$

then $q = 0$ a.e. They also presented a generalization for matrix-valued potentials. Yang [60] 2006 found an extension for general (separated or coupled) boundary conditions. Finally let me mention the work of Pivovartschik [53] 2005, where an Ambarzumian theorem is obtained on a graph with Schrödinger operators on its edges.

The proof of the above statements relies upon the extremal properties of the first eigenvalue and its eigenfunction. If we consider the one-dimensional Dirac-operator, however, there is no first eigenvalue. The

Dirac operator looks like

$$Lu = \begin{pmatrix} V(x)+m & d/dx \\ -d/dx & V(x)-m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad x \in [0,\pi] \tag{2.12}$$

where V is the potential and m is the mass of the particle.

The Neumann boundary conditions are defined by

$$u_1(0) = u_1(\pi) = 0. \tag{2.13}$$

It is known that in this case the eigenvalues come from $-\infty$ and go to ∞:

$$\lim_{n\to-\infty} \lambda_n = -\infty, \quad \lim_{n\to\infty} \lambda_n = \infty. \tag{2.14}$$

Theorem 2.1. *Horváth* [21]

Let $0 < m \le 1/2$. If $V \in C[0,\pi]$ produces the same spectrum as $V = 0$ (but with the same m), then $V = 0$.

In the proof I used the following lemma of moments with alternating signs.

Lemma 2.2. *Let $0 \le f \in C[a,b]$ and $g \in L_1[a,b]$. If the generalized moments*

$$\mu_k = \int_a^b f^k g \quad k = 0, 1, \dots \tag{2.15}$$

have alternating signs i.e. $(-1)^k \mu_k \ge 0$ then actually $\mu_k = 0$ for $k \ge 1$.

In 2004 Márton Kiss [37] removed the condition $m \le 1/2$ from the above theorem and gave a generalization for selfadjoint $n \times n$ matrix-valued operators.

Concerning the multidimensional case recall the following result of Kuznetsov [39]. Let Ω be a two- or three-dimensional bounded domain with smooth boudary. Denote by λ_n, $n \ge 0$ the eigenvalues of the Schrödinger operator with Neumann boundary conditions:

$$-\Delta u + q(x)u = \lambda u \quad \text{on } \Omega, \qquad \frac{\partial u}{\partial n}\Big|_\Gamma = 0 \tag{2.16}$$

and let $\lambda_{n,0}$ be the same set of eigenvalues with $q = 0$. Now we have

$$\lambda_0 = \lambda_{0,0}, \ \sum(\lambda_n - \lambda_{n,0}) \text{ convergent } \Rightarrow q = 0 \text{ a.e.} \tag{2.17}$$

Once $\int_\Omega q = 0$ is obtained, $q = 0$ easily follows from the minimization of the quadratic form $(-\Delta u + qu, u)$ as we have seen in the one-dimensional case. So the key point is how to deduce $\int_\Omega q = 0$ from the convergence of $\sum(\lambda_n - \lambda_{n,0})$.

Finally we mention the paper [19] of Harrell, where Schrödinger operators on spheres of dimension $m \geq 1$ is considered. It is proved that if the real potential belongs to L_∞ and if $\lambda_n - \lambda_{n,0} \to 0$ then the potential vanishes a.e.

2.2. **Uniqueness theorems using several spectra.**

The Ambarzumian theorem states that the zero potential can be identified by its Neumann spectrum. Is it true that any potential can be recovered from one single spectrum? The answer is clearly negative, since the potential reflected to the midpoint gives the same eigenvalues. More precisely the substitution $x \mapsto \pi - x$ gives that

$$\sigma(\alpha, \beta; q) = \sigma(-\beta, -\alpha; q(\pi - x)) \tag{2.18}$$

and consequently $\sigma(\alpha, \pi - \alpha)$ is the same for $q(x)$ and $q(\pi - x)$ (in particular the Dirichlet spectrum and the Neumann spectrum is invariant to this reflexion of the potential to the midpoint). On the other hand for symmetrical potentials the knowledge of $\sigma(\alpha, \pi - \alpha)$ is enough for uniqueness of the potential. It is proved for the Dirichlet and Neumann spectrum in Borg [8], the general case is given in Levinson [44].

The above mentioned paper of Borg is of fundamental importance. His main discovery is the statement that in most cases two spectra are needed for the unique identification of a (non-symmetric) potential. In other words, the Ambarzumian theorem describes an exceptional case. In what follows by the knowledge of $\lambda \in \sigma(\alpha, \beta)$ we mean the knowledge of the value λ and the parameters $\alpha,\ \beta$.

It is worth mentioning that the asymptotic distribution of the eigenvalues gives some (not complete) information about the boundary conditions. It is known that for $0 < \alpha, \beta < \pi$

$$\lambda_n = n^2 + \frac{2}{\pi}(\cot\beta - \cot\alpha) + \frac{1}{\pi}\int_0^\pi q + \mathbf{o}(1),\ n \geq 0,\ n \to \infty; \tag{2.19}$$

for $\alpha = 0$, $0 < \beta < \pi$ we have

$$\lambda_n = (n + 1/2)^2 + \frac{2}{\pi}\cot\beta + \frac{1}{\pi}\int_0^\pi q + \mathbf{o}(1),\ n \geq 0,\ n \to \infty; \tag{2.20}$$

and for $\alpha = \beta = 0$

$$\lambda_n = n^2 + \frac{1}{\pi}\int_0^\pi q + \mathbf{o}(1),\ n \geq 0,\ n \to \infty. \tag{2.21}$$

So we can decide from the knowledge of a whole spectrum whether Dirichlet boundary condition is applied in both ends, in one of the endpoints or in none of the endpoints.

