SPECTRAL SHIFT FUNCTIONS IN THE FIXED ENERGY INVERSE SCATTERING

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Abstract. In this paper the notion of the Krein spectral shift function is extended to the radial Schrödinger operator with fixed energy. Then we analyze the connections between the tail of the potential and the decay rate of the fixed-energy phase shifts. Finally we extend former results on the uniqueness of the fixed-energy inverse scattering problem to a general class of potentials.

1. Introduction

The notion of the spectral shift function, introduced by M. G. Krein has become an important tool in the inverse spectral theory of Schrödinger and other operators. The interested reader can consult the review paper of Birman and Yafaev [6] and many other publications, e.g. Simon [27], Gesztesy and Simon [11], [12], Gesztesy and Holden [9], Gesztesy and Makarov [10] and so on. In the present paper we consider analogous notions for the three-dimensional inverse potential scattering with fixed energy in case of spherically symmetrical potentials. This is described by the radial Schrödinger operator

$$(\tau f)(r) = -(r^2 f'(r))' - \frac{1}{4} f(r) + r^2(q(r) - 1) f(r). \tag{1.1}$$

The potential $q(r)$ is supposed to be real-valued and

$$\int_0^1 r|q(r)|\,dr < \infty, \quad \int_1^\infty |q(r)|\,dr < \infty \tag{1.2}$$

unless other condition is explicitly stated. Since τ is in the limit circle case at infinity, we obtain a selfadjoint operator by introducing a

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boundary condition at infinity. That is, if $\theta \in [0, \pi]$ is a fixed parameter, τ is selfadjoint with the domain

$$D_\theta = \{f \in L_2(0,\infty) : f, f' \in \mathrm{AC}^{\mathrm{loc}}(0,\infty), \tau f \in L_2(0,\infty), \lim_{r\to\infty} W_R(f, \frac{\cos(r+\theta)}{r}) = 0\} \tag{1.3}$$

where $W_R(f,g) = r^2(fg' - f'g)$ is the weighted (or radial) Wronskian, see e.g. [17].

Consider the (unique) solution $\varphi(r) = \varphi(r,\nu)$ of $\tau\varphi = -\nu^2\varphi$, $\Re\nu > 0$ satisfying

$$\varphi(r) = r^{\nu-1/2}(1 + \mathbf{o}(1)), \quad r \to 0+. \tag{1.4}$$

Then we have

$$\varphi(r) = \frac{c(\nu)}{r}\sin(r - (\nu - 1/2)\pi/2 + \delta(\nu)) + \mathbf{o}(1/r), \quad r \to \infty. \tag{1.5}$$

The quantity $\delta(\nu)$ is the phase shift corresponding to the spectral parameter ν. It is defined by (1.5) only modulo π but it can be defined as an analytic function of ν, $\Re\nu > 0$ and then $\delta(\nu)$ becomes unique if we suppose that it tends to zero for positive parameters $\nu \to \infty$.

If $\varphi \in D_\theta$ then $\lambda = -\nu^2$ is an eigenvalue of τ. It is known that the eigenvalues $\lambda_n(\theta)$ are negative and tend to $-\infty$ by the rate

$$\lambda_n(\theta) = -(2n - 1/2 - 2\theta/\pi + \mathbf{o}(1))^2. \tag{1.6}$$

The phase shifts are related to the scattering amplitude by the well-known formula

$$A(t) = \sum_{n=0}^{\infty}(2n+1)A_nP_n(t), \quad A_n = \frac{1}{2i}(e^{2i\delta(n+1/2)} - 1) \tag{1.7}$$

where $P_n(t)$ are the Legendre polynomials. Thus the physical phase shifts $\delta_n = \delta(n+1/2)$ are of primary interest in the applications. However Regge [25] proposed the investigation of $\delta(\nu)$ with complex ν as early as in 1959. The Gelfand-Levitan inverse spectral theory has been extended to this complex setting e.g. in Loeffel [17] and Burdet, Giffon and Predazzi [8]. The idea of taking complex phase shifts proved to be fruitful in investigating inverse scattering problems, see e.g. [17]. In the present paper two uniqueness results are based on the Regge uniqueness theorem, using $\delta(\nu)$, $\Re\nu > 0$; details are given later.

Concerning the notion of the spectral shift function belonging to the operators τ the following statements will be proved.

Theorem 1.1. *Let $0 \le \theta_1 < \theta_2 < \pi$ and suppose (1.2). If the domain (1.3) of the operator τ_j is defined by the parameter θ_j, $j = 1, 2$, then there exists a measurable function $0 \le \xi_{\theta_1,\theta_2}(t) \le 1$ such that*

$$\mathrm{Tr}[(\tau_2+\nu^2)^{-1} - (\tau_1+\nu^2)^{-1}] = \frac{1}{4\nu}\frac{d}{d\nu}\ln\frac{\cos^2(\delta(\nu)-\pi/2(\nu-1/2)-\theta_2)}{\cos^2(\delta(\nu)-\pi/2(\nu-1/2)-\theta_1)} = -\int_{-\infty}^{\infty}\frac{\xi_{\theta_1,\theta_2}(t)}{(t+\nu^2)^2}dt, \quad \Re\nu > 0,\ \Im\nu < 0.$$

The range of $(\tau_2+\nu^2)^{-1} - (\tau_1+\nu^2)^{-1}$ consists of the constant multiples of $\varphi(.,\nu)$. Under the additional condition

$$\int_0^1 r^{1-\varepsilon_0}|q(r)|\,dr < \infty \text{ for some } \varepsilon_0 > 0 \tag{1.8}$$

the spectral shift function $\xi_{\theta_1,\theta_2}(t)$ uniquely determines θ_1, θ_2 and the potential $q(r)$ a.e. in $[0,\infty)$.

