ON THE STABILITY IN AMBARZUMIAN THEOREMS

MIKLÓS HORVÁTH

ABSTRACT. We provide extensions of the classical Ambarzumian theorem for bounded C^3 domains of any dimension. The simple proof is based on classical spectral function asymptotics. We prove a stability property by showing that if the perturbation of the eigenvalues of the zero potential is small in some sense then the L_2-norm of the potential is also small.

1. INTRODUCTION

The classical theorem of Ambarzumian [1] states that if the eigenvalues $\lambda_n(q)$ of the problem $-y" + q(x)y = \lambda y$ on $(0, \pi)$, $y'(0) = y'(\pi) = 0$ with the potential $q \in C[0, \pi]$ are identical to the eigenvalues $\lambda_n(0)$ corresponding to the zero potential $q = 0$ then $q = 0$. This statement generated a lot of research in the inverse spectral theory, see e.g. in [8]. However these are mainly one-dimensional results. Very little is known in multidimensional situations. The following extension to two- and three-dimensional Schrödinger operators is due to Kuznetsov 1962 [9]: Let Ω be a bounded domain of sufficiently smooth boundary in $\mathbf{R^2}$ or $\mathbf{R^3}$. Consider the Neumann eigenvalue problem

$$-\Delta u + q(x)u = \lambda u \text{ on } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1.1}$$

with a (real) potential $q \in L_\infty(\Omega)$. If

$$\lambda_1(q) = \lambda_1(0) = 0 \text{ and } \sum_n (\lambda_n(q) - \lambda_n(0)) \text{ is a convergent series} \tag{1.2}$$

then $q = 0$ a.e. The 3D case has been reconsidered in Ramm and Stefanov [11] in 1992: Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a C^3-smooth boundary $\partial\Omega \in C^3$ and let $q \in \mathrm{Lip}_\beta(\overline{\Omega})$ be a real-valued Hölder continuous potential with some $0 < \beta \leq 1$. Now if the eigenvalues of (1.1) satisfy

$$\lambda_1(q) = \lambda_1(0) = 0 \text{ and } |\lambda_n(q) - \lambda_n(0)| \leq \frac{c}{n^\alpha} \tag{1.3}$$

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for some $\alpha > 0$ then $q = 0$. The first main result of the present paper is the following common generalization and extension of [9] and [11] to any dimensions:

Theorem 1.1. *Let $d \geq 1$ and $\Omega \in C^3$ be a bounded domain in $\mathbb{R}^d$ (or Ω be a finite interval if $d = 1$). If $q \in L_\infty(\Omega)$ and if the eigenvalues of (1.1) satisfy*

$$\lambda_1(q) = \lambda_1(0) = 0 \text{ and } \frac{1}{n}\sum_{k=1}^{n}(\lambda_k(q) - \lambda_k(0)) \to 0 \quad (n \to \infty) \tag{1.4}$$

then $q = 0$ a.e.

The key point of the proof is the following statement which says, roughly speaking, that the average shift between the eigenvalues of the potentials q and q^* tends to the average shift of the potentials:

Theorem 1.2. *Let $d \geq 1$ and $\Omega \in C^3$ be a bounded domain in $\mathbb{R}^d$ (or Ω be a finite interval if $d = 1$). If q, $q^* \in L_\infty(\Omega)$ then for the eigenvalues of (1.1) we have*

$$\frac{1}{n}\sum_{k=1}^{n}(\lambda_k(q^*) - \lambda_k(q)) \to \frac{1}{|\Omega|}\int_\Omega (q^* - q) \quad (n \to \infty). \tag{1.5}$$

This theorem is not new: in Grinberg [6] the three-dimensional case has been proved under the slightly stronger conditions $\Omega \in C^3$, q, $q^* \in \mathrm{Lip}_\beta$ for some $0 < \beta \leq 1$. Grinberg applied Green functions in the proof. The one-dimensional case follows easily from the known eigenvalue asymptotics

$$\lambda_n(q) = (n-1)^2 + \frac{1}{\pi}\int_0^\pi q + \mathbf{o}(1) \quad n \to \infty.$$

The case $d \neq 1, 3$ was not known before. Remark that for compact symmetric spaces like spheres an analogous statement can be found in Harrell [7].

