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A BOUND FOR RATIOS OF EIGENVALUES OF SCHRÖDINGER OPERATORS ON THE REAL LINE

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Abstract. We give upper estimates of ratios of eigenvalues of Schrödinger operators with nonnegative single-well potentials tending to infinity for large $|x|$, corresponding to previous estimates on a finite interval.

1. **Introduction.**

Consider the Schrödinger operator

$$-y'' + q(x)y = \lambda y \tag{1.1}$$

on the real line. A solution $y \neq 0$ is called an eigenfunction of (1.1) if it tends to zero as $|x|$ tends to infinity. If q tends to infinity for large $|x|$, then the spectrum consists of a growing sequence of real eigenvalues $\lambda_1, \lambda_2, \ldots$, see for example in [4]. Moreover, if $q(x)$ is nonnegative then $\lambda_1 > 0$ (see Remark after Theorem 1.1).

On a finite interval exact estimates have already been known concerning eigenvalue ratios of Schrödinger operators. We only mention here the fundamental result of Ashbaugh and Benguria [1] who proved the bound

$$\frac{\lambda_n}{\lambda_m} \leq \left\lceil \frac{n}{m} \right\rceil^2 \text{ for } m < n \tag{1.2}$$

for nonnegative potentials (with Dirichlet boundary conditions), where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Horváth and Kiss showed [2] that for nonnegative single-well potentials

$$\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m} \right)^2 \tag{1.3}$$

is also true. Single-well means that there is a point $a \in [0, \pi]$ such that q is decreasing in $[0, a]$ and increasing in $[a, \pi]$ (see in [1]). In this paper we prove a corresponding statement for Schrödinger operators on the real line, namely, if $\lim_{|x|\to\infty} q(x) = +\infty$ and q is nonnegative and single-well, then

$$\frac{\lambda_n}{\lambda_m} < \left(\frac{n}{m} \right)^2. \tag{1.4}$$

The following theorem summarizes the properties of the solutions y that we are going to use:

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Theorem 1.1. *Consider the Schrödinger equation*

$$-y'' + q(x)y = \lambda y \tag{1.5}$$

on the half-line $(-\infty, 0]$ *(and suppose that* $q > 0$ *is decreasing and* $q(x) \to +\infty$ *if* $x \to -\infty$*). Then for each* $\lambda \in \mathbb{R}$ *there exists exactly one solution* y *tending to zero at* $-\infty$ *such that* $y > 0$ *near* $-\infty$ *and* $\int_{-\infty}^{0} y^2 = 1$*. This solution has the following properties:*

- $y(x)$ *and* $y'(x)$ *are positive if* $q(x) \ge \lambda$*,*
- *if* $x \to -\infty$*,* $\frac{y}{y'}$ *tends to zero.*

The proof will be given in Section 3.

Remark. Let y be an eigenfunction of (1.1). If it is positive near $-\infty$ and $\lambda \le 0$, then through the sign of y'' it must be convex. So y is increasing and positive in $\mathbb{R}$ which is impossible for an eigenfunction, thus $\lambda_1 > 0$.

2. **The Main Statement.**

Let $0 < \lambda = z^2$ and denote by $y(x,z)$ the solution of (1.1) mentioned in Theorem 1.1. Let us introduce Prüfer-type variables:

$$y(x,z) = \frac{r(x,z)}{z} \sin \varphi(x,z), \tag{2.1}$$

$$y'(x,z) = r(x,z) \cos \varphi(x,z), \tag{2.2}$$

where $r(x,z) > 0$, and we denote by prime the derivative with respect to x (and by dot the derivative with respect to z). According to the last statement of Theorem 1.1 we can assume

$$\lim_{x \to -\infty} \varphi(x,z) = 0. \tag{2.3}$$

Define further

$$\psi = \frac{\varphi}{z}. \tag{2.4}$$

An easy computation shows that for these variables the following equations hold:

$$\varphi' = z - \frac{q}{z} \sin^2 \varphi, \tag{2.5}$$

$$\frac{r'}{r} = \frac{q}{z} \sin \varphi \cos \varphi. \tag{2.6}$$

Remark. These formulae hold in the usual sense at the continuity points of q and in both half-sided senses at the jumps of q: $\varphi'_{\pm}(x,z) = z - \frac{q(x \pm 0)}{z} \sin^2 \varphi(x,z)$, and analogously for r.