Theorem *Borg* [8]
a. If $0 < \alpha < \pi$ *then* $\sigma(0,0) \cup \sigma(\alpha,0)$ *uniquely identifies the potential* $q \in L_1(0,\pi)$ *and no proper subset has the same property.*
b. We call reduced spectrum $\sigma_R(\alpha,\beta)$ *the set after deleting the first eigenvalue of* $\sigma(\alpha,\beta)$ *Now if* $0 < \alpha, \beta < \pi$ *, then* $\sigma(0,\beta) \cup \sigma_R(\alpha,\beta)$ *uniquely identifies* $q \in L_1$ *and no proper subset has the same property.*

The non-uniqueness means that if we apply a small perturbation on one of the eigenvalues from the union, having the other eigenvalues unchanged, this corresponds to another L_1-potential. The uniqueness part has been extended in

Theorem *Levinson* [43]
If $\sin(\alpha_2 - \alpha_1) \neq 0$ *then* $\sigma(\alpha_1,\beta) \cup \sigma(\alpha_2,\beta)$ *uniquely determines* $q \in L_1$.

As we see from point b) of the above cited Theorem of Borg, the knowledge of the first eigenvalue is crucial in the Ambarzumian theorem. Indeed, if $\lambda_0 = 0$ is known then any infinite subset of the Neumann spectrum $\sigma(\pi/2,\pi/2)$ is enough to identify the zero potential; if λ_0 is unknown, then we need $\sigma(0,\pi/2) \cup \sigma_R(\pi/2,\pi/2)$ and less is not enough. Thus, the first Neumann eigenvalue carries the same amount of information as almost two spectra!

The statement that two spectra are needed to define the potential suggests that one spectrum defines "half" of the potential. For example, if half of the potential is known, one spectrum defines the other half of q. This is indeed true:

Theorem *Hochstadt and Lieberman* [20]
The potential q *on* $[0,\pi/2]$ *and* $\sigma(\alpha,\beta)$ *uniquely determines* $q \in L_1(0,\pi)$.

Later on, Hald [18] observed that if the left half of q is given then the right boundary condition can depend on q:

Theorem *Hald* [18]
$\sigma(\alpha, \beta_1; q_1) = \sigma(\alpha, \beta_2; q_2)$ and $q_1 = q_2$ on $[0, \pi/2]$ implies $q_1 = q_2$ a.e. on $[0, \pi]$.

The left boundary condition, however, must be independent of q:

Theorem *del Rio* [14]
Let $0 < \varepsilon < \pi$. Then there are different potentials q_1, $q_2 \in L_1(0, \pi)$ such that $\sigma(\alpha_1, 0; q_1) = \sigma(\alpha_2, 0; q_2)$ and $q_1 = q_2$ on $[0, \pi - \varepsilon]$.

So in this context one spectrum determines much less than one half of the potential. On the other hand two spectra with potential dependent boundary conditions can be enough for uniqueness:

Theorem *Marchenko* [48]
Let $\sin(\alpha_1 - \alpha_1') \neq 0$, $\sin(\alpha_2 - \alpha_2') \neq 0$. Now if $\sigma(\alpha_1, \beta_1; q_1) = \sigma(\alpha_2, \beta_2; q_2)$ and $\sigma(\alpha_1', \beta_1; q_1) = \sigma(\alpha_2', \beta_2; q_2)$ then $q_1 = q_2$ a.e. (and $\beta_1 = \beta_2$, $\alpha_1 = \alpha_2$, $\alpha_1' = \alpha_2'$).

We see that in every uniqueness result mentioned the boundary condition in one of the endpoints if fixed. And indeed, the knowledge of $\sigma(\alpha_1, \beta_1)$ and $\sigma(\alpha_2, \beta_2)$ does not imply uniqueness in general. It is observed in Pierce [52] that in many cases there is an uncountable family of potentials q having the same Dirichlet and Neumann spectra.

Return to the original concept, where the knowledge of eigenvalues includes the knlowledge of the corresponding boundary parameters. Some recent results are obtained in Gesztesy and Simon 2000 [16] and del Rio, Gesztesy and Simon 1997 [15], along the philosophy of the Hochstadt-Lieberman theorem:

Theorem [16], [15]
a) The knowledge of more that one half of the potential requires proportionally less information from one spectrum: if $q \in L_1(0, \pi)$ is known on $(0, a)$ with some $\pi/2 < a < \pi$ and if $S \subset \sigma = \sigma(\alpha, \beta)$ contains $2(1 - a/\pi)$-th part of σ in the sense that

$$\#\{\lambda \in S : \lambda \leq t\} \geq 2(1 - a/\pi)\#\{\lambda \in \sigma : \lambda \leq t\} + a/\pi - 1/2$$

for large t then q on $(0, a)$ and S determine q on $(0, \pi)$.