Consider now the question of stability in Ambarzumian theorems. Suppose that instead of (1.4) we only know that $|\lambda_1(q)| < \delta$ and $\frac{1}{n}|\sum_{k=1}^{n}(\lambda_k(q) - \lambda_k(0))| < \delta$ for sufficiently large n. Our aim is to estimate the $L_2(\Omega)$-norm of q. We illustrate by the following example that $\|q\|_{L_2(\Omega)} \leq c\delta^{1/2}$ is the best possible upper estimate even for bounded smooth potentials. Indeed, let $\Omega' \subset \Omega$ be a small subdomain of volume δ and let $q = \chi_{\Omega'}$ be the characteristic function of Ω'. Then $\|q\|_{L_2(\Omega)} = \delta^{1/2}$ and by Theorem 1.2 $\frac{1}{n}|\sum_{k=1}^{n}(\lambda_k(q) - \lambda_k(0))| < c\delta$ for sufficiently large n. Finally consider the Rayleigh minimum principle for the first Neumann eigenvalue

$$\lambda_1(q) = \min_{0\neq u\in H^1(\Omega)} \frac{\int_\Omega (|\nabla u|^2 + qu^2)}{\int_\Omega u^2}. \tag{1.6}$$

From $q \geq 0$ we get $\lambda_1(q) \geq 0$, and taking $u = 1$ in (1.6) we obtain $\lambda_1(q) \leq \frac{1}{|\Omega|}\int_\Omega q = c\delta$. Thus, if smoothness is not required, $\|q\|_{L_2(\Omega)} \leq c\delta^{1/2}$ is the best possible stability bound. Remark that smoothing by convolution gives similar examples $q \in C^\infty$ but the norm of the derivatives will not be bounded.

The following result guarantees a stability estimate of order $\delta^{1/2-\varepsilon}$ for sufficiently smooth domain and potential.

Theorem 1.3. *Let $k \geq [d/2] + 3$, $\Omega \in C^k$ be a bounded domain in $\mathbb{R}^d$, $q \in C^{k-2}(\overline{\Omega})$, $\|q\|_{C^{k-2}} \leq K$ and suppose*

$$|\lambda_1(q)| < \delta, \quad \lim_{n\to\infty} \left| \frac{1}{n} \sum_{k=1}^{n} (\lambda_k(q) - \lambda_k(0)) \right| < \delta. \tag{1.7}$$

Then

$$\|q\|_{L_2} \leq c\delta^{\frac{k-2}{2(k-1)}}, \quad c = c(k, K, d, \Omega). \tag{1.8}$$

2. Proof of the Ambarzumian theorem

Introduce the spectral function $e_\lambda(x, y)$ of the Schrödinger operator $Lu = -\Delta u + qu$ on Ω with Neumann boundary condition by

$$e_\lambda(x, y) = \sum_{\lambda_n<\lambda} u_n(x)u_n(y) \tag{2.1}$$

where the eigenfunctions u_n of (1.1) form an orthonormal basis in $L_2(\Omega)$. Then e_λ is the kernel of the spectral measure E_λ of the differential operator corresponding to (1.1). We also consider the counting function of the eigenvalues

$$n(\lambda) = \sum_{\lambda_n<\lambda} 1 = \int_\Omega e_\lambda(x, x)\, dx. \tag{2.2}$$

The asymptotic behavior of $e_\lambda(x, x)$ and $n(\lambda)$ for large λ is a classical topic starting from the early works of Weil and Courant. We need the following variant due to Beals [2], Theorem B' and C (remark that the operator in (1.1) is selfadjoint with the domain $\{u \in H^2(\Omega) : \partial u/\partial\nu = 0\}$, see e.g. in Mizohata [10], 3.16). Let

$$c_0 = \frac{1}{(2\sqrt{\pi})^d\Gamma(d/2+1)} = \frac{\omega_d}{(2\pi)^d}$$

where ω_d is the volume of the unit ball in $\mathbf{R}^d$. Then by Beals [2]

$$\sum_{\lambda_k<\lambda} u_k^2(x) = c_0\lambda^{d/2}(1+\mathbf{o}(1)), \quad \lambda\to\infty, \text{ locally uniformly in } x\in\Omega, \tag{2.3}$$

$$\sum_{\lambda_k<\lambda} u_k^2(x) \le K\lambda^{d/2}, \quad \lambda\to\infty, \text{ uniformly in } x\in\Omega, \tag{2.4}$$

and consequently, by integration in x,

$$n(\lambda) = c_0|\Omega|\lambda^{d/2}(1+\mathbf{o}(1)), \quad \lambda\to\infty. \tag{2.5}$$