Now let θ be fixed, τ be the operator with domain (1.3) and for a constant $r_0 > 0$ let τ_{r_0} be the selfadjoint operator with domain

(1.9)

$$D_{r_0} = \{f \in L_2(0,\infty) : f, f' \in \mathrm{AC}^{\mathrm{loc}}(0,r_0) \cap \mathrm{AC}^{\mathrm{loc}}(r_0,\infty), f(r_0 \pm 0) = 0, \tau f \in L_2(0,\infty), \lim_{r\to\infty} W_R(f, \frac{\cos(r+\theta)}{r}) = 0\}.$$

Let ψ be the solution of $\tau f = -\nu^2 f$ with $\psi(r) = \cos(r+\theta)/r + \mathbf{o}(1/r)$, $r \to \infty$. Consider the function

$$F(z) = \frac{\varphi'(r_0,\nu)}{\varphi(r_0,\nu)} - \frac{\psi'(r_0,\nu)}{\psi(r_0,\nu)}, \quad \Im z > 0,\ z = -\nu^2,\ \Re\nu > 0,\ \Im\nu < 0.$$

Theorem 1.2. *Suppose (1.2). For $\Im\nu > 0$ the range of $(\tau-z)^{-1} - (\tau_{r_0} - z)^{-1}$ consists of functions parallel with φ on $(0,r_0)$, with ψ on (r_0,∞) and continuous at r_0. The function $F(z)$ is Herglotz (i.e. $\Im z > 0$ implies $\Im F(z) > 0$) and there exists a measurable function $0 \le \xi_{r_0}(t) \le 1$ called Krein spectral shift function such that*

$$\mathrm{Tr}[(\tau-z)^{-1} - (\tau_{r_0}-z)^{-1}] = -\frac{d}{dz}\ln F(z) = -\int_{-\infty}^{\infty}\frac{\xi_{r_0}(t)}{(t-z)^2}dt.$$

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In case of Schrödinger operators there are explicit formulae (called trace formulae) expressing special values of the potential by the Krein function, see e.g. Gesztesy and Simon [12], Gesztesy and Holden [9] and Rybkin [26]. The counterpart of these formulae in the situation of Theorem 1.2 is

Theorem 1.3. *Suppose (1.2). If q is right and left Lebesgue continuous at r_0 then*

(1.10)
$$\frac{q(r_0-0)+q(r_0+0)}{2} = 1+r_0^{-2}\lim_{z\to i\infty}\int_{-\infty}^{\infty}\frac{z^2}{(t-z)^2}(\chi_{(0,\infty)}(t)-2\xi_{r_0}(t))\,dt.$$

In case of Theorem 1.1 we can state the vague claim that the characteristics of the behavior of the potential at infinity can be expressed from the Krein function. Since the phase shifts can be expressed by the Krein function, we formulate the results directly for the phase shifts $\delta(\nu)$. Below we obtain an explicit asymptotics for the phase shifts with an estimation of the remainder.

Theorem 1.4. *Suppose that $\nu > 1/3$ is sufficiently large:*

$$2\pi\int_0^\infty r|q(r)|\,dr \le \nu^{-1/2}. \tag{1.11}$$

Then

$$\left|e^{i\delta(\nu)}\sin\delta(\nu)+\frac{\pi}{2}\int_0^\infty rq(r)J_\nu^2(r)\,dr\right| \tag{1.12}$$
$$\le \frac{c_0}{\sqrt{\nu}}\int_0^\infty r|q(r)|J_\nu^2(r)\,dr,\quad c_0=\pi^2\int_0^\infty r|q(r)|\,dr.$$

For large ν the Bessel function J_ν is extremely small in fixed finite intervals, so (1.12) implies that the asymptotical properties of $\delta(\nu)$ are mostly influenced by the tail of the potential. Some special cases are listed below, where the tail of q has polynomial, exponential or superexponential decay and where q has compact support. In all cases listed below the asymptotics of the potential can be reconstructed from the asymptotics of phase shifts of large indices; in the last case the support $[0,a]$ and $q(a-0)$ can be recovered.

Corollary 1.5. *Under the conditions of the previous Theorem:*

- *If $q(r) = cr^{-s}(1+\mathbf{o}(1))$ as $r\to\infty$ with some constants $c\neq 0$ and $s>2$ then*

$$\delta(\nu) = -c\frac{\pi\Gamma(s-1)}{2^{s+1}\Gamma(s/2)^2}\nu^{1-s}(1+\mathbf{o}(1)). \tag{1.13}$$

- *If $q(r)=\frac{c}{r}e^{-ar}(1+\mathbf{o}(1))$ with $c\neq 0$ and $a>0$ then*

$$\delta(\nu) = -c\sqrt{\frac{\pi}{8\sinh\eta}}\nu^{-1/2}e^{-\eta\nu}(1+\mathbf{o}(1)),\ \cosh\eta = a^2/2+1. \tag{1.14}$$

- *If* $q(r) = ce^{-a^2r^2}(1+\mathbf{o}(1))$ *with* $c \neq 0$ *and* $a > 0$ *then*

$$\delta(\nu) = -c\frac{\pi}{4a^2}e^{-\frac{1}{2a^2}}\frac{1}{\sqrt{2\pi\nu}}\left(\frac{e}{4a^2\nu}\right)^{\nu}(1+\mathbf{o}(1)) \tag{1.15}$$

- *If* $q = 0$ *for* $r > a$ *and* q *is left Lebesgue continuous in* a *in the sense that*

$$\lim_{h\to 0+}\frac{1}{h}\int_{a-h}^{a} q = q(a-0) \tag{1.16}$$

then

$$\delta(\nu) = -\left(\frac{ae}{2\nu}\right)^{2\nu+2}\left(\frac{q(a-0)}{2e^2}+\mathbf{o}(1)\right). \tag{1.17}$$

Our last topic is the uniqueness of the inverse scattering problem with fixed energy and spherically symmetrical potentials. That is, the scattering amplitude is known at a fixed energy and we have to identify uniquely the potential. In Newton [19], Ch. 20.4 constructions are given to suggest that there exist potentials oscillating and decaying at infinity at the rate $r^{-3/2}$ producing no scattering whatsoever, that is, for which all physical phase shifts $\delta_n = \delta(n+1/2)$ vanish. Thus uniqueness can fail for slowly decaying potentials. Regge [25] proved uniqueness if all the (nonphysical) phase shifts $\delta(\nu)$ are known and formulated some hints how to prove uniqueness from the scattering amplitude. Loeffel [17] made rigorous some considerations of Regge and proved uniqueness from the scattering amplitude for the potentials of compact support. This result has been considerably strengthened by Ramm [23] who proved that knowledge of a very sparse subsequence of the phase shifts δ_n is enough to ensure uniqueness: the sum of reciprocals of the indices of known phase shifts must be infinite. In [15] it is shown that this condition is also necessary. Martin and Targonski [18] showed uniqueness for Yukawa-like potentials (special analytic potentials of exponential decay). If we remove the condition of spherical symmetry of the potential, uniqueness from the fixed-energy scattering amplitude is proved in Henkin and Novikov [14] if the potential is exponentially decaying and has sufficiently small norm. This smallness condition has been removed later in Novikov [20]. For potentials of compact support uniqueness has been obtained independently by Novikov [21] and Ramm [22]. Weder [29] proved that if two potentials q_1 and q_2 have a decay of order $|x|^{-3-\varepsilon}$ and $q_2 - q_1$ has compact support and the fixed-energy scattering amplitudes are the same then $q_1 = q_2$. Ramm and Stefanov [24] proved uniqueness for potentials decaying faster than any exponential. Weder and Yafaev [30] proved that uniqueness holds for C^∞-potentials which are (for large $|x|$) finite linear combinations of homogeneous functions of order $-\rho_j$, $\rho_j > 3$. Finally remark that concerning 2D inverse scattering problems Grinevich [13] constructed transparent potentials (with no scattering at a given energy) of decay $\mathbf{O}(|x|^{-2})$.

