ON THE FIRST TWO EIGENVALUES OF STURM-LIOUVILLE OPERATORS

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ABSTRACT. Among the Schrödinger operators with single-well potentials defined on $(0, \pi)$ with transition point at $\frac{\pi}{2}$, the gap between the first two eigenvalues of the Dirichlet problem is minimized when the potential is constant. This extends former results of Ashbaugh and Benguria with symmetric single-well potentials. An analogous result is given for the Dirichlet problem of vibrating strings with single-barrier densities for the ratio of the first two eigenvalues.

1. INTRODUCTION

Consider the Dirichlet problem for the Schrödinger operator acting on $(0, \pi)$:

$$-y'' + V(x)y = \lambda y \quad \text{on} \quad [0, \pi], \tag{1}$$

$$y(0) = y(\pi) = 0. \tag{2}$$

Let $0 \leqslant a \leqslant \pi$ be fixed. After Ashbaugh and Benguria [4] we call the function V a *single-well* function if V is decreasing in $[0, a]$ and increasing in $[a, \pi]$. Analogously V is called a *single-barrier* function if it is increasing in $[0, a]$ and decreasing in $[a, \pi]$. The point a is the *transition point* in both cases.

Ashbaugh and Benguria proved in [4] that if V is a symmetric single-well potential then the first two eigenvalues of (1)–(2) satisfy

$$\lambda_2 - \lambda_1 \geqslant 3 \tag{3}$$

with equality if and only is V is constant. They formulated the conjecture that the hypothesis on the symmetry of V should be somehow eliminated e. g. (3) should be true for (nonsymmetric) convex potentials. This conjecture has been proved later by Lavine [10]. Our first goal is to investigate another natural way to remove symmetry. We show that (3) still holds for nonsymmetric single-well potentials if the transition point remains the midpoint:

Theorem 1.1. *Let V be a (not necessarily symmetric) single-well potential on $[0,\pi]$ with transition point $a=\frac{\pi}{2}$. Then the first two eigenvalues of the Dirichlet problem (1)–(2) satisfy (3) with equality if and only if V is constant. If $a\neq\frac{\pi}{2}$, there are single-well potentials V with $\lambda_2-\lambda_1<3$.*

Remark 1.2. It is also proved in [3] that the reversed inequality of (3) holds for symmetric single-barrier potentials. However this result can not be extended to any class of nonsymmetric single-barrier potentials with fixed transition point, including fixing it at the midpoint. Counterexamples are the stepfunctions mentioned at the end of the proof of Theorem 1.1.

Recently, M-J. Huang [7] gave the statements corresponding to the above results of [4] and of Lavine for the case of vibrating strings. More precisely, consider the Dirichlet problem

$$u''+\lambda\varrho(x)u=0 \qquad \text{in } \ [0,\pi], \tag{4}$$

$$u(0)=u(\pi)=0 \tag{5}$$

for the vibrating string; here the density function $\varrho(x)$ is supposed to be positive. Huang proved that in the class of concave densities or symmetric single-barrier densities the first two eigenvalues of (4)–(5) satisfy

$$\frac{\lambda_2}{\lambda_1}\geqslant 4 \tag{6}$$

and in the class of symmetric single-well densities we have

$$\frac{\lambda_2}{\lambda_1}\leqslant 4. \tag{7}$$

In (6) resp. (7) equality occurs if and only if ϱ is constant. Here we can also drop the condition of symmetry:

Theorem 1.3. *Let ϱ be a (not necessarily symmetric) single-barrier density function on $[0,\pi]$ with transition point $a=\frac{\pi}{2}$. Then the first two eigenvalues of the Dirichlet problem (4)–(5) satisfy (6) with equality if and only if ϱ is constant. If the transition point $a\neq\frac{\pi}{2}$, there are single-barrier densities ϱ for which $\frac{\lambda_2}{\lambda_1}<4$.*

Remark 1.4. Theorem 1.3 is not true with single-barrier replaced by single-well, (6) replaced by (7) and with anywhere fixed transition point; counterexamples are given at the end of the proof.