One important idea of [2] was to show that the monotonicity of $\psi(x,z)$ in z implies (1.4). It proves to be useful again:

Theorem 2.1. *Let* $\lim_{x \to -\infty} q(x) = +\infty$ *and* $q(x) \ge 0$ *be monotone decreasing in* $(-\infty, x_0]$*. Then* $\dot{\psi}(x_0, z) > 0$*, so* $\psi(x_0, z)$ *is a strictly monotone increasing function in* $z > 0$*.*

The proof will be given in Section 4.

The main statement of this paper reads as follows:

Theorem 2.2. *Consider equation (1.1) with the boundary conditions*

$$\lim_{|x|\to\infty} y(x) = 0. \tag{2.7}$$

If the potential q is nonnegative, single-well and $\lim_{x\to\pm\infty} q(x) = +\infty$ then for the m-th and n-th eigenvalues with $m < n$

$$\frac{\lambda_n}{\lambda_m} < \frac{n^2}{m^2}. \tag{2.8}$$

The proof will be given in Section 5.

Remark. A well-known example of a Schrödinger operator with discrete spectrum arises when $q(x) = x^2$; then

$$-y'' + x^2 y = (2n+1)y, \quad y = e^{-x^2/2} H_n(x) \quad (n \geq 0)$$

where the H_n are the Hermite polynomials. In this case $\lambda_n = 2n - 1$, $n \geq 1$ and (2.8) is obvious.

Remark. A weaker form of estimate (2.8), with $\leq$ instead of $<$, has a simpler proof based on the previous result [2] concerning finite intervals. As it is discussed in the proof of Theorem 4.1. in Ashbaugh and Benguria [1], the eigenvalues of the problem with Dirichlet boundary condition imposed at $(-b, b)$ tend to the eigenvalues of the problem in $\mathbb{R}$ as $b \to \infty$, thus (1.3) must hold. This and the above example shows that it would be an interesting problem to find the exact upper estimate for the eigenvalue ratios. The authors express their thanks to the referees for this observation.

3. **The proof of Theorem 1.1.** It is known that there exists a solution y of (1.1), unique up to a constant multiple, satisfying $y(-\infty) = 0$ and $y \in L^2(-\infty, 0]$, see e.g. [3]. Suppose that $y(x_0) = 0$ for some x_0 with $q(x_0) \geq \lambda$. Then $y'(x_0) \neq 0$; for example let $y'(x_0) < 0$. Since $q \geq \lambda$ for $x \leq x_0$, we see from (1.1) that y is positive, convex and decreasing in $(-\infty, x_0]$, hence $y(-\infty) = +\infty$; a contradiction. So in the half-line $q \geq \lambda$ y has no zeros. We can suppose that $y > 0$ here and then it is convex; from $y(-\infty) = 0$ we get that y is strictly increasing i.e. y' is positive.

To prove the second property in Theorem 1.1 introduce the function $h = \frac{y'}{y}$. An easy calculation shows that the following equation holds:

$$h'(x, z) = q(x) - z^2 - h^2(x, z) \tag{3.1}$$

Let $N > 0$ be arbitrary. If $x_0 < -K$ with K large enough, then $q(x) - z^2 > (N+1)^2 + 1$ and $y(x) > 0$, $y'(x) > 0$ for $x < x_0$, so $h(x) > 0$. If $h(x) < N + 1$ for some $x < x_0$ then $h'(x) > 1$. Consequently $h(x') < N + 1$ and $h'(x') > 1$ for all $x' < x$; this is clearly incompatible with $h(x') > 0$. In other words, $h(x) \geq N + 1$ for all $x < x_0$ and then $h(-\infty) = +\infty$ as asserted. □

4. **The proof of Theorem 2.1.**

Lemma 4.1. *If $q(x)$ is monotone decreasing in $(-\infty, x_0]$, then (for $z > 0$) $\varphi(x_0, z)$ is a strictly monotone increasing function of x in $(-\infty, x_0]$. Moreover, $\varphi'_\pm(x, z) > 0$ for $z > 0$.*

Proof.