Below we prove uniqueness for 3D spherically symmetrical potentials where instead of exponential decay we only suppose that $rq(r)$ is integrable at infinity:

Theorem 1.6. *Suppose (1.8) and*

$$\int_1^\infty r|q(r)|\,dr < \infty. \tag{1.18}$$

Then the scattering amplitude (1.7) uniquely determines $q(r)$ a.e.

2. Proof of the results about the Krein function

In what follows we borrow some ideas and notations from Loeffel [17]. Let $\psi_\pm(r,\nu)$ be the solutions of $\tau\psi = -\nu^2\psi$ such that

$$\psi_\pm(r,\nu) = \frac{e^{\pm ir}}{r}(1+\mathbf{o}(1)), \quad r \to \infty.$$

By the quantities in (1.5) define the functions

$$\alpha(\nu) = \frac{c(\nu)}{2i}e^{i(\delta(\nu)-\pi/2(\nu-1/2))}, \; \beta(\nu) = -\frac{c(\nu)}{2i}e^{-i(\delta(\nu)-\pi/2(\nu-1/2))},$$

Then

$$\varphi = \alpha\psi_+ + \beta\psi_-. \tag{2.1}$$

Let further

$$\begin{aligned} u &= \alpha e^{-i\theta} + \beta e^{i\theta} = c(\nu)\sin(\delta(\nu) - \pi/2(\nu-1/2) - \theta), \\ v &= i[\alpha e^{-i\theta} - \beta e^{i\theta}] = c(\nu)\cos(\delta(\nu) - \pi/2(\nu-1/2) - \theta). \end{aligned} \tag{2.2}$$

We see that $v = 0$ if and only if $\varphi = c\psi$ for the function ψ in Theorem 1.2, that is, if $\lambda = -\nu^2$ is an eigenvalue of the operator τ with domain (1.3).

The Green function of τ, i.e. the kernel of $(\tau-\lambda)^{-1}$, is

$$G_\lambda(r,r') = \frac{\varphi(r_<)\psi(r_>)}{v}, \quad \lambda = -\nu^2,\ \Re\nu > 0,\ \Im\nu \neq 0 \tag{2.3}$$

see [17].

Lemma 2.1. *a. If $\Re\nu_1, \Re\nu_2 > 0$ then*

$$(\nu_2^2-\nu_1^2)\int_0^\infty \frac{\varphi(r,\nu_1)\varphi(r,\nu_2)}{c(\nu_1)c(\nu_2)}\,dr = \sin\Big(\frac{\pi}{2}(\nu_2-\nu_1)+\delta(\nu_1)-\delta(\nu_2)\Big). \tag{2.4}$$

b.

$$\delta'(\nu) = \frac{\pi}{2} - 2\nu \int_0^\infty \frac{\varphi^2}{c(\nu)^2}, \quad \Re\nu > 0. \tag{2.5}$$

c.

$$\int_0^\infty \frac{|\varphi|^2}{|c(\nu)|^2} = \frac{\sinh 2\Im(\frac{\pi}{2}\nu - \delta(\nu))}{4\Re\nu\Im\nu}, \quad \Re\nu > 0. \tag{2.6}$$

Remark that (2.5) is given in Regge [25].

Proof. Denote $\varphi_i = \varphi(r, \nu_i)$, then

$$[\varphi_1 \cdot r^2\varphi_2' - \varphi_2 \cdot r^2\varphi_1']' = \varphi_1(r^2\varphi_2')' - \varphi_2(r^2\varphi_1')' = (\nu_2^2 - \nu_1^2)\varphi_1\varphi_2$$

hence

$$(\nu_2^2 - \nu_1^2)\int_0^\infty \varphi_1\varphi_2 = [r^2(\varphi_1\varphi_2' - \varphi_2\varphi_1')]_0^\infty.$$

The asymptotics (1.4) and (1.5) can be differentiated in r (see [17]), so the limit at zero vanishes ($r^{\nu_1-1/2}r^{\nu_2-3/2}r^2 \to 0$) and the main term of the asymptotics at infinity gives, apart from the factor $c(\nu_1)c(\nu_2)$,

$$\begin{aligned} &\sin(r - \frac{\pi}{2}(\nu_1 - 1/2) + \delta(\nu_1))\cos(r - \frac{\pi}{2}(\nu_2 - 1/2) + \delta(\nu_2)) \\ -&\cos(r - \frac{\pi}{2}(\nu_1 - 1/2) + \delta(\nu_1))\sin(r - \frac{\pi}{2}(\nu_2 - 1/2) + \delta(\nu_2)) \\ =&\sin(\frac{\pi}{2}(\nu_2 - \nu_1) + \delta(\nu_1) - \delta(\nu_2)). \end{aligned}$$

This shows (2.4). Dividing it by $\nu_2 - \nu_1$, with $\nu_1 = \nu$, $\nu_2 \to \nu$ we get (2.5). Since q is real and $\overline{r^{\nu-1/2}} = r^{\overline{\nu}-1/2}$, the formulae $\overline{\varphi(r,\nu)} = \varphi(r, \overline{\nu})$, $\overline{c(\nu)} = c(\overline{\nu})$ and $\overline{\delta(\nu)} = \delta(\overline{\nu})$ follow; putting $\nu_1 = \nu$, $\nu_2 = \overline{\nu}$ in (2.4) we finally get (2.6). □

As in [17], introduce the function

$$m(\lambda) = \frac{u(\nu)}{v(\nu)} = \tan(\delta(\nu) - \pi/2(\nu - 1/2) - \theta), \quad \lambda = -\nu^2, \ \Re\nu > 0, \ \Im\nu \neq 0. \tag{2.7}$$