Remark that in case of multiple eigenvalues $\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k_1} < \lambda_{k_1+1}$ the substitutions $\lambda = \lambda_k$ and $\lambda = \lambda_k + 0$ yield

$$\frac{k_1}{k} = 1+\mathbf{o}(1) \tag{2.6}$$

and then (2.5) gives the eigenvalue asymptotics

$$\lambda_k = [c_0|\Omega|]^{-2/d}k^{2/d}(1+\mathbf{o}(1)). \tag{2.7}$$

Lemma 2.1. *Let $\Omega \in C^3$ be a bounded domain and $q \in L_\infty(\Omega)$ (or suppose any conditions ensuring (2.3) and (2.4) . Then for every function $f \in L_1(\Omega)$*

$$\int_\Omega f\frac{u_1^2+\cdots+u_n^2}{n} \to \frac{1}{|\Omega|}\int_\Omega f. \tag{2.8}$$

Proof. By (2.6) it is enough to show that

$$\frac{1}{n(r)}\int_\Omega f\sum_{\lambda_k<r} u_k^2 \to \frac{1}{|\Omega|}\int_\Omega f.$$

Indeed, if $\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k_1} < \lambda_{k_1+1}$ then for $k < k_0 < k_1$

$$\frac{u_1^2+\cdots+u_{k_0}^2}{k_0} - \frac{u_1^2+\cdots+u_{k_1}^2}{k_1} = (u_1^2+\cdots+u_{k_0}^2)(\frac{1}{k_0}-\frac{1}{k_1}) - \frac{u_{k_0+1}^2+\cdots+u_{k_1}^2}{k_1} = I_1 - I_2.$$

From (2.4), (2.5) (2.6) it follows that $I_1 \to 0$ uniformly, and then $\int_\Omega fI_1 \to 0$. Taking (2.3) into account we see that $|I_2| \le K$ and $I_2 \to 0$

locally uniformly, so $\int_\Omega f I_2 \to 0$ follows from the Lebesgue dominated convergence theorem.

Introduce the functions

$$g_r(x) = \frac{1}{n(r)} \sum_{\lambda_k < r} u_k^2(x),$$

then we have to check that

$$\int_\Omega \Big(g_r - \frac{1}{|\Omega|} \Big) f \to 0, \quad r \to \infty. \tag{2.9}$$

Since

$$|g_r| \le K, \quad g_r \to \frac{1}{|\Omega|} \text{ locally uniformly}$$

hence (2.9) follows again by dominated convergence. □

Proof of Theorem 1.2 Let $A = -\Delta + q$ and $A^* = -\Delta + q^*$ be the operators defined by Neumann boundary conditions and let u_k denote the orthonormal system of eigenfunctions of A. By definition,

$$\langle A^* u_k, u_k \rangle = \lambda_k(q) + \int_\Omega (q^* - q) u_k^2.$$

From the orthonormality of $u_1, \dots, u_n$ we have

$$\langle A^* u_1, u_1 \rangle + \dots + \langle A^* u_n, u_n \rangle \ge \lambda_1(q^*) + \dots + \lambda_n(q^*).$$

Consequently

$$\frac{1}{n} \sum_{k=1}^{n} (\lambda_k(q^*) - \lambda_k(q)) \le \int_\Omega (q^* - q) \frac{u_1^2 + \dots + u_n^2}{n} = \frac{1}{|\Omega|} \int_\Omega (q^* - q) + \mathbf{o}(1)$$

that is,

$$\frac{1}{n} \sum_{k=1}^{n} (\lambda_k(q^*) - \lambda_k(q)) - \frac{1}{|\Omega|} \int_\Omega (q^* - q) \le \mathbf{o}(1).$$

Interchanging q and q^* gives (1.5). □

Proof of Theorem 1.1 It is standard after Theorem 1.2. Indeed, from (1.6) we see by taking the constant function $u = 1$ that

$$\lambda_1(q) \le \frac{1}{|\Omega|} \int_\Omega q$$

and in case of equality the function $u = 1$ is an eigenfunction and then $q = \lambda_1(q)$ a.e. In our case $\lambda_1(q) = 0$ and by Theorem 1.2 $\int_\Omega q = 0$, hence $q = 0$ a.e, as asserted. □

3. Proof of the stability theorem

In this section we adopt notations and use several classical results from the monograph of Gilbarg and Trudinger [5]. First we need a stronger form of the fact that the first Neumann eigenfunctions are positive:

Lemma 3.1. *Let $k \geq [d/2]+3$, $\Omega \in C^k$ be a bounded domain in $\mathbb{R}^d$, $q \in C^{k-2}(\overline{\Omega})$, $\|q\|_{C^1} \leq K$. Then the eigenfunction u corresponding to the first Neumann eigenvalue is positive and*

$$\max_{\overline{\Omega}} u \leq c \min_{\overline{\Omega}} u \tag{3.1}$$

with a constant $c = c(k, K, d, \Omega)$ independent of q.