In [2], Lemma 3.1 we proved that there exists no interval (x_1, x_2) such that $0 < \varphi < \frac{\pi}{2}$, $\varphi'_- > 0$ on (x_1, x_2) and $\varphi'_-(x_2) = 0$. By the monotonicity of q we see that $\varphi'_+ \geq \varphi'_-$. Suppose that $\varphi_-(x^*, z) \leq 0$ for some $x^* < x_0$. Now (2.5) implies

$z^2 \le q(x^* - 0)$. Thus $q \ge z^2$ and then $0 < \varphi < \pi/2$ on $(-\infty, x^*)$. There must be an $x_1 < x^*$ with $\varphi'_-(x_1, z) > 0$; otherwise φ would be decreasing on $(-\infty, x_1)$. If x increases, φ'_- is either continuous or has upward jumps (at the discontinuities of q). Now if

$$x_2 = \sup\{t \in [x_1, x^*] : \varphi'_- > 0 \text{ on } [x_1, t)\}$$

then $\varphi'_- > 0$ on (x_1, x_2) and $\varphi'_-(x_2, z) = 0$. Since this contradicts the result of [2] quoted at the beginning of this proof, Lemma 4.1 is proved.

□

In the following formulae we sometimes write $\varphi(x)$ instead of $\varphi(x, z)$.

Lemma 4.2. *For every $x_1 < x_2$*

$$\dot\varphi(x_2) - \dot\varphi(x_1) e^{-\int_{x_1}^{x_2} \frac{q}{z} \sin 2\varphi} = \int_{x_1}^{x_2} (1 + \frac{q(t)}{z^2} \sin^2 \varphi(t)) e^{-\int_t^{x_2} \frac{q}{z} \sin 2\varphi} \, dt, \tag{4.1}$$

and the integrand on the right-hand side is in $L^1(-\infty, x_2]$.

Proof.

Differentiate equation (2.5) with respect to z:

$$\dot\varphi'(x, z) = 1 + \frac{q(x)}{z^2} \sin^2 \varphi(x) - \frac{q(x)}{z} \sin 2\varphi(x, z) \dot\varphi(x, z). \tag{4.2}$$

This is a linear differential equation in $x \to \dot\varphi(x, z)$. Multiplying both sides by $e^{-\int_x^{x_2} \frac{q}{z} \sin 2\varphi}$, we have:

$$(\dot\varphi(x, z) e^{-\int_x^{x_2} \frac{q}{z} \sin 2\varphi})' = (1 + \frac{q(x)}{z^2} \sin^2 \varphi(x, z)) e^{-\int_x^{x_2} \frac{q}{z} \sin 2\varphi}. \tag{4.3}$$

The right-hand side of this equation is in $L^1(-\infty, x_2]$, since

$$r^2(x_2)(1 + \frac{q(x)}{z^2} \sin^2 \varphi(x, z)) e^{-\int_x^{x_2} \frac{q}{z} \sin 2\varphi} = (1 + \frac{q(x)}{z^2} \sin^2 \varphi(x)) r^2(x)$$

$$= r^2 + qy^2 = z^2 y^2(x) + y'^2(x) + q(x) y^2(x) = (y'(x) y(x))' + 2z^2 y^2(x).$$

By convexity at infinity both $y(x)$ and $y'(x)$ tend to zero, so $y'(x)y(x)$ is bounded and positive for $x \to -\infty$, finally $(y'y)' = y''y + y'^2 > 0$, hence $(y'y)' \in L^1$ follows. From Theorem 1.1 $y^2 \in L^1$ also holds. Integrating (4.3) from x_1 to x_2, we get (4.1). □

Lemma 4.3. *Let $z_1 < z_2$. If $x < -K$ where K is sufficiently large, then $\varphi(x, z_1) < \varphi(x, z_2)$.*

Proof. We can suppose that $q(x) > z_1^2$, $q(x) > z_2^2$ for $x < -K$ and then $0 < \varphi(x, z_1) < \frac{\pi}{2}$ and $0 < \varphi(x, z_2) < \frac{\pi}{2}$. If $\varphi(x, z_1) - \varphi(x, z_2) \ge 0$ then $(\varphi(x, z_1) - \varphi(x, z_2))' = z_1 - z_2 - q(x)(\sin^2 \varphi(x, z_1)/z_1 - \sin^2 \varphi(x, z_2)/z_2) \le z_1 - z_2 < 0$, thus $\lim_{x \to -\infty}(\varphi(x, z_1) - \varphi(x, z_2))$ cannot be zero. This contradiction proves Lemma 4.3. □

Lemma 4.4. *$\varphi(x, z)$ is strictly monotone increasing in z.*

Proof.