Lemma 2.2. *The functions $\delta(\nu) - \pi\nu/2$ and $m(\lambda)$ are Herglotz in the variable $\lambda = -\nu^2$, $\Re\nu > 0$, $\Im\nu < 0$.*

Proof. From (2.6) we get that $\Im\nu < 0$ implies $\Im(\delta(\nu) - \pi/2\nu) > 0$ which is the Herglotz property of the first function. The identity

$$\Im \tan z = \frac{(1 + \tan^2 x)\tanh y}{1 + \tan^2 x \tanh^2 y}, \quad z = x + iy$$

shows that $\Im z > 0$ if and only if $\Im \tan z > 0$. Thus by (2.7) $m(\lambda)$ is also Herglotz. □

Lemma 2.3. *If m_j and G_j denote the m-function and the Green function corresponding to the parameter θ_j then*

$$G_2 - G_1 = \frac{m_2 - m_1}{c(\nu)^2}\varphi(r)\varphi(r'). \tag{2.8}$$

Proof. From

$$m = \frac{u}{v} = \frac{\alpha + \beta e^{2i\theta}}{i(\alpha - \beta e^{2i\theta})}$$

we find

$$e^{2i\theta} = \frac{\alpha(im-1)}{\beta(im+1)}.$$

Thus,

$$\frac{\psi}{v} = \frac{1}{2i}\frac{\psi_+ e^{2i\theta} + \psi_-}{\alpha - \beta e^{2i\theta}} = \frac{1}{4i}\left[\frac{\psi_+}{\beta}(im-1) + \frac{\psi_-}{\alpha}(im+1)\right]$$

and then

$$G(r,r') = \frac{1}{4i}\varphi(r_<)\left[\frac{\psi_+(r_>)}{\beta}(im-1) + \frac{\psi_-(r_>)}{\alpha}(im+1)\right]. \tag{2.9}$$

Now a subtraction gives

$$\begin{aligned} G_2 - G_1 &= \frac{\varphi(r_<)}{4}(m_2 - m_1)\left[\frac{\psi_+(r_>)}{\beta} + \frac{\psi_-(r_>)}{\alpha}\right] = \frac{m_2 - m_1}{4}\varphi(r_<)\frac{\varphi(r_>)}{\alpha\beta} \\ &= (m_2 - m_1)\frac{\varphi(r)\varphi(r')}{c(\nu)^2} \end{aligned}$$

as asserted. □

Proof of Theorem 1.1 By (2.8) the operator $(\tau_2 + \nu^2)^{-1} - (\tau_1 + \nu^2)^{-1}$ is of rank one,

$$\begin{aligned} (\tau_2 + \nu^2)^{-1} f - (\tau_1 + \nu^2)^{-1} f &= \int_0^\infty (G_2 - G_1)(r,r') f(r')\, dr' \\ &= \frac{m_2 - m_1}{c(\nu)^2}\varphi \int_0^\infty f\varphi \end{aligned}$$

and its trace is

$$\mathrm{Tr}[(\tau_2 + \nu^2)^{-1} - (\tau_1 + \nu^2)^{-1}] = \frac{m_2 - m_1}{c(\nu)^2}\int_0^\infty \varphi^2. \tag{2.10}$$

Thus there exists a spectral shift function $\xi_{\theta_1,\theta_2}(t)$ such that

$$\mathrm{Tr}[(\tau_2 + \nu^2)^{-1} - (\tau_1 + \nu^2)^{-1}] = -\int_{-\infty}^{\infty} \frac{\xi_{\theta_1,\theta_2}(t)}{(t+\nu^2)^2} dt, \quad \Re\nu > 0,\ \Im\nu < 0.$$

Since

$$\begin{aligned}4m_1\nu\int_0^\infty \frac{\varphi^2}{c(\nu)^2} &= 2\tan(\delta(\nu)-\pi/2(\nu-1/2)-\theta_1)(\pi/2-\delta'(\nu))\\ &= \frac{d}{d\nu}\ln[\cos^2(\delta(\nu)-(\nu-1/2)\pi/2-\theta_1)]\\ &= -4\nu\frac{d}{dz}\ln[\cos(\delta(\nu)-(\nu-1/2)\pi/2-\theta_1)],\quad z=-\nu^2\end{aligned}$$

we get that

$$\mathrm{Tr}[(\tau_2+\nu^2)^{-1}-(\tau_1+\nu^2)^{-1}] = -\frac{d}{dz}\ln H(z),$$

where

$$H(z) = \frac{\cos(\delta(\nu)-(\nu-1/2)\pi/2-\theta_2)}{\cos(\delta(\nu)-(\nu-1/2)\pi/2-\theta_1)},\quad z=-\nu^2. \tag{2.11}$$

The function $H(z)$ is Herglotz since m_1 is Herglotz and

$$H(z) = \cos(\theta_2-\theta_1)+m_1\sin(\theta_2-\theta_1),\quad 0<\theta_2-\theta_1<\pi. \tag{2.12}$$

Consequently by the Aronszajn-Donoghue theorem [4] there is a measurable function $0\le\xi(t)\le 1$ with

$$H(z) = \exp(c+\int_{-\infty}^\infty\left[\frac{1}{t-z}-\frac{t}{1+t^2}\right]\xi(t)\,dt)$$

and

$$\xi(t) = \pi^{-1}\Im[\ln H(t+i0)]. \tag{2.13}$$

Now

$$-\frac{d}{dz}\ln H(z) = -\int_{-\infty}^\infty\frac{\xi(t)}{(t-z)^2}dt,\quad z=-\nu^2$$

shows that $\xi=\xi_{\theta_1,\theta_2}$.