Finally we mention some papers related to these topics. In [2] Ashbaugh and Benguria proved (7) for the Dirichlet problem of Schrödinger operators with nonnegative potentials. Generalizations for other eigenvalue ratios $\left(\text{e. g. } \frac{\lambda_n}{\lambda_1} \leqslant n^2\right)$ are given in [3]. These results are then extended in [5] for general Sturm–Liouville operators, see also Y-L. Huang and Law [8]. Most of these results provide upper bounds for some eigenvalue ratios. Worse lower bounds for $\frac{\lambda_2}{\lambda_1}$ than (6) for a larger class of densities than in Theorem 1.2. above are given in [5]. For other lower estimates see Gentry and Banks [6], Keller [9] and Mahar and Willner [11]. The first gap of eigenvalues for some symmetric double-well situations is investigated in Abramovich [1].

The proofs of Theorems 1.1, 1.3 consist of two parts. First we reduce the problem to the search of minimum among the stepfunction potentials (or densities), being constant in $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. In such cases the eigenfunctions can be expressed by trigonometric functions, so finding the minimum of $\lambda_2 - \lambda_1$ $\left(\text{or } \frac{\lambda_2}{\lambda_1}\right)$ requires the study of nonlinear equations containing the tangent functions. This is given in Section 2, while in Section 3 we finish the proofs of Theorems 1.1 and 1.3. The use of trigonometric functions appears in several papers, e. g. in [8], [3], [11], [9], [5].

2. Solutions of trigonometric equations

In this part of the paper we prove some statements necessary for later purposes. In the proof of Theorem 1.1. we need

Lemma 2.1. *Define $f(t) = \sqrt{t}\cot\left(\sqrt{t}\frac{\pi}{2}\right)$ for real t and let $m > 0$. Then the first two real solutions of the equation $f(t) = -f(t-m)$ satisfies*

$$t_2 - t_1 > 3.$$

Proof.
Step 1. The function $f(t)$ is strictly decreasing in the intervals $(-\infty, 4)$, $(4, 16)$ and in general in $(4n^2, 4(n+1)^2)$, $n \geqslant 1$. This follows from the formulae

$$f'(t) = \frac{\sin(\sqrt{t}\,\pi) - \sqrt{t}\,\pi}{4\sqrt{t}\sin^2\left(\sqrt{t}\frac{\pi}{2}\right)} < 0 \quad \text{for} \quad t > 0, \ \ t \neq 4n^2$$

$$f'(t) = \frac{\sqrt{-t}\pi - \sinh(\sqrt{-t}\pi)}{4\sqrt{-t}\sinh^2\left(\sqrt{-t}\frac{\pi}{2}\right)} < 0 \quad \text{for} \quad t < 0.$$

The monotonity then implies that

$$t_1 \in (1, 4), \ \ t_2 \in (4, 16), \ \ f(t_1) < 0. \tag{8}$$

Step 2. The functions $t_1(m)$ and $t_2(m)$ are strictly increasing functions of $m > 0$. Indeed, if $t(m)$ denotes any of the solutions of $f(t) = -f(t-m)$, continuous in m, then

$$f\left(t(m)\right) = -f\big(t(m) - m\big) > -f\big(t(m) - m - \Delta m\big)$$
$$f\big(t(m+\Delta m)\big) = -f\big(t(m+\Delta m) - m - \Delta m\big)$$

implies $t(m + \Delta m) > t(m)$.

Step 3. We can suppose

$$0 < m < 8, \quad 4 < t_2 < 9, \quad f(t_2) > 0. \tag{9}$$

Indeed, for $m = 8$ we have $t_2 = 9$, so $m \geqslant 8$ implies $t_2 \geqslant 9$, whence $t_2 - t_1 > 5 > 3$. From $m < 8$ we get that $t_2 < 9$ and consequently $f(t_2) > 0$.

Step 4. The statement of Lemma 2.1 will follow from the estimates

$$\frac{dt_1}{dm} < \frac{1}{2} < \frac{dt_2}{dm} \quad \text{for} \quad 0 < m < 8, \tag{10}$$

since for $m = 0$ we have $t_2 - t_1 = 3$. To verify (10) we use the formula

$$\frac{1 - t_i'(m)}{t_i'(m)} = \frac{f'(t_i(m))}{f'(t_i(m) - m)} \quad i = 1, 2 \tag{11}$$

which is obtained by differentiating $f(t_i(m)) = -f(t_i(m) - m)$ in m. Since f' is negative, (10) is equivalent to