Let $z_1 < z_2$, $F(x, \varphi) = z_1 - \frac{q(x)}{z_1} \sin^2 \varphi$ and $G(x, \varphi) = z_2 - \frac{q(x)}{z_2} \sin^2 \varphi$. If $x < -K$, K large enough, then $\varphi(x, z_1) < \varphi(x, z_2)$. Using

$$\varphi'(x, z_1) = F(x, \varphi(x, z_1)), \tag{4.4}$$

$$\varphi'(x, z_2) = G(x, \varphi(x, z_2)) \tag{4.5}$$

and

$$F(x, \varphi) < G(x, \varphi), \tag{4.6}$$

by comparison theorems we get $\varphi(x, z_1) < \varphi(x, z_2)$ for all x. □

Comparing this with Lemma 4.2, we get:

Corollary 4.5.

$$\dot{\varphi}(x) \geq \int_{-\infty}^{x} (1 + \frac{q(t)}{z^2} \sin^2 \varphi(t)) e^{-\int_t^x \frac{q}{z} \sin 2\varphi} \, \mathrm{d}t. \tag{4.7}$$

□

Remark. From (2.6) we can rewrite (4.7):

$$\dot{\varphi}(x) \geq \int_{-\infty}^{x} (1 + \frac{q(t)}{z^2} \sin^2 \varphi(t)) \frac{r^2(t)}{r^2(x)} \, \mathrm{d}t. \tag{4.8}$$

Corollary 4.6.

$$\dot{\psi}(x) \geq \frac{2}{r^2(x) z^2} \int_{-\infty}^{x} r^2 (\frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi). \tag{4.9}$$

Proof.

$$\dot{\psi}(x) = \frac{\dot{\varphi}(x)}{z} - \frac{\varphi(x)}{z^2} \geq$$

$$\geq \frac{1}{r^2(x) z^2} \left\{ \int_{-\infty}^{x} r^2(t) [2(z - \varphi'(t)) + \varphi'(t)] \, \mathrm{d}t - r^2(x) \varphi(x) \right\} =$$

$$= \frac{2}{r^2(x) z^2} \int_{-\infty}^{x} [r^2(t)(z - \varphi'(t)) - r(t) r'(t) \varphi(t)] \, \mathrm{d}t =$$

$$= \frac{2}{r^2(x) z^2} \int_{-\infty}^{x} r^2 (\frac{q}{z} \sin^2 \varphi - \frac{q}{z} \varphi \sin \varphi \cos \varphi).$$

□

Proof of Theorem 2.1.
Let

$$h(t) = r^2(t) \left(\frac{q(t)}{z} \sin^2 \varphi(t) - \frac{q(t)}{z} \varphi(t) \sin \varphi(t) \cos \varphi(t) \right)$$

be the integrand in (4.9); we have to show that $\int_{-\infty}^{x_0} h > 0$ if $q \geq 0$ is decreasing in $(-\infty, x_0]$. If $\varphi(x_0) \leq \pi$ then $h \geq 0$ on $(-\infty, x_0)$ and $h > 0$ for $x \to -\infty$ since $\sin^2 \varphi > \varphi \sin \varphi \cos \varphi$ for $0 < \varphi < \pi$. Thus $\dot{\psi} > 0$ follows in this case. Now suppose that

$$\varphi(x_0) = k\pi + \pi/2 + D, \quad k \geq 0 \text{ integer}, \, 0 \leq D < \pi$$

and consider the decomposition

$$\int_{-\infty}^{x_0} h = \int_{-\infty}^{\varphi^{-1}(\pi/2)} h + \sum_{j=1}^{k} \int_{\varphi^{-1}(j\pi - \pi/2)}^{\varphi^{-1}(j\pi + \pi/2)} h + \int_{\varphi^{-1}(k\pi + \pi/2)}^{x_0} h. \tag{4.10}$$