b)Two third of three spectra implies uniqueness: let $\sigma_i = \sigma(\alpha_i, \beta)$ *; if* $S \subset \sigma_1 \cup \sigma_2 \cup \sigma_3$ *satisfies* $\#\{\lambda \in S : \lambda \leq t\} \geq 2/3\#\{\lambda \in \sigma_1 \cup \sigma_2 \cup \sigma_3 : \lambda \leq t\}$ *for large* t *then the set* S *and the knowledge of* β *uniquely determine* q*.*

c) Let $\sigma_N = \sigma(\pi/2, \beta)$, $\sigma_D = \sigma(0, \beta)$, $S \subset \sigma_N \cup \sigma_D$*. Suppose that the relative densitiy of* S $\sigma_N \cup \sigma_D$ *is at least* $1 - a/\pi$ *i.e.* $\#\{\lambda \in S : \lambda \leq t\} \geq (1 - a/\pi)\#\{\lambda \in \sigma_N \cup \sigma_D : \lambda \leq t\}$ *for large* t*. Then* S *and* q *on* $(0, a)$ *uniquely determine* q *a.e. on* $(0, \pi)$*.*

d) $\sigma(\alpha_1, \beta)$*, half of* $\sigma(\alpha_2, \beta)$ *and* q *on* $(0, \pi/4)$ *determines* q*.*

A similar recent statement of Ramm, formulated to the segment $[0, 1]$ rather than $[0, \pi]$:

Theorem *Ramm* [58]
$\lambda_{m(n)} \in \sigma(0, 0)$ *and* q *on* $(b, 1)$ *implies uniqueness of* q *on* $(0, 1)$ *if*

$$\frac{m(n)}{n} = \frac{1}{2b} + \gamma_n, \quad \sum |\gamma_n| < \infty.$$

Two results on eigenvalues of infinitely many spectra:

Theorem *McLaughlin and Rundell* [49]
The first (or k-th) eigenvalues from infinitely many spectra $\sigma(\alpha_i, 0)$ *determine the potential* $q \in L_2(0, \pi)$

Theorem *del Rio, Gesztesy, Simon* [15]
Denote $x_+ = \max(x, 0)$*. If* q, $q^* \in L_1(0, \pi)$ *and there are common eigenvalues* $\lambda_n \in \sigma(\alpha_n, \beta; q) \cap \sigma(\alpha_n, \beta; q^*)$ *such that*

$$\sum \frac{(\lambda_n - n^2/4)_+}{1 + n^2} < \infty$$

then the potential can uniquely recovered by the eigenvalues λ_n*.*

The above list of statements consists mainly of various sufficient conditions on some sets of eigenvalues which imply the unique recovery of the operator. The following necessary and sufficient condition gives a unified treatment of the problem and most of the above-listed results appear as a special case of this statement.

Theorem 2.3 (Horváth [23]). *Let $1 \le p \le \infty$, $q \in L_p(0,\pi)$, $0 \le a < \pi$ and $\lambda_n : n \ge 1$ be arbitrary real numbers with $\lambda_n \not\to -\infty$. Then the following statements are equivalent:*

a) There are no different potentials q_1, $q_2 \in L_p(0,\pi)$ for which $q_1 = q_2$ on $(0,a)$ and

$$\lambda_n \in \sigma(\alpha_n, 0; q_1) \cap \sigma(\alpha_n, 0; q_2) \quad \forall n \ge 1$$

holds with some $\alpha_n \in \mathbf{R}$

b) The exponential system

$$e(\Lambda) = \left\{ e^{\pm 2i\mu x}, e^{\pm 2i\sqrt{\lambda_n}x} : n \ge 1 \right\}$$

is closed in $L_p(a-\pi, \pi-a)$, where $\mu \ne \pm\sqrt{\lambda_n}$ is arbitrary.

Remarks

a) A system $\{\varphi_n : n \ge 0\} \subset L_{p'}$, $1/p + 1/p' = 1$ is called closed in L_p, if $h \in L_p$, $\int_0^\pi h\varphi_n = 0$ implies $h = 0$. If $p > 1$, it is equivalent to the completeness of the φ_n in $L_{p'}$

b) Every value $\lambda_n \in \mathbf{R}$ can be considered as an eigenvalue from some spectra $\sigma(\alpha_n, 0)$. Indeed, the initial conditions $v(\pi) = 0$, $v'(\pi) = -1$ uniquely define a solution $v(x)$ of $-v" + qv = \lambda_n v$; now let

$$\alpha_n = -\text{arc ctg}\frac{v'(0)}{v(0)} \quad (\alpha_n = 0 \text{ if } v(0) = 0).$$

Then $\lambda_n \in \sigma(\alpha_n, 0; q)$.

c) The knowledge of the boundary parameter $\beta = 0$ is essential in the above theorem; however there is no need to fix the parameters α_n. If the α_n are unknown, every value λ_n can be an eigenvalue for any $q \in L_p$; but the statement that λ_n is a common eigenvalue of q_1 and q_2 carries real information, namely that the parameter α_n must be the same for q_1 and q_2.