$$f'(t_1 - m) > f'(t_1) \tag{12}$$

and

$$f'(t_2 - m) < f'(t_2). \tag{12$'$}$$

Step 5. We show (12'). We apply the formula

$$f'(t) = \frac{1}{2\sqrt{t}} \cot\left(\sqrt{t}\frac{\pi}{2}\right) - \frac{\pi}{4}\left(1 + \cot^2\left(\sqrt{t}\frac{\pi}{2}\right)\right) \tag{13}$$

and express $\cot\left(\sqrt{t-m}\frac{\pi}{2}\right)$ from $f(t) = -f(t-m)$ to obtain

$$\begin{aligned} &f'(t_2 - m) - f'(t_2) \\ &= \frac{1}{\sqrt{t_2}} \frac{1}{t_2 - m} \cot\left(\sqrt{t_2}\frac{\pi}{2}\right) \cdot \left[\frac{m - 2t_2}{2} - \frac{\pi}{4} m f(t_2)\right]. \end{aligned} \tag{14}$$

We know $t_2 > 1 + m$ since for $t = 1 + m$ we have $-f(t-m) = 0$ and $-f(t_2 - m) = f(t_2) > 0$. So $t_2 - m > 0$, $f(t_2) > 0$, $m - 2t_2 < 0$ and (14) proves indeed (12').

Step 6. We prove (12) by showing that

$$f'(t_1 - m) > -\frac{\pi}{4} \tag{15}$$

and

$$f'(t_1) < -\frac{\pi}{4}. \tag{15'}$$

Applying again (13) we get

$$f'(t_1) + \frac{\pi}{4} = \frac{1}{2\sqrt{t_1}} \cot\left(\sqrt{t_1}\frac{\pi}{2}\right) - \frac{\pi}{4}\cot^2\left(\sqrt{t_1}\frac{\pi}{2}\right) < 0$$

since $f(t_1) < 0$. This proves (15'). To show (15) we express again $\cot\left(\sqrt{t_1 - m}\frac{\pi}{2}\right)$ from $f(t_1) = -f(t_1 - m)$ and we get

$$f'(t_1 - m) + \frac{\pi}{4} = \frac{-f(t_1)}{2}\frac{1}{t_1 - m}\left[1 + \frac{\pi}{2}f(t_1)\right]. \tag{16}$$

Analysing the sign of the right-hand side of (16) we remark first that $t_1 - m$ is positive for $m = 0$ and negative for $m = 8$ so there must be a value m^* with $t_1(m^*) = m^*$. For such values m^* we have $1 + \frac{\pi}{2}f\left(t_1(m^*)\right) = 0$ since the derivative in (16) must be finite. But t_1 is strictly increasing in m, $1 + \frac{\pi}{2}f(t_1)$ is strictly decreasing in m so there is only one value m^*, and

$$t_1 - m > 0, \quad 1 + \frac{\pi}{2}f(t_1) > 0 \quad \text{if} \quad 0 < m < m^*$$
$$t_1 - m < 0, \quad 1 + \frac{\pi}{2}f(t_1) < 0 \quad \text{if} \quad m^* < m < 8.$$

With $f(t_1) < 0$ this implies (15). So (12), (12') and then (10) is verified, the proof of Lemma 2.1. is complete. □

In proving Theorem 1.3 we need

Lemma 2.2. *Let $m > 1$, then the first two positive solutions of $f(\lambda) = -f(\lambda m)$ satisfies $\frac{\lambda_2}{\lambda_1} > 4$. As solutions we allow values λ for which $f(\lambda)$ and $f(\lambda m)$ are infinite; $f(\lambda)$ is defined in Lemma 2.1.*

Proof. Again we argue in steps. By the substitutions $t = \sqrt{\lambda}\frac{\pi}{2}$, $d = \sqrt{m}$ the equation $f(\lambda) = -f(\lambda m)$ can be transformed into the form

$$\tan t = -\frac{1}{d}\tan(td).$$

We have to show that the first two solutions t_1, t_2 satisfy $t_2 > 2t_1$ if $d > 1$.