Finally consider the uniqueness claim in Theorem 1.1. We need

Lemma 2.4. *The eigenvalues of the operators τ_1 and τ_2 separate each other. Moreover, if $t<0$ increases, the function $\xi_{\theta_1,\theta_2}(t)$ jumps from 0 to 1 at the eigenvalues of τ_1, from 1 to 0 at the eigenvalues of τ_2 and stays constant in between.*

Proof. For $t<0$ $H(t)$ is real (since $\nu>0$), so by (2.13) $\xi_{\theta_1,\theta_2}(t)=0$ if $H(t)>0$ and $\xi_{\theta_1,\theta_2}(t)=1$ if $H(t)<0$. From (2.5) we see that $\delta(\nu)-(\nu-1/2)\pi/2$ is a strictly decreasing function of $\nu>0$. The eigenvalues are $\lambda_{k,j}=-\nu_{k,j}^2$, where $\delta(\nu_{k,j})-(\nu_{k,j}-1/2)\pi/2-\theta_j=-(k-1/2)\pi$ and $\nu_{k,j}>0$, hence $\nu_{k,2}<\nu_{k,1}<\nu_{k+1,2}$, i.e. $\lambda_{k,2}>\lambda_{k,1}>\lambda_{k+1,2}$ so the eigenvalues separate each other. If ν passes through $\nu_{k,1}$ increasingly then the denominator in (2.11) goes from negative to positive for k

even and from positive to negative for k odd. The same is true for the numerator if ν passes through $\nu_{k,2}$. This means that $H(t) < 0$ in $(\lambda_{k,1}, \lambda_{k,2})$ and $H(t) > 0$ in $(\lambda_{k+1,2}, \lambda_{k,1})$. This proves the lemma. □

By this lemma the spectral shift function ξ_{θ_1,θ_2} gives the eigenvalues of τ_1 and τ_2. From the formula

$$\lambda_{k,j} = -(2k - 1/2 + (\delta(\nu_{k,j}) - \theta_j)2/\pi)^2, \quad \delta(\nu_{k,j}) \to 0$$

we get the values $0 \le \theta_1 < \theta_2 < \pi$. By the trace formula ξ_{θ_1,θ_2} gives $\frac{d}{dz} \ln H(z)$. For $t \to +\infty$, $H(-t + i0)$ almost equals to

$$\frac{\cos((\sqrt{t} - 1/2)\pi/2 + \theta_2)}{\cos((\sqrt{t} - 1/2)\pi/2 + \theta_1)}.$$

Thus from $\frac{d}{dz} \ln H(z)$ we get $H(z)$ and then by (2.12) we obtain $m(z)$ and finally $e^{2i\delta(\nu)}$. By the uniqueness theorem of Regge [25] (see also [17]) the knowledge of $e^{2i\delta(\nu)}$, $\Re\nu > 0$ implies the knowledge of q a.e. under the conditions (1.2) and (1.8). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 It is based on the observation that the radial Schrödinger operator τ is unitarily equivalent to the 1D Schrödinger operator $Ly = -y" + Q(x)y$ on the real line with the potential

$$Q(x) = r^2(q(r) - 1), \quad x = \ln(r_0/r) \tag{2.14}$$

where $0 < r_0 < \infty$ is fixed. Indeed, let

$$U : L_2(0, \infty) \to L_2(\mathbf{R}), \quad U(\varphi(r)) = \sqrt{r}\varphi(r)|_{r=r_0e^{-x}} \tag{2.15}$$

with the inverse

$$U^* : L_2(\mathbf{R}) \to L_2(0, \infty), \quad U^*(y(x)) = y(\ln(r_0/r))r^{-1/2}. \tag{2.16}$$

U is unitary since

$$\int_{-\infty}^{\infty} |y(x)|^2 dx = \int_0^{\infty} |y(\ln(r_0/r))|^2 \frac{dr}{r} = \int_0^{\infty} |\varphi(r)|^2 dr.$$

Straightforward differentiations of $y(\ln(r_0/r)) = r^{1/2}\varphi(r)$ with respect to r give

$$U^*LU\varphi = U^*Ly = U^*(-y"+Qy) = \frac{1}{\sqrt{r}}(-y"(\ln(r_0/r)+Q(\ln(r_0/r)))) = \tau\varphi.$$

The radial and ordinary Wronskians are connected by

$$W_R(\varphi(r), \varphi^*(r)) = -W(y(x), y^*(x)), \quad y = U\varphi,\ y^* = U\varphi^*.$$

Indeed,

$$\begin{aligned}&y(\ln(r_0/r))y^{*\prime}(\ln(r_0/r)) - y'(\ln(r_0/r))y^*(\ln(r_0/r))\\ =&r^{1/2}\varphi \cdot (-r^{1/2}\varphi^*/2 - r^{3/2}\varphi^{*\prime})\\ &+(-r^{1/2}\varphi/2 - r^{3/2}\varphi') \cdot r^{1/2}\varphi^* = r^2(\varphi'\varphi^* - \varphi\varphi^{*\prime}).\end{aligned}$$

Thus the boundary condition

$$\lim_{r\to\infty} W_R(\varphi, \frac{\cos(r+\theta)}{r}) = 0$$

in (1.3) is transformed into the form

$$\lim_{x\to-\infty} W(f, e^{x/2}\cos(r_0 e^{-x} + \theta)) = 0.$$

Applying this condition L becomes selfadjoint and unitarily equivalent to τ. Analogously if we define the operator L_0 by inserting Dirichlet condition at $x = 0$, then $U^* L_0 U = \tau_{r_0}$. Thus

$$\mathrm{Tr}[(\tau - z)^{-1} - (\tau_{r_0} - z)^{-1}] = \mathrm{Tr}[(L - z)^{-1} - (L_0 - z)^{-1}]$$

hence the Krein functions are the same. The function $y = U\psi$ satisfies the transformed boundary condition at $x \to -\infty$ and

$$\frac{y'(0)}{y(0)} = -\frac{1}{2} - r_0\frac{\psi'(r_0)}{\psi(r_0)},$$

so

$$\frac{\psi'(r_0)}{\psi(r_0)} = \frac{1}{r_0}(m_- + 1/2)$$

where m_- is the m-function of the left hand side of L. Analogously $U\varphi$ is the Weyl solution at $x \to +\infty$ and

$$\frac{\varphi'(r_0)}{\varphi(r_0)} = \frac{1}{r_0}(m_+ + 1/2).$$

Thus

$$F(z) = \frac{\varphi'(r_0,\nu)}{\varphi(r_0,\nu)} - \frac{\psi'(r_0,\nu)}{\psi(r_0,\nu)} = \frac{1}{r_0}(m_+ - m_-).$$

Since m_+ and $-m_-$ are Herglotz, $F(z)$ is Herglotz, too, and $\mathrm{Tr}[(L - z)^{-1} - (L_0 - z)^{-1}] = -\frac{d}{dz}\ln F(z)$, see Gesztesy and Simon [11] It is also known that $(L-z)^{-1} - (L_0 - z)^{-1}$ is of rank one. The description of the range of $(\tau - z)^{-1} - (\tau_{r_0} - z)^{-1}$ can be given as follows. If $f = (\tau - z)h = (\tau_0 - z)h_0$ then $h - h_0$ is in the (two-dimensional) kernel of the differential expression $\tau - z$ on $(0, r_0)$ and on (r_0, ∞). Since $h - h_0$ is regular at 0 and satisfies the boundary condition of parameter θ at infinity, it must be parallel to φ on $(0, r_0)$, to ψ on (r_0, ∞) and continuous at r_0. Thus all statements of Theorem 1.2 is verified. □