Remark that the condition $q \in C^{k-2}(\overline{\Omega})$ is needed only to ensure $u \in C^{2,1/2}(\overline{\Omega})$.

Proof. From Theorems 8.10 and 9.19 of [5] we know that the weak eigenfunction $u \in H^1(\Omega)$ is in fact in $H^k(\Omega)$, see also [12], Theorem 15.2. By Theorem 5.6.6 of Evans [4], $H^k(\Omega) \subset C^{k-[d/2]-1,1/2}(\overline{\Omega})$ and hence $u \in C^{2,1/2}(\overline{\Omega})$. It is easy to check that for $f \in H^1$ $|f|$ is also in H^1 and

$$\nabla |f| = \begin{cases} \nabla f \text{ a.e. if } f > 0 \\ 0 \text{ a.e. if } u = 0 \\ -\nabla f \text{ a.e. if } f < 0. \end{cases}$$

That is, if $u \in H^1$ minimizes the Rayleigh quotient (1.6) then $|u|$ does the same. Thus, $|u|$ is an eigenfuntion, and then $|u| \in C^{2,1/2}$. Now the Harnack inequality (e.g. Theorem 8.20 in [5]) implies $|u| > 0$ in Ω. Since Ω is connected, u has constant sign; we will assume that $u > 0$ in Ω. Next we show that $u > 0$ also on the boundary. Indirectly suppose that $u(x_0) = 0$ for some $x_0 \in \partial\Omega$. Apply the Hopf lemma (6.4.2 in [4]) stating that if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $0 \leq q \in C(\overline{\Omega})$, $-\Delta u + qu \geq 0$, $u > 0$ on Ω, $u(x_0) = 0$ for some $x_0 \in \partial\Omega$ and if there exists a ball $B \subset \Omega$ with $x_0 \in \partial B$ (which is automatically fulfilled for $\Omega \in C^2$) then $\partial u / \partial \nu (x_0) < 0$. In our case $-\Delta u + (q - \lambda_1)_+ u = (q - \lambda_1)_- u \geq 0$, hence $\partial u/\partial\nu(x_0) < 0$ which contradicts to the Neumann boundary condition. Thus, u have positive lower and upper bounds. Suppose indirectly that there are potentials q_i satisfying the conditions of the Lemma and that the eigenfunctions $u_i > 0$ corresponding to the first Neumann eigenvalue $\lambda_i = \lambda_1(q_i)$ satisfy

$$\frac{\max_{\overline{\Omega}} u_i}{\min_{\overline{\Omega}} u_i} \to \infty. \tag{3.2}$$

We will suppose that $\|u_i\|_{C^{2,1/2}(\overline{\Omega})} = 1$. The embeddings $C^1(\overline{\Omega}) \subset C(\overline{\Omega})$ and $C^{2,1/2}(\overline{\Omega}) \subset C^2(\overline{\Omega})$ are compact (see e.g. [5], Lemma 6.36), thus taking subsequences we can assume that

$$q_i \to q^* \text{ in } C(\overline{\Omega}) \text{ and } u_i \to u^* \text{ in } C^2(\overline{\Omega}).$$

By a global Schauder estimate (Theorem 6.30 in [5], see also Theorem 7.3 in [12])

$$1 = \|u_i\|_{C^{2,1/2}} \leq c(k, K, d, \Omega)\|u_i\|_{L_\infty} \to c(k, K, d, \Omega)\|u^*\|_{L_\infty}$$

which implies that $u^* \neq 0$. Next we verify that u^* is the first Neumann eigenfunction of q^*. Introduce the Rayleigh quotients

$$J_i(u) = \frac{\int_\Omega (|\nabla u|^2 + q_i u^2)}{\int_\Omega u^2}, \quad J^*(u) = \frac{\int_\Omega (|\nabla u|^2 + q^* u^2)}{\int_\Omega u^2}.$$

We know that J_i is minimal at $u = u_i$ and that J_i tends uniformly to J^*, since

$$|J_i(u) - J^*(u)| = \left| \frac{\int_\Omega (q_i - q^*)u^2}{\int_\Omega u^2} \right| \leq \|q_i - q^*\|_{L_\infty} \to 0$$