Step 1. If $d > 1$ is increasing then t_1 decreases and $t_1 d$ increases. To see this we remark first that $\frac{\pi}{2d} < t_1 < \min\left(\frac{3\pi}{2d}, \frac{\pi}{2}\right)$. On the other hand, $\frac{\partial}{\partial d}\left[\frac{1}{d}\tan(t_1 d)\right] = \frac{2t_1 d - \sin 2t_1 d}{2d^2\cos^2(t_1 d)} > 0$ shows that $-\frac{1}{d}\tan(t_1 d)$ strongly decreases in d. Hence for $d < d'$

$$\tan t_1 = -\frac{1}{d}\tan(t_1 d) > -\frac{1}{d'}\tan(t_1 d'), \tan t_1' = -\frac{1}{d'}\tan(t_1' d')$$

so $t_1 \leqslant t_1'$ would lead to a contradiction:

$$\tan t_1 > -\frac{1}{d'}\tan(t_1 d') \geqslant -\frac{1}{d'}\tan(t_1' d') = \tan t_1' \geqslant \tan t_1.$$

The monotonity of $t_1 d$ is analogously derived from the fact that $\tau = t_1 d$ is the first positive solution of $\tan\tau = -d\tan\frac{\tau}{d}$.
Step 2. For $1 < d \leqslant 3$ we have $t_1 < \frac{3\pi}{4d}$. Since $t_1 d$ is increasing, it is enough to check it for $d = 3$. In this case $t_2 = \frac{\pi}{2}$ and $t_1 \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ is the solution of $\tan t_1 = -\frac{1}{3}\tan(3t_1) = \frac{\frac{1}{3}\tan^3 t_1 - \tan t_1}{1 - 3\tan^2 t_1}$. Solving this equality we get $\tan t_1 = \sqrt{\frac{3}{5}} < 1$, hence $t_1 < \frac{\pi}{4} = \frac{3\pi}{4d}$.
Step 3. For $1 < d \leqslant 3$ we have $t_2 > 2t_1$. We have seen this for $d = 3$. If $d < 3$ then $t_1, t_2 \in \left(\frac{\pi}{2d}, \frac{3\pi}{2d}\right)$ and $t_2 > \frac{\pi}{2}$. Hence if $t_1 \leqslant \frac{\pi}{4}$ then $t_2 > 2t_1$, only the case $t_1 > \frac{\pi}{4}$ remains, when (by Step 2) $2t_1$, $t_2 \in \left(\frac{\pi}{2}, \frac{3\pi}{2d}\right)$. On this interval $\tan t$ and $-\frac{1}{d}\tan(td)$ are strongly monotonous, hence

$$t_2 > 2t_1 \Leftrightarrow \tan(2t_1) < -\frac{1}{d}\tan(2t_1 d). \tag{17}$$

We solve the inequality on the right by applying the formulae $\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$, $\tan t_1 = -\frac{1}{d}\tan(t_1 d)$ as follows:

$$\frac{2\tan t_1}{1 - \tan^2 t_1} < -\frac{1}{d}\,\frac{-2d\tan t_1}{1 - d^2\tan^2 t_1}.$$

From $\tan t_1 > 0$ we finally get

$$\frac{1}{1 - \tan^2 t_1} < \frac{1}{1 - d^2\tan^2 t_1}. \tag{18}$$

Since $t_1 > \frac{\pi}{4}$, the denominator on the left is negative and then the other denominator is also negative. Thus (18) means $\tan^2 t_1 < d^2\tan^2 t_1$ which is of course valid. This proves by (17) the statement of Step 3.
Step 4. For $d > 3$ we also have $t_2 > 2t_1$. To see this, we observe first that $t_1, t_2 \in \left(0, \frac{\pi}{2}\right)$, $t_1, \in \left(\frac{\pi}{2d}, \frac{3\pi}{2d}\right)$, $t_2 \in \left(\frac{3\pi}{2d}, \frac{5\pi}{2d}\right)$. If $2t_1 \leqslant \frac{3\pi}{2d}$, then $t_2 > 2t_1$ follows from $t_2 > \frac{3\pi}{2d}$, so we can suppose $2t_1 > \frac{3\pi}{2d}$. Since $t_1 < \frac{\pi}{4}$ for $d = 3$ and t_1 is decreasing in d, $t_1 < \frac{\pi}{4}$ still holds for $d > 3$, too. Consequently we have $2t_1, t_2 \in \left(\frac{3\pi}{2d}, \frac{\pi}{2}\right)$. On this interval $\tan t$ and $-\frac{1}{d}\tan(td)$ are strongly monotonic, so we can state (17) and reduce its right inequality to the form (18). From $\tan t_1 = -\frac{1}{d}\tan(t_1 d) < 0$ we get $t_1 d < \pi$; with $t_1 d > \frac{3\pi}{4}$ this means that $d^2\tan^2 t_1 = \tan^2(t_1 d) < 1$.