Proof of Theorem 1.3 In the sense of the previous proof, this is a simple transformation of Theorem 6.1 in Rybkin [26]. □

3. Proof of Theorem 1.4 and its Corollary

We start from the known integral equation

(3.1)
$$r\varphi(r,\nu) = e^{-i\delta(\nu)}u(r,\nu)c(\nu) + \int_0^\infty u(r_<,\nu)w(r_>,\nu)q(r')r'\varphi(r',\nu)\,dr'.$$

Here $u(r,\nu) = \sqrt{\pi r/2}J_\nu(r)$ and $w(r,\nu) = -i\sqrt{\pi r/2}H_\nu^{(1)}(r)$. At $r \to +\infty$ we have $u = \sin(r-(\nu-1/2)\pi/2) + \mathbf{o}(1)$, $w = -e^{i(r-(\nu-1/2)\pi/2)} + \mathbf{o}(1)$ and $u\int_r^\infty r'wq\varphi = \mathbf{o}(1)$. Hence we get from (3.1) that

$$r\varphi(r) = e^{-i\delta}\sin(r-(\nu-1/2)\pi/2)c(\nu) - e^{i(r-(\nu-1/2)\pi/2)}\int_0^\infty r'uq\varphi + \mathbf{o}(1).$$

Comparing this with (1.5) gives that

$$\sin(\delta(\nu)) = -\int_0^\infty ru(r)q(r)\varphi(r)\,dr/c(\nu). \tag{3.2}$$

We borrow some estimation tricks from Alfaro, Regge [3]. The bound

(3.3)
$$|J_\nu(r)H_\nu^{(1)}(R)| \le (\nu^2 - 1/16)^{-1/4} \le 2\nu^{-1/2}, \quad 0 < r \le R,\ \nu > 1/3$$

is given in [3], Appendix D. Applying it in (3.1) yields

$$\left|r\varphi(r) - e^{-i\delta(\nu)}u(r)c(\nu)\right| \le \pi\sqrt{\frac{r}{\nu}}\int_0^\infty r'^{3/2}|q(r')\varphi(r')|\,dr' \tag{3.4}$$

Multiply it by $r^{1/2}|q(r)|$ and integrate in r to obtain

$$\int_0^\infty r^{3/2}|q(r)\varphi(r)|\,dr \le e^{\Im\delta}|c(\nu)|\int_0^\infty r^{1/2}|u(r)q(r)|\,dr$$
$$+\pi\nu^{-1/2}\int_0^\infty r|q(r)|\,dr\int_0^\infty r^{3/2}|q(r)\varphi(r)|\,dr.$$

Now it follows from (1.11)

$$\int_0^\infty r^{3/2}|q(r)\varphi(r)|\,dr \le 2e^{\Im\delta}|c(\nu)|\int_0^\infty r^{1/2}|u(r)q(r)|\,dr.$$

Substitute it back to (3.4) to obtain

(3.5)
$$\left|r\varphi(r) - e^{-i\delta(\nu)}u(r)c(\nu)\right| \le 2\pi\sqrt{\frac{r}{\nu}}e^{\Im\delta(\nu)}|c(\nu)|\int_0^\infty r^{1/2}|q(r)u(r)|\,dr.$$

for large ν. Now from (3.2) we get

$$\big|\sin\delta(\nu)e^{i\delta(\nu)} + \int_0^\infty qu^2\big| = \big|\int_0^\infty qu(r\varphi/c(\nu)e^{i\delta(\nu)} - u)\big|$$

$$\leq c\nu^{-1/2}\left(\int_0^\infty r^{1/2}|q(r)u(r)|\,dr\right)^2 \leq c\nu^{-1/2}\int_0^\infty r|q(r)|\,dr\int_0^\infty |q(r)u(r)^2|\,dr$$

which proves Theorem 1.4. □

Proof of Corollary 1.5 Remark first that (1.12) implies that

$$\begin{aligned}&\big|\delta(\nu) + \frac{\pi}{2}\int_0^\infty rq(r)J_\nu^2(r)\,dr\big|\\ &\leq \frac{c}{\sqrt{\nu}}\int_0^\infty r|q(r)|J_\nu^2(r)\,dr + c\left(\int_0^\infty r|q(r)|J_\nu^2(r)\,dr\right)^2.\end{aligned}$$

From the estimate (4.3) below and from (1.8), (1.18) we get that

$$\int_0^{\nu^{1/3}} r|q(r)|J_\nu^2(r)\,dr \leq c\left(\frac{e}{2\nu^{2/3}}\right)^{2\nu}.$$

In the first case consider the identity [5],7.7.4(30)

$$\begin{aligned}\int_0^\infty J_\nu^2(r)r^{1-s}dr &= \frac{\Gamma(s-1)\Gamma(\nu+1-s/2)}{2^{s-1}\Gamma(s/2)\Gamma(\nu+s/2)}\\ &= \frac{\Gamma(s-1)}{2^{s-1}\Gamma(s/2)}\nu^{1-s}(1+\mathbf{o}(1)), \quad 2\nu+1 > s-1 > 0.\end{aligned}$$

Taking into account the estimate

$$\int_0^{\nu^{1/3}} r^{1-s}J_\nu^2(r)\,dr \leq c\left(\frac{e}{2\nu}\right)^{2\nu}\int_0^{\nu^{1/3}} r^{2\nu+1-s}dr \leq c\left(\frac{e}{2\nu^{2/3}}\right)^{2\nu}\nu^{-(1+s)/3}$$

the above considerations prove (1.13). Now consider the formula [28], 13.22(2) or [1], 10.22.66

$$\int_0^\infty e^{-ar}J_\nu^2(r)\,dr = \frac{1}{\pi}Q_{\nu-1/2}(1+a^2/2),$$

where $Q_\nu = Q_\nu^0$ is the associated Legendre function of the second kind. From [1], 14.3.10, 14.15.14 and 10.25.3 we know that

$$Q_{\nu-1/2}(\cosh\eta) = \sqrt{\frac{\eta}{\sinh\eta}}K_0(\nu\eta)(1+\mathbf{o}(1)) = \sqrt{\frac{\pi}{2\nu\sinh\eta}}e^{-\nu\eta}(1+\mathbf{o}(1))$$

which proves (1.14). The formula (1.15) can be similarly proved on the basis of

$$\int_0^\infty re^{-a^2r^2}J_\nu^2(r)\,dr = \frac{1}{2a^2}e^{-1/(2a^2)}I_\nu(1/(2a^2))$$
$$= \frac{1}{2a^2}e^{-1/(2a^2)}\frac{1}{\sqrt{2\pi\nu}}\left(\frac{e}{4a^2}\right)^\nu(1+\mathbf{o}(1))$$

see [1], 10.22.67 and 10.41.1. Finally suppose that $q(r)=0$ for $r>a$. We need

Lemma 3.1. *If (1.16) holds then*

$$\int_0^a r^{2\nu+1}q(r)\,dr = \frac{a^{2\nu+2}}{2\nu+2}(q(a-0)+\mathbf{o}(1)),\quad \nu\to+\infty.$$