Suppose indirectly that J^* is not minimal at u^*, that is, there exists $\tilde{u} \in H^1(\Omega)$ with $J^*(\tilde{u}) < J^*(u^*)$. Denote $\delta = J^*(u^*) - J^*(\tilde{u}) > 0$. From the uniform convergence it follows that for large i

$$J_i(\tilde{u}) + \delta/2 < J_i(u^*). \tag{3.3}$$

Consider the decomposition

$$J_i(u_i) - J_i(u^*) = \frac{\int_\Omega \left(|\nabla u_i|^2 - |\nabla u^*|^2 + q_i(u_i^2 - u^{*2})\right)}{\int_\Omega u_i^2} + \int_\Omega \left(|\nabla u^*|^2 + q_i u^{*2}\right)\left(\frac{1}{\int_\Omega u_i^2} - \frac{1}{\int_\Omega u^{*2}}\right) = I_1 + I_2.$$

Since the convergence of $u_i^2 \to u^{*2} \neq 0$, $\nabla u_i \to \nabla u^*$ and $q_i \to q^*$ are uniform on $\overline{\Omega}$, we get that $J_i(u_i) - J_i(u^*) \to 0$. Comparing this with (3.3) gives that $J_i(\tilde{u}) < J_i(u_i)$ for large i, a contradiction. Consequently J^* is minimal at u^* and then $0 \leq u^* \in C^2(\overline{\Omega})$ is the first Neumann eigenfunction for q^*. Again by the Harnack inequality and the Hopf lemma we get $u^* > 0$ on $\overline{\Omega}$. But this is impossible: the uniform convergence of u_i to u^* and (3.2) imply that u^* must have a zero on $\overline{\Omega}$. The contradiction proves the Lemma. □

Proof of Theorem 1.3 Let $u > 0$ be the first Neumann eigenfunction corresponding to q normalized by $\max_{\overline{\Omega}} u = 1$. Using an integration by parts and taking into account Theorem 1.2 we get

$$\int_\Omega |\nabla u|^2 \le \int_\Omega \frac{|\nabla u|^2}{u^2} = \int_\Omega \Delta u \cdot \frac{1}{u} = \int_\Omega (q-\lambda_1) < 2|\Omega|\delta. \tag{3.4}$$

We have seen that $u \in H^k(\Omega)$. We need the estimate

$$\|u\|_{H^k} \le c(k,K,d,\Omega)\|u\|_{L_2} \le c(k,K,d,\Omega)$$

see e.g. Theorem 15.2 in [12] or repeated use of Theorem 9.26 in [3]. We will apply a compactness estimate (Theorem 7.28 in [5]) to $u_{x_i} \in H^{k-1}(\Omega)$:

$$\|u_{x_i x_i}\|_{L_2} \le \varepsilon\|u_{x_i}\|_{H^{k-1}} + c(k,\Omega)\varepsilon^{\frac{1}{1-(k-1)}}.$$

Summing up in i gives

$$\|\Delta u\|_{L_2} \le d\varepsilon\|u\|_{H^k} + c(k,\Omega)\varepsilon^{\frac{1}{2-k}} \sum_i \|u_{x_i}\|_{L_2}$$

Putting here (3.4) and the boundedness of $\|u\|_{H^k}$ we get

$$\|\Delta u\|_{L_2} \le c(k,K,d,\Omega)(\varepsilon + \varepsilon^{\frac{1}{2-k}}\|\nabla u\|_{L_2}) \le c(k,K,d,\Omega)(\varepsilon + \varepsilon^{\frac{1}{2-k}}\sqrt{\delta}). \tag{3.5}$$

The right hand side is minimal if the summands are equal, that is, if $\varepsilon = \delta^{\frac{k-2}{2(k-1)}}$ and then

$$\|\Delta u\|_{L_2} \le c(k,K,d,\Omega)\delta^{\frac{k-2}{2(k-1)}}.$$

On the other hand by Lemma 3.1 we have

$$\|\Delta u\|_{L_2}^2 = \int_\Omega (\Delta u)^2 = \int_\Omega (q-\lambda_1)^2 u^2 \ge c\int_\Omega (q-\lambda_1)^2$$

and then

$$\|q\|_{L_2} \le \|q-\lambda_1\|_{L_2} + \|\lambda_1\|_{L_2} \le c(\|\Delta u\|_{L_2} + \delta) \le c\delta^{\frac{k-2}{2(k-1)}}$$

with $c = c(k,K,d,\Omega)$. The proof is complete. □

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Department of Mathematical Analysis
Institute of Mathematics
Budapest University of Technology and Economics
H 1111 Budapest, Műegyetem rkp. 3-9.
E-mail address: horvath@math.bme.hu