Thus the denominators in (18) are positive and then (18) reduces to the trivial inequality $\tan^2 t_1 < d^2 \tan^2 t_1$. This proves Step 4 and the whole Lemma 2.2. $\square$

3. The proof of Theorems 1.1 and 1.3

We recall some known properties of the eigenfunctions of the Dirichlet problem(1)-(2), given either in Lavine [10] or in classical monographs on eigenvalue problems. We norm the eigenfunctions y_n such that $y_n(x)$ be positive for small $x > 0$ and

$$\int_0^\pi y_n^2 = 1. \tag{19}$$

Then y_1 is positive in $(0, \pi)$; y_2 has an inner root $0 < x_0 < \pi$, y_2 is positive in $(0, x_0)$, negative in (x_0, π). Furthermore there exist two values $0 \leqslant x_- < x_0 < x_+ \leqslant \pi$ such that

$$y_2^2 - y_1^2 \begin{cases} > 0 & \text{on} \quad (0, x_-) \cup (x_+, \pi) \\ < 0 & \text{on} \quad (x_-, x_+) \end{cases} \tag{20}$$

and both sets in (20) are nonempty. Recall the following known formula for the derivatives of the eigenvalues. Let V_0, V_1 be single-well potentials and

$$V(x,t) = tV_1(x) + (1-t)V_0(x).$$

Then the derivative of λ_n with respect to t is

$$\dot{\lambda}_n = \int_0^\pi \dot{V} y_n^2 = \int_0^\pi (V_1(x) - V_0(x))\, y_n^2(x,t)dx, \tag{21}$$

hence

$$(\lambda_2 - \lambda_1\dot{)} = \int_0^\pi (V_1(x) - V_0(x)) \left(y_2^2(x,t) - y_1^2(x,t)\right) dt. \tag{22}$$

Proof of Theorem 1.1 Denote for $M > 0$
$A_M = \{V : 0 \leqslant V \leqslant M,\, V \text{ is single-well with transition point at } \frac{\pi}{2}\}$. It is known that there exists an optimal potential $V_0 \in A_M$ giving the minimal value $\lambda_2 - \lambda_1$ over A_M. Therefore it is enough to show that for large M the optimal V_0 must be constant since every single-well potential with transition point $\frac{\pi}{2}$ can be shifted into A_M if M is large. Consider some cases:
A) $x_- \leqslant \frac{\pi}{2} < x_+$ (the case $\frac{\pi}{2} = x_+$ is similar). Let

$$V_1(x) = \begin{cases} V_0(x_-) & \text{on} \quad \left(0, \frac{\pi}{2}\right) \\ V_0(x_+) & \text{on} \quad \left(\frac{\pi}{2}, \pi\right), \end{cases}$$

we see that $V_1 \in A_M$ and

$$V_1 - V_0 \begin{cases} \leqslant 0 & \text{on} \quad (0, x_-) \cup (x_+, \pi) \\ \geqslant 0 & \text{on} \quad (x_-, x_+)\,. \end{cases} \tag{23}$$

Define

$$V(x,t) = tV_1(x) + (1-t)V_0(x),$$

it belongs to A_M if $0 \leqslant t \leqslant 1$. Thus by the optimality of V_0 the derivative $(\lambda_2 - \lambda_1)^{\cdot}$ must be nonnegative at $t = 0$:

$$0 \leqslant (\lambda_2 - \lambda_1)^{\cdot} = \int_0^\pi (V_1(x) - V_0(x)) \left(y_2^2(x,0) - y_1^2(x,0)\right) dx.$$

By (20) and (23) the product is nonpositive, its integral is nonnegative. This is possible only when $V_1 = V_0$ (except for the points $x = 0, \frac{\pi}{2}, \pi$) i.e. the optimal V_0 must be a stepfunction with the only jump at $\frac{\pi}{2}$.
B) $\frac{\pi}{2} < x_-$ (the case $x_+ < \frac{\pi}{2}$ is similar). Now let

$$V_1(x) = \begin{cases} V_0\left(\frac{\pi}{2}\right) & \text{on} \quad (0,\, x_-) \\ V_0\left(x_+\right) & \text{on} \quad (x_-, \pi). \end{cases}$$