Proof. We have to show that

$$\int_0^a r^{2\nu+1}(q(r)-q(a-0))\,dr$$
$$= (2\nu+1)\int_0^a r^{2\nu}\int_r^a (q-q(a-0))\,dr = \mathbf{o}(a^{2\nu}/\nu).$$

Take the number $0<\delta<a$ with $(a-\delta)^{2\nu}\nu^2 = a^{2\nu}$. Then $\delta\to 0$ as $\nu\to+\infty$. Now

$$\left|(2\nu+1)\int_0^{a-\delta} r^{2\nu}\int_r^a (q-q(a-0))\,dr\right| \le c\nu\int_0^{a-\delta} r^{2\nu}dr \le c(a-\delta)^{2\nu+1} \le c\nu^{-2}a^{2\nu}$$

and

$$\left|(2\nu+1)\int_{a-\delta}^a r^{2\nu}\int_r^a (q-q(a-0))\,dr\right| = \mathbf{o}\left(\nu\int_{a-\delta}^a r^{2\nu}(a-r)\,dr\right)$$
$$= \mathbf{o}\left(\frac{\nu}{2\nu+1}(a^{2\nu+2}-(a-\delta)^{2\nu+1}a) - \frac{\nu}{2\nu+2}(a^{2\nu+2}-(a-\delta)^{2\nu+2})\right) = \mathbf{o}(a^{2\nu}/\nu)$$

which proves the Lemma. □

From the power series expansion of the Bessel functions ([5], 7.2(2)) we see that

$$J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+1)}(1+\mathbf{O}(\nu^{-1})),\quad \nu\to+\infty$$

uniformly on $[0,a]$. Hence

$$\begin{aligned}\int_0^a rq(r)J_\nu^2(r)\,dr =&\frac{1}{2^{2\nu}\Gamma(\nu+1)^2}\cdot\\ &\cdot\left[\int_0^a r^{2\nu+1}q(r)\,dr+\mathbf{O}(\nu^{-1})\int_0^a r^{2\nu+1}|q(r)|\,dr\right]\\ =&\frac{1}{2^{2\nu}\Gamma(\nu+1)^2}\frac{a^{2\nu+2}}{2\nu}[q(a-0)+\mathbf{o}(1)]\\ =&\left(\frac{ae}{2\nu}\right)^{2\nu+2}\left(\frac{q(a-0)}{\pi e^2}+\mathbf{o}(1)\right)\end{aligned}$$

which implies (1.17) □

4. Proof of Theorem 1.6

We will apply the Regge uniqueness theorem in [17], mentioned in the proof of Theorem 1.1 stating that the knowledge of $e^{2i\delta(\nu)}$, $\Re\nu>0$ implies uniqueness of q under the conditions (1.2) and (1.8). The phase shifts $\delta(n+1/2)$ can be expressed from the scattering amplitude, hence we have to reconstruct the function $\delta(\nu)$ from its values at $\nu=n+1/2$. It is possible only if we have information about the growth of $\delta(\nu)$ for large ν. We need the following

Lemma 4.1. *Let $D>0$. If $\Re\nu\ge D^{-1}$ and $|\Im\nu|\le D$ then*

$$|J_\nu(r)|\le c|\nu|^{-1/3},\quad r\ge0 \tag{4.1}$$

and

$$\int_0^\infty\frac{|J_\nu(r)|}{r}dr\le c|\nu|^{-1/3}. \tag{4.2}$$

The constants $c=c(D)$ are independent of ν and r.

Remark that the growth order of $J_\nu(\nu)$ is $\nu^{-1/3}$ for real $\nu\to+\infty$ (see e.g. [2], 9.3.31), so the estimate (4.1) is sharp. On the other hand, in (4.2) the better bound $c\nu^{-1/2}$ is proved in [3] for real ν.

Proof. The representation

$$J_\nu(r)=\frac{(r/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)}\int_0^\pi\cos(r\cos t)\sin^{2\nu}t\,dt$$

([2], 9.1.20) shows that

$$|J_\nu(r)|\le\pi^{1/2}\frac{(r/2)^{\Re\nu}}{|\Gamma(\nu+1/2)|}\le c(r/2)^{\Re\nu}|(e/\nu)^\nu|\le c\left(\frac{er}{2|\nu|}\right)^{\Re\nu}. \tag{4.3}$$

For large values r we consider another representation
(4.4)

$$\pi J_\nu(r) = \int_0^\infty e^{-\nu t}\sin(r\cosh t - \nu\pi/2)\,dt + \int_0^{\pi/2}\cos(r\sin t - \nu t)\,dt =: I_1 + I_2,$$

see [5], 7.12(16). Let now $t_0 > r^{-1/2}$, then

$$\int_{t_0}^\infty \sin(r\cosh t)\,dt = \int_{t_0}^\infty \frac{1}{r\sinh t}\cdot r\sinh t\sin(r\cosh t)\,dt$$
$$= \left[\frac{-1}{r\sinh t}\cos(r\cosh t)\right]_{t_0}^\infty - \int_{t_0}^\infty \frac{\cosh t}{r\sinh^2 t}\cos(r\cosh t)\,dt$$
$$= \mathbf{O}(r^{-1/2}) + \int_1^\infty \mathbf{O}(r^{-1}e^{-t})\,dt + \int_{t_0}^1 \mathbf{O}(r^{-1}t^{-2})\,dt = \mathbf{O}(r^{-1/2}).$$