d) There are two quite easy ways to check closedness of $e(\Lambda)$. A sufficient condition of Levinson says that if

$$N(r) = \int_1^r \frac{n(t)}{t} dt, \quad n(r) = \sum_{|\mu_n| \le r} 1$$

then for $p < \infty$

$$\overline{\lim_{r\to\infty}} \left[N(r) - 2d/\pi r + 1/p' \ln r \right] > -\infty$$

implies the closedness of $\{e^{i\mu_n x}\}$ in $L_p(-d, d)$. A necessary and sufficient condition for the closedness checks the basis property of a subset of $e(\Lambda)$. For example, if the λ_n run over $\sigma(0,0) \cup \sigma(\alpha, 0)$ then

$\sqrt{\lambda_n} = n/2 + \mathbf{o}(1)$ $n \geq 1$, thus $\left\{e^{\pm 2i\mu x}, e^{\pm 2i\sqrt{\lambda_n}x} : n \geq 1\right\}$ is Riesz-basis in $L_2(-\pi,\pi)$; for more details see [23].

e) The non-uniqueness part is proved in the following stronger form: if $e(\Lambda)$ is not closed, then for every $q_1 \in L_p$ there is a different $q_2 \in L_p$ such that $q_1 = q_2$ on $(0,a)$ and $\lambda_n \in \sigma(\alpha_n, 0; q_1) \cap \sigma(\alpha_n, 0; q_2)$ for some values α_n.

f) The closedness of $e(\Lambda)$ in $L_p(a-\pi, \pi-a)$ is equivalent to the closedness of

$$C(\Lambda) = \{\cos 2\mu x, \cos 2\sqrt{\lambda_n}x : n \geq 1\}$$

in $L_p(0, \pi - a)$, see in [23].

Heuristic proof of Theorem 2.3 (in case $a = 0$)

Let the λ_n be common eigenvalues of q and q^*. This means that

$$-y_n'' + qy_n = \lambda_n y_n, \quad -y_n^{*}{}'' + q^* y^* = \lambda_n y_n^*.$$

Multiply the first equation by y_n^*, the second one by y_n, substract and integrate over $[0,\pi]$:

$$\int_0^\pi (q - q^*) y_n y_n^* = \int_0^\pi (y_n'' y_n^* - y_n y_n^{*}{}'') = 0 \tag{2.22}$$

since y_n and y_n^* satisfy the same boundary conditions. It is known that for large n both y_n and y_n^* behave asymptotically like $\sin\sqrt{\lambda_n}x/\sqrt{\lambda_n}$. If we substitute in (2.22) y_n and y_n^* by $\sin\sqrt{\lambda_n}x/\sqrt{\lambda_n}$, we obtain

$$0 = \int_0^\pi (q^*(x) - q(x)) 2\sin^2\sqrt{\lambda_n}x\,dx = \int_0^\pi (q^*(x) - q(x))(1 - \cos 2\sqrt{\lambda_n}x)\,dx.$$

Supposing $\lambda_n \to +\infty$ we get from the Riemann-lemma that

$$0 = \int_0^\pi (q^* - q), \quad 0 = \int_0^\pi (q^*(x) - q(x)) \cos 2\sqrt{\lambda_n}x\,dx.$$

Thus the uniqueness (i.e. that $q^* - q$ must be zero) means that the system $\{1, \cos 2\sqrt{\lambda_n}x : n \geq 1\}$ is closed in $L_p(0,\pi)$.

The rigorous proof starts with the Povzner-Levitan representation

$$y_n(x) = \frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}} + \int_0^x K(x,t)\frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}dt$$

The main tool in proving the uniqueness part is the Weyl-Titchmarsh m-function. In the non-uniqueness part the existence of different potentials having some common eigenvalues is based on the following

Lemma 2.4. *Let B_1 and B_2 be Banach spaces. For every $q \in B_1$ a continuous linear operator*

$$A_q : B_1 \to B_2$$

be defined so that for some $q_0 \in B_1$

$$A_{q_0} : B_1 \to B_2 \text{ is an (onto) isomorphism,} \tag{2.23}$$

and the mapping $q \to A_q$ be Lipschitzian in the sense that

$$\|(A_{q^*} - A_q)h\| \leq c(q_0)\|q^* - q\|\|q\|\forall h, q, q^* \in B_1, \|q\|, \|q^*\| \leq 2\|q_0\|, \tag{2.24}$$

the constant $c(q_0)$ being independent of q, q^* and h. Then the set $\{A_q(q - q_0) : q \in B_1\}$ contains a ball in B_2 with center at the origin.

So far we supposed $\beta = 0$ i.e. at $x = 0$ we imposed Dirichlet boundary condition. If $\sin\beta \neq 0$ then the situation is more complicated. I was not able to find a necessary and sufficient condition for the unique recovery of the potential. The following statement is a sufficient condition (which implies most of the formerly known results of this type and which is best possible in some sense) and a weaker necessary condition. The gap between the two is the topic of further study.

Theorem 2.5. *Let $1 \leq p \leq \infty$, $q \in L_p(0, \pi)$, $\sin\beta \neq 0$, $\lambda_n \in \sigma(q, \alpha_n; \beta)$, $\lambda_n \not\to -\infty$ and $0 \leq a < \pi$. If the set*

$$e_0(\Lambda) = \left\{e^{\pm 2i\sqrt{\lambda_n}x} : n \geq 1\right\} \tag{2.25}$$

is closed in $L_p(a - \pi, \pi - a)$ then q on $(0, a)$ and the eigenvalues λ_n determine q in L_p.