It is shown in [2] that if $q \geq 0$ is decreasing on $[0, x_0]$ and $\varphi(0) = 0$ then all the summands in (4.10) are nonnegative (with $\int_0^{\varphi^{-1}(\pi/2)}$ instead of $\int_{-\infty}^{\varphi^{-1}(\pi/2)}$) and any of them can be zero only if the potential is zero on the (open) interval of integration. In our setting, where $q \geq 0$ is decreasing in $(-\infty, x_0]$ and $\varphi(-\infty) = 0$, the same arguments show that the first summand of (4.10) is positive and the others are nonnegative. (The reason of this decomposition is that each term can be estimated from below with the help of the alteration of r. This function, as it is proved in Lemma 3.7. of [2], has the following properties: $r(\varphi^{-1}(k\pi + 3\frac{\pi}{2})) \leq r(\varphi^{-1}(k\pi + \frac{\pi}{2}))$, if $k = 0, 1, 2, \ldots$. Moreover, the function r is monotone increasing between $\varphi^{-1}(k\pi)$ and $\varphi^{-1}(k\pi + \frac{\pi}{2})$ and is monotone decreasing between $\varphi^{-1}(k\pi + \frac{\pi}{2})$ and $\varphi^{-1}((k+1)\pi)$. This leads to the desired nonnegativity. For details see [2] from Lemma 3.4. to the end of the section.) Thus $\dot{\psi} > 0$ holds also in this case. $\square$

5. **The proof of Theorem 2.2.** Let the potential $q(x)$ be monotone decreasing in $(-\infty, a]$ and monotone increasing in $[a, \infty)$. Denote by $\tilde{q}(x)$ the reverse of the potential, i.e., $\tilde{q}(x) = q(2a - x)$. Denote $y(x, z)$ the solution described in Theorem 1.1. Moreover, (with $z_n = \sqrt{\lambda_n}$) define

$$\tilde{y}(x, z_n) = (-1)^{n+1} y(2a - x, z_n), \tag{5.1}$$

$$\tilde{r}(x, z_n) = r(2a - x, z_n) \tag{5.2}$$

and

$$\tilde{\varphi}(x, z_n) = n\pi - \varphi(2a - x, z_n). \tag{5.3}$$

Then $\tilde{y}(x, z_n)$ solves (1.1) with $\tilde{q}$ instead of q and $\tilde{y}(\pm\infty) = 0$. It is also simple that

$$\tilde{y}(x, z_n) = \frac{\tilde{r}(x, z_n)}{z_n} \sin \tilde{\varphi}(x, z_n), \tag{5.4}$$

$$\tilde{y}'(x, z_n) = \tilde{r}(x, z_n) \cos \tilde{\varphi}(x, z_n). \tag{5.5}$$

Lemma 5.1.

$$\tilde{y} > 0 \textit{ near } -\infty \quad \textit{and} \quad \tilde{\varphi}(-\infty, z_n) = 0. \tag{5.6}$$

Proof. We know ([3]) that $y(x, z_n)$ has exactly $n - 1$ (simple) zeros in $\mathbb{R}$ so it has the sign $(-1)^{n-1}$ near $+\infty$ i.e. $\tilde{y}$ is positive near $-\infty$. The second statement in (5.6) can be reformulated as

$$\varphi(+\infty, z_n) = n\pi.$$

And indeed, since y has $n-1$ zeros and since $\varphi(x, z) = d\pi, , d \in \mathbf{N}$ implies by (2.5) that $\varphi'(x, z) > 0$, we get $(n-1)\pi < \varphi < n\pi$ for large x. Since $y(+\infty) = 0$, this means that $\varphi \to (n-1)\pi + 0$ or $\varphi \to n\pi - 0$ at infinity. The first case cannot be true because then y and y' would have the same sign for large x and then $y \to 0$ is impossible. $\square$

We proved that $\tilde{\varphi}$ and const$\cdot\tilde{r}$ are the Prüfer-variables for $\tilde{y}$; the constant depends on the $L^2(0, \infty)$-norm of y. According to (2.4), let $\tilde{\psi} = \frac{\tilde{\varphi}}{z}$.

Proof of Theorem 2.2.

Consider the function $\Psi(z) = \psi(a, z) + \tilde{\psi}(a, z)$. This is, by Theorem 2.1, the sum of two strictly increasing functions. By (5.3),

$$z_n \Psi(z_n) = n\pi. \tag{5.7}$$

Let m be less than n. Then $\frac{m\pi}{z_m} = \Psi(z_m) < \Psi(z_n) = \frac{n\pi}{z_n}$, thus $\frac{z_n}{z_m} < \frac{n}{m}$, and $\frac{\lambda_n}{\lambda_m} < \frac{n^2}{m^2}$. $\square$

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