Obviously

$$\int_0^{r^{-1/2}} \sin(r\cosh t)\,dt = \mathbf{O}(r^{-1/2})$$

hence

$$\int_{t_0}^\infty \sin(r\cosh t)\,dt = \mathbf{O}(r^{-1/2}), \quad t_0 \ge 0$$

and analogously for $\cos(r\cosh t)$ and $\sin(r\cosh t - \nu\pi/2)$. Consequently in (4.4)

$$I_1 = \left[-e^{-\nu t_0}\int_{t_0}^\infty \sin(r\cosh t - \nu\pi/2)\,dt\right]_0^\infty$$
$$-\nu\int_0^\infty e^{-\nu t_0}\int_{t_0}^\infty \sin(r\cosh t - \nu\pi/2)\,dt dt_0$$

we have

$$|I_1| \le cr^{-1/2} + cr^{-1/2}|\nu|\int_0^\infty e^{-Re\nu t_0}dt_0 \le cr^{-1/2}. \tag{4.5}$$

Concerning I_2, we exclude from $[0,\pi/2]$ the subinterval where $|r\cos t - \nu| < r^{1/3} + 2D$. It has a length at most $\mathbf{O}(r^{-1/3})$, so on the excluded interval the integral of $\cos(r\sin t - \nu t)$ is of order $r^{-1/3}$, too. In the remaining intervals $[a,b]$ we have

$$\int_a^b \cos(r\sin t - \nu t)\,dt = \left[\frac{1}{r\cos t - \nu}\sin(r\sin t - \nu t)\right]_a^b$$
$$-\int_a^b \frac{r\sin t}{(r\cos t - \nu)^2}\sin(r\sin t - \nu t)\,dt = \mathbf{O}(r^{-1/3}) + \mathbf{O}\left(\int_a^b \frac{r\sin t}{(r\cos t - \Re\nu)^2}dt\right)$$
$$= \mathbf{O}(r^{-1/3}).$$

That is, $|I_2| \le cr^{-1/3}$, $|I_1| \le cr^{-1/2}$ and then

$$|J_\nu(r)| \le cr^{-1/3}. \tag{4.6}$$

In particular for $r \ge (2-\varepsilon)|\nu|/e$ we get $|J_\nu(r)| \le c|\nu|^{-1/3}$. Now for $r \le (2-\varepsilon)|\nu|/e$ we apply (4.3):

$$|J_\nu(r)| \le c\left(\frac{er}{2|\nu|}\right)^{\Re\nu} \le c(1-\varepsilon/2)^{\Re\nu} \le c|\nu|^{-1/3},$$

so (4.1) is proved. Finally we use (4.3) for $r \le 2|\nu|/e$ and (4.6) for $r \ge 2|\nu|/e$ to obtain

$$\int_0^{2|\nu|/e} \frac{|J_\nu(r)|}{r}\, dr \le c\int_0^{2|\nu|/e} \left(\frac{er}{2|\nu|}\right)^{\Re\nu} \frac{dr}{r} = \frac{c}{\Re\nu}$$

and

$$\int_{2|\nu|/e}^\infty \frac{|J_\nu(r)|}{r}\, dr \le c\int_{2|\nu|/e}^\infty r^{-4/3} = c|\nu|^{-1/3}$$

which gives (4.2). □

Corollary 4.2. *Under the conditions of the previous Lemma,*

$$|J_\nu(r)H_\nu^{(1)}(R)| \le c|\nu|^{-1/3}, \quad 0 < r \le R.$$

Proof. Consider the formula ([28], 13.7(2) and [3], Appendix D)

$$J_\nu(r)H_\nu^{(1)}(R) = \frac{1}{i\pi}\int_0^{i\infty} \exp(t/2 - \frac{r^2+R^2}{2t})I_\nu(\frac{rR}{t})\frac{dt}{t}$$

where $I_\nu(-ix) = \exp(-i\nu\pi/2)J_\nu(x)$, $x > 0$ is the modified Bessel function of index ν. Thus

$$\begin{aligned} J_\nu(r)H_\nu^{(1)}(R) &= \frac{1}{\pi}e^{-i\nu\pi/2}\int_0^\infty \exp(i/2(t + \frac{r^2+R^2}{t}))J_\nu(\frac{rR}{t})\frac{dt}{t} \\ &= \frac{1}{\pi}e^{-i\nu\pi/2}\int_0^\infty \exp(i/2(\frac{rR}{t} + \frac{r^2+R^2}{rR}t))J_\nu(t)\frac{dt}{t} \end{aligned}$$

and then

$$|J_\nu(r)H_\nu^{(1)}(R)| \le c\int_0^\infty |J_\nu(t)|\frac{dt}{t} \le c|\nu|^{-1/3}.$$

□

Proof of Theorem 1.6

Repeating the estimation procedure in the proof of Theorem 1.4 we see that

$$\left|\sin\delta(\nu)e^{i\delta\nu}\right| \le \int_0^\infty |qu^2| + c|\nu|^{-1/3}\int_0^\infty |q(r)u(r)^2|\,dr$$
$$\le c\int_0^\infty r|q(r)J_\nu^2(r)|dr \le c|\nu|^{-2/3}$$

i.e.

$$|\delta(\nu)| \le c|\nu|^{-2/3}, \quad \Re\nu \ge 1/D,\ |\Im\nu| \le D. \tag{4.7}$$

Recall the following estimate for the growth of Herglotz functions (Levin [16], Ch.I, §7): if $f(z)$ is Herglotz, then

$$|f(z)| < 5|f(i)|\frac{|z|^2}{\Im z}.$$

By Lemma 2.2 we can apply this estimate to the function $f(z) = \delta(\nu) - \pi\nu/2$, $z = -\nu^2$, $\Re\nu > 0$, $\Im\nu < 0$. Thus

$$|\delta(\nu) - \pi\nu/2| \le c\frac{|\nu|^4}{\Re\nu|\Im\nu|}, \quad Re\nu > 0,\ \Im\nu < 0.$$

Taking into account (4.7) and the fact that $\delta(\overline{\nu}) = \overline{\delta(\nu)}$ we see that $\delta(\nu)$ is of polynomial growth on the half-plane $\Re\nu \ge 1/D$. The Carlson uniqueness theorem (see e.g. in Fuchs [7]) says that if $g(\nu)$ is regular in $\Re\nu \ge 1/D$ and $|g(\nu)| \le c\exp(a|\nu|)$ for some $a < \pi$ then the values $g(n+1/2)$, $n \ge n_0$ uniquely determine $g(\nu)$. Since the knowledge of the scattering amplitude means the knowledge of $\delta(n+1/2)$, we get $\delta(\nu)$ by the Carlson theorem and then q by the Regge uniqueness theorem, mentioned earlier. This completes the proof. □

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