Theorem 2.6. *Let $\sin\beta \neq 0$, $0 \leq a < \pi$, $1 \leq p \leq \infty$ and λ_n, $n \geq 1$ be different real numbers with $\lambda_n \not\to -\infty$. Suppose that there exists $h \in L_p(a, \pi)$ such that*

$$\int_a^\pi h \neq 0 \quad \text{but} \quad \int_a^\pi h(x)[v^2(x, \lambda_n) - 1/2\sin^2\beta]\,dx = 0 \quad \forall n.$$

and that the system

$$e(\Lambda) = \left\{e^{\pm 2i\mu x}, e^{\pm 2i\sqrt{\lambda_n}x}\right\}$$

is not closed in $L_p(a-\pi, \pi-a)$*, where* $\mu \neq \pm\sqrt{\lambda_n}$*. Then for every* $q_1 \in L_p(0,\pi)$ *there exists a different* $q_2 \in L_p(0,\pi)$ *such that* $q_1 = q_2$ *on* $(0,a)$ *and* $\lambda_n \in \sigma(\alpha_n, \beta; q_1) \cap \sigma(\alpha_n, \beta; q_2)$*.*

Remark

The sufficient condition given above can not be weakened. Indeed, if q_1 and q_2 are the characteristic function of the left and right half-interval, then for the set of all common eigenvalues of q_1 and q_2, the system $e_0(\Lambda)$ has deficiency 1 in $L_p(-\pi,\pi)$, $1 \leq p < \infty$. In other words, the system $e_1(\Lambda) = \left\{e^{2i\mu x}, e^{\pm 2i\sqrt{\lambda_n}x} : n \geq 1\right\}$ with $\mu \neq \pm\sqrt{\lambda_n}$ is closed in $L_p(-\pi,\pi)$. On the other hand, in the standard situation of the classical Borg theorem i.e. if $\{\lambda_n : n \geq 1\} = \sigma(0,\beta) \cup \sigma_R(\alpha,\beta)$ the system $e_0(\Lambda)$ is not closed, it has codimension 1. Thus the Borg theorem can not be proved within this general framework.

2.3. **The case of infinite interval.**

The eigenvalues are defined by the system

$$(2.26)\quad -y" + qy = \lambda y, \quad y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y \in L_2(0,\infty).$$

The set of eigenvalues is denoted by

$$\sigma_p(\alpha) = \sigma_p(\alpha; q)$$

and is called the point spectrum of the operator. As the notation suggests, σ_p is not the whole spectrum in general (in contrast with the finite interval case). For example $q \geq -K$, $q(x) \to +\infty$ yields discrete spectrum while for integrable potentials the spectrum is never purely discrete. The second difference compared with the finite interval case is that for the operator on a finite interval, there is a simple asymptotical distribution for the eigenvalues and eigenfunctions: they behave for large λ as the eigenvalues and eigenfunctions of the free operator ($q = 0$). If the spectrum is discrete, such a reference potential does not exists in the half-line case and it is hard to describe the distribution of the eigenvalues. Due to these difficulties the inverse Sturm-Liouville theory is much less elaborated on infinite intervals. The fact that two spectra implies uniqueness on the half-line is given in

Theorem *Marchenko* [48]

Let $q_1, q_2 \in L_1^{loc}(0,\infty)$*,* $q_1 \geq -K$ $q_2 \geq -K$ *for some* $K > 0$*,* $q_1(x) \to +\infty$*,* $q_2(x) \to +\infty$ *as* $x \to +\infty$*. If* $\sigma(\alpha_1; q_1) = \sigma(\alpha_2; q_2)$ *and* $\sigma(\beta_1; q_1) = \sigma(\beta_2; q_2)$ *for some* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *with* $\sin(\alpha_1 - \beta_1) \neq 0$*,* $\sin(\alpha_2 - \beta_2) \neq 0$ *then* $q_1 = q_2$ *a.e. (and* $\alpha_1 = \alpha_2$*,* $\beta_1 = \beta_2$*).*

On the whole line e.g. $q \geq -K$, $q(x) \to +\infty$ if $|x| \to +\infty$ implies the discreteness of the spectrum. Since $q(x)$ and $q(-x)$ generate the same spectrum, we have no uniqueness from the knowledge of the eigenvalues. The idea that the eigenvalues determine half of the potential is neither correct since in **R** the midpoint can not be defined. For example let $q(x) = q(-2-x)$ for $x > 0$ and q be arbitrary (nonsymmetrical) in $(-2, 0)$. Now the different potentials $q(x)$ and $q(-2-x)$ are identical for $x > 0$ and generate the same spectrum. On the other hand we have

Theorem *Khodakovsky* [35]
Let $q \in L_1^{loc}(\mathbf{R})$ be given such that the operator $Ly = -y" + qy$ is bounded from below and

$$q(-x) \geq q(x) \quad x \geq 0.$$

Then q on $(0, \infty)$ and the spectrum of L uniquely determine q on $\mathbf{R}$.

For a former version see Gesztesy and Simon [16].

Now consider operators on the half-line with not purely discrete spectrum. Suppose that the potential is decaying at infinity in the sense that

$$\int_0^\infty x|q(x)|\,dx < \infty.$$

It is known that in this case the operator defined by

$$-y" + qy = \lambda y, \quad y(0)\cos\alpha + y'(0)\sin\alpha = 0$$

has only finitely many eigenvalues (if any) and they are all negative. If only $q \in L_1(0, \infty)$ is known then the eigenvalues are still negative and can accumulate only at zero, see e.g. Neumark [50].

Concerning the inverse eigenvalue problem we have the following statements:

Theorem 2.7 (Horváth [24]). *Let $q \in L_1(0, \infty)$ and consider different numbers $\lambda_n = -k_n^2$, $\inf k_n > 0$. If*

$$\sum_n \frac{1}{k_n} = \infty \tag{2.27}$$

then the eigenvalues uniquely determine q a.e. on $(0, \infty)$ i.e. there are no different potentials $q_1 \neq q_2$ in $L_1(0, \infty)$ such that $\lambda_n \in \sigma(\alpha_n; q_1) \cap \sigma(\alpha_n; q_2)$ $\forall n$ with some parameters α_n.

It is not known whether the condition (2.27) is necessary. However, if we require a small exponential decay in the potential, this condition becomes necessary and sufficient:

Theorem 2.8 (Horváth [24]). *For $\delta > 0$ define*

$$C_\delta = \{q : \int_0^\infty |q(x)|e^{\delta x} dx < \infty\}.$$

Consider the numbers $\lambda_n = -k_n^2$, $\inf k_n > 0$. If

$$\sum_n \frac{1}{k_n} < \infty$$

then there is no uniqueness: for every $q_1 \in C_\delta$ there is a different $q_2 \in C_\delta$ such that $\lambda_n \in \sigma(\alpha_n; q_1) \cap \sigma(\alpha_n; q_2)$ $\forall n$ with some parameters α_n.

Remark The condition (2.27) holds if and only if the system

$$\{e^{i\sqrt{\lambda_n}x}\} = \{e^{-k_n x}\}$$

is closed in $L_1(0, \infty)$. Thus there is a close analogy with the finite interval case: uniqueness is again described in terms of closedness of exponential systems.

2.4. **Extremal properties of the eigenvalues.**

Consider the Dirichlet eigenvalue problem on the interval $[0, \pi]$:

$$-y" + q(x)y = \lambda y \text{ on } [0, \pi], \quad y(0) = y(\pi) = 0.$$

As it has already been remarked, the eigenvalues satisfy

$$\lambda_n = n^2 + \frac{1}{\pi} \int_0^\pi q + \mathbf{o}(1) \quad n \geq 1, \ n \to \infty$$

whenever $q \in L_1(0, \pi)$.

This means that we know pretty well the position of the large eigenvalues. The question considered here is the distribution of the first few eigenvalues. For general potentials no restrictions can be true about the first N eigenvalues, N being arbitrarily large. Indeed, the Dirichlet spectra of all potentials $q \in L_2(0, \pi)$ run over the set of all strictly increasing sequences $\lambda_1 < \lambda_2 < \dots$ having the representation

$$\lambda_n = n^2 + c + \gamma_n, \quad c \in \mathbf{R} \text{ fixed }, \ \sum \gamma_n^2 < \infty,$$

see in Pöschel and Trubowitz [54]. Consequently for every strictly increasing sequence of N real numbers $\lambda_1, \dots, \lambda_N$ there are L_2-potentials for which $\lambda_1, \dots, \lambda_N$ are the first N Dirichlet eigenvalues. On the other hand, under some conditions on the potential we find interesting relations between the eigenvalues.

The first eigenvalue gap

A function $f : [0, \pi] \to \mathbf{R}$ is called single-well if there exists a point $0 \leq a \leq \pi$ such that f is decreasing in $[0, a]$ and increasing in $[a, \pi]$. The function f is single-barrier if it is first increasing, then decreasing. The point a is the transition point.

Concerning the first eigenvalue gap $\lambda_2 - \lambda_1$ a lower bound is proved in

Theorem *Ashbaugh and Benguria* [4]
If the potential q is symmetric single-well then for the first Dirichlet eigenvalue gap

$$\lambda_2 - \lambda_1 \geq 3. \tag{2.28}$$

If single-well is substituted by single-barrier then

$$\lambda_2 - \lambda_1 \leq 3. \tag{2.29}$$

In both cases equality occurs if and only if the potential is constant.

They conjectured that the symmetry requirement can be removed, e.g. convexity of the potential would imply (2.28). Later this conjecture has been verified by Lavine [41], who also proved the estimate $\mu_2 - \mu_1 \geq 1$ for the first two Neumann-eigenvalues, if the potential is convex. Another extension is given in Horváth [22]: for nonsymmetric single-well potentials (2.28) remains true if the transition point is the midpoint, $a = \pi/2$. Abramovich [1] proved (2.29) (resp. (2.28)) provided that the potential is symmetric on $[0, \pi]$ and symmetric single-well (resp. single-barrier) on $[0, \pi/2]$. Equality occurs only for constant potentials. For another results of this type see Huang and Tsai [28].

Remark that if $\Omega \subset \mathbf{R}^n$ is a bounded convex domain and V is a convex potential then for the first gap of the problem

$$-\Delta u + Vu = \lambda u \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

the inequalities

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} \quad d = \text{ diameter of } \Omega$$

is proved in Yu and Zhong [61].

Eigenvalue ratios

Consider again the Dirichlet eigenvalue problem

$$-y" + q(x)y = \lambda y \text{ on } [0, \pi], \quad y(0) = y(\pi) = 0.$$

If $q \geq 0$ then $\lambda_1 \geq 1$ and it turns out that $\lambda_2/\lambda_1 \leq 4$, $\lambda_n/\lambda_1 \leq n^2$ and, in the most general form

Theorem *Ashbaugh and Benguria* [5]
If $q \geq 0$ then the Dirichlet eigenvalues satisfy

$$\frac{\lambda_n}{\lambda_m} \leq \left\lceil \frac{n}{m} \right\rceil^2 \quad n \geq m \geq 1 \tag{2.30}$$

where $\lceil x \rceil$ is the ceiling function: the smallest integer not smaller than x. If $m|n$, equality occurs only for the zero potential; if n is not a multiple of m, we have strict inequalities in (2.30) but the upper bounds are best possible.

Concerning extensions of this result to general Sturm-Liouville operators $-(py')' + qy = \lambda wy$ and/or to general boundary conditions see Huang and Law [29], Huang [27], Chen [11], Ashbaugh and Benguria [7], Huang and Law [30] and Kiss [38]. For other estimates of eigenvalue ratios see e.g. Keller [34] and Mahar and Willner [47].

In connection with the above theorem Ashbaugh and Benguria formulated the conjecture that if the potential is nonnegative and convex then the upper bounds in (2.30) can be substituted by $(n/m)^2$. The following statement verifies this conjecture (and more):

Theorem 2.9 (Horváth and Kiss [26])**.** *If $q \geq 0$ is single-well then*

$$\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2 \quad n \geq m \geq 1 \tag{2.31}$$

For any pair $n > m$ equality occurs only for the zero potential.

The starting point of the proof, as in many papers about eigenvalue inequalities, is the use of the Prüfer variables r and φ defined by

$$y = \frac{r}{z} \sin \varphi, \quad y' = r \cos \varphi$$

where y is the solution of $-y" + qy = z^2 y$, $z > 0$, $y(0) = 0$, $y'(0) = 1$. The key point in proving (2.31) is the discovery of new monotonicity properties of the Prüfer variables: if $q \geq 0$ is decreasing then

a. $x \mapsto \varphi(x, z)$ is strictly increasing
b. $x \mapsto r(x, z)$ is increasing between $\varphi^{-1}(k\pi)$ and $\varphi^{-1}(k\pi + \pi/2)$,

decreasing between $\varphi^{-1}(k\pi+\pi/2)$ and $\varphi^{-1}(k\pi+\pi)$ and the consecutive local maxima $r(\varphi^{-1}(k\pi+\pi/2))$ are decreasing
c. $z \mapsto \varphi(x,z)/z$ is increasing.

Remark that (2.31) follows directly from property c.

The counterpart of Theorem 2.9 in case of the real line is given in

Theorem 2.10. *Horváth and Kiss* [25]

Consider the Schrödinger operator on the real line. If the potential q is nonnegative, single-well and

$$\lim_{|x|\to\infty} q(x) = +\infty$$

then the spectrum is discrete and the eigenvalues satisfy

$$\frac{\lambda_n}{\lambda_m} < \left(\frac{n}{m}\right)^2 \quad n \geq m \geq 1. \tag{2.32}$$

So the inequalities are always strict; it is not known whether these estimates are exact.

In the multidimensional case much less is known. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and consider the eigenvalues of the Dirichlet Laplacian on Ω:

$$-\Delta u = \lambda \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

In [40] Payne, Pólya and Weinberger formulated (among others) the famous conjecture that the first eigenvalue ratio λ_2/λ_1 is maximal if and only if Ω is a ball. The conjecture has been proved in Ashbaugh and Benguria [6]. In case $n = 2$ this is related to the famous question of Kac [33], namely that one can hear whether the shape of a drum is a circle. Another developments in this direction can be found e.g. in Ashbaugh [3]

3. Inverse scattering with fixed energy

In this section the fundamental notations and ideas described in Example 3, Subsection 1.2 of Section 1 are used. We suppose that the spherically symmetrical potential

$$V(x) = q(r), \quad r = |x|$$

satisfies

$$\int_0^\infty r|q(r)|\,dr < \infty.$$

This means that the potential decays at infinity, hence for sufficiently large $a > 0$ the effect of $q(r)$, $r > a$ to the sequence of phase shifts is negligible, much less than the effect of noise in the input data. In other words, it is hopeless to get information about the "tail" of the potential if the phase shifts are taken from real measurements. Thus we suppose that the potential is of compact support,

$$q(r) = 0 \text{ if } r > a.$$

Using the notation $\varphi_n(r) = \sqrt{r}\psi_n(r)$ (see Section 1), we get for $k = 1$:

$$\varphi_n" - \frac{(n+1)n}{r^2}\varphi_n + (1 - q(r))\varphi_n = 0, \tag{3.1}$$

$$\varphi_n(r) = c_n r^{n+1}(1 + \mathbf{o}(1)), \quad r \to 0+ \tag{3.2}$$

$$\varphi_n(r) = d_n \sin(r - n\pi/2 + \delta_n) + \mathbf{o}(1), \quad r \to \infty. \tag{3.3}$$

Concerning the inverse problem of identifying the operator from the phase shifts, Ramm [56] realized that a subset $\{\delta_n : n \in L\}$ is enough for the unique recovery of q with $rq(r) \in L_2(0, a)$ and $q(r) = 0$ for $r > a$ if

$$\sum_{n\in L,\, n\neq 0} \frac{1}{n} = \infty. \tag{3.4}$$

This statement has been extended in

Theorem 3.1. *Horváth* [24]
Let $q(r) = 0$ for $r > a$.
a. If $rq(r) \in L_1(0, a)$ then (3.4) implies the unique recovery of q.
b. If $r^{1-\varrho}q(r) \in L_1(0, a)$ for some $\varrho > 0$ then (3.4) is necessary and sufficient for uniqueness.

The method of the proof is essentially new: it is based on a connection found between this inverse scattering problem and an inverse eigenvalue problem. Consider the variable substitution

$$x = \ln \frac{a}{r}. \tag{3.5}$$

This transforms the segment $r \in (0, a]$ onto the half-line $x \in [0, \infty)$. The differential equation (3.1) (for $r \leq a$) is transformed into the form

$$-y_n"(x) + Q(x)y_n(x) = -(n + 1/2)^2 y_n(x) \quad x \in [0, \infty) \tag{3.6}$$

with

$$y_n(x) = r^{-1/2}\varphi_n(r), \quad Q(x) = r^2(q(r) - 1). \tag{3.7}$$

Now $rq(r) \in L_1(0, a)$ means that $Q(x) \in L_1(0, \infty)$. Since φ_n behaves like r^{n+1} at $r \to 0+$, this gives an exponential decay $e^{-(n+1/2)x}$ for $y_n(x)$ at infinity. This means by (3.6) that y_n is an eigenfunction of $Ly = -y" + Qy$ with eigenvalue $-(n + 1/2)^2$ i.e.

$$\lambda_n = -(n + 1/2)^2 \in \sigma_p(\alpha_n; Q), \quad \alpha_n = -\text{arc ctg}\, \frac{y_n'(0)}{y_n(0)}. \tag{3.8}$$

Finally, α_n can be explicitly expressed by the phase shift δ_n. Indeed, for $r > a$ the potential is zero, thus φ_n is a linear combination of $\sqrt{r}J_{n+1/2}(r)$ and $\sqrt{r}Y_{n+1/2}(r)$ where J and Y are Bessel functions. Taking into account the asymptotical expressions

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos(r - \nu\pi/2 - \pi/4) + \mathbf{O}(r^{-3/2}) \quad r \to \infty, \tag{3.9}$$

$$Y_\nu(r) = \sqrt{\frac{2}{\pi r}} \sin(r - \nu\pi/2 - \pi/4) + \mathbf{O}(r^{-3/2}) \quad r \to \infty \tag{3.10}$$

we get

$$\varphi_n(r) = c_n\sqrt{r} \left[\cos\delta_n \cdot J_{n+1/2}(r) - \sin\delta_n \cdot Y_{n+1/2}(r)\right], \; r \geq a. \tag{3.11}$$

By this representation we are able to express $\varphi_n'(a)/\varphi_n(a)$ and hence $y_n'(0)/y_n(0)$ from δ_n. So knowledge of the phase shift δ_n means the knowledge of the boundary parameter α_n in (3.8). Thus, reconstructing q from the δ_n, $n \in L$ and reconstructing Q from the eigenvalues $\lambda_n = -(n + 1/2)^2 \in \sigma_p(\alpha_n; Q)$ are the same problem. In this context the Müntz-type condition (3.4) appears as the closedness of the exponential system

$$\{e^{i\sqrt{\lambda_n}x} : n \in L\} = \{e^{-(n+1/2)x} : n \in L\}$$

in $L_1(0, \infty)$, so Theorems 2.7 and 2.8 are simply transformed into Theorem 3.1.

Remark Various procedures are known for the retrieval of the potential from the phase shifts. These methods are based on some a priori constraints (called Ansatz in the literature) on the potential or on the so-called input kernel in a Gelfand-Levitan type integral equation; see for example in the monography of Chadan and Sabatier [9]. No reconstruction procedure is available if only an arbitrary subset $\delta_n,\ n \in L$ is known.

Remark The fixed-energy inverse scattering problem is quite unstable. There are constructions for different stepfunction potentials having almost the same phase shifts, see e.g. Airapetyan, Ramm and Smirnova [55]. If the scattering amplitude has a perturbation $< \varepsilon$ then an estimate of order $|log\, \varepsilon|^{-1}$ for the Fourier transform of the potential perturbation is given in Ramm [57], see also [58]. This logarithmic bound tends to zero very slowly, which is another illustration of the difficulties in stability issues. However, under special a priori conditions e.g. on the potential the problem may become stable; these phenomena are not yet thoroughly investigated.

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