Then (23) remains valid and just like as above we can prove that $V_1 = V_0$. Our second choice is

$$V_2(x) = \begin{cases} 0 & \text{on} \quad (0,\, x_-) \\ M & \text{on} \quad (x_-, \pi). \end{cases}$$

From the definition of x_- and from (19) we obtain

$$\int_0^{x_-} \left(y_2^2(x,0) - y_1^2(x,0)\right) dx > 0,$$

$$\int_{x_-}^{\pi} \left(y_2^2(x,0) - y_1^2(x,0)\right) dx < 0.$$

This gives by the optimality of V_0 that

$$\begin{aligned} 0 \leqslant (\lambda_2 - \lambda_1)^{\cdot} &= \int_0^\pi (V_2(x) - V_0(x)) \left(y_2^2(x,0) - y_1^2(x,0)\right) dx \\ &= - V_0\left(\frac{\pi}{2}\right) \cdot \int_0^{x_-} \left(y_2^2(x,0) - y_1^2(x,0)\right) dx \\ &\quad + (M - V_0(x_+)) \int_{x_-}^{\pi} \left(y_2^2(x,0) - y_1^2(x,0)\right) dx \end{aligned}$$

which is only possible when $V_0\left(\frac{\pi}{2}\right) = 0$ and $M = V_0(x_+)$ i.e. the optimal V_0 must be of the form V_2. But this is impossible for large M.

Indeed, the second eigenfunction of the potential V_2 can be expressed by

$$y_2(x) = \begin{cases} c\sin\left(\sqrt{\lambda_2}x\right) & \text{on } (0, x_-) \\ d\sin\left(\sqrt{\lambda_2 - M}(\pi - x)\right) & \text{on } (x_-, \pi). \end{cases}$$

The only inner zero x_0 must lie between x_- and x_+, so on $\left(0, \frac{\pi}{2}\right)$ $y_2 \neq 0$. This is only possible when $\sqrt{\lambda_2}\frac{\pi}{2} < \pi$ i.e. when $\lambda_2 < 4$. Now for $M \geqslant 4$ we have $\lambda_2 - M < 0$ and then $0 = y_2(x_0) = d\sin\left(\sqrt{\lambda_2 - M}(\pi - x_0)\right)$ is impossible. The contradiction shows that the case B) doesn't occur for $M \geqslant 4$. Summing up the above considerations we have seen that for $M \geqslant 4$ the optimal V_0 must have the form

$$V_0 = \begin{cases} 0 & \text{on } \left(0, \frac{\pi}{2}\right) \\ m & \text{on } \left(\frac{\pi}{2}, \pi\right) \end{cases} \quad \text{or} \quad V_0 = \begin{cases} m & \text{on } \left(0, \frac{\pi}{2}\right) \\ 0 & \text{on } \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

with some $m \geqslant 0$. The two potentials have the same eigenvalues so we deal only with the first form of V_0. In this case an eigenfunction corresponding to the eigenvalue λ can be expressed as

$$y(x) = \begin{cases} c\sin\left(\sqrt{\lambda}x\right) & \text{on } \left(0, \frac{\pi}{2}\right) \\ d\sin\left(\sqrt{\lambda - M}(\pi - x)\right) & \text{on } \left(\frac{\pi}{2}, \pi\right). \end{cases}$$

The constants c, d have to be chosen such that $y(x)$ be C^1-smooth at $\frac{\pi}{2}$. This can be done if and only if the quotients $\frac{y'}{y}$, counted in $\frac{\pi}{2}$ from both sides are the same i.e. when

$$\sqrt{\lambda}\cot\left(\sqrt{\lambda}\frac{\pi}{2}\right) = -\sqrt{\lambda - M}\cot\left(\sqrt{\lambda - M}\frac{\pi}{2}\right). \tag{24}$$

The eigenvalues are the real solutions λ of (24); we allow cases when both sides are infinite. But Lemma 2.1 states that in this case $\lambda_2 - \lambda_1 > 3$ if $m > 0$. Hence the optimal V_0 (where $\lambda_2 - \lambda_1 = 3$) must be constant. This proves Theorem 1.1 in the case when the transition point a is at $\frac{\pi}{2}$. Suppose finally that $0 < a < \frac{\pi}{2}$ (the case $a > \frac{\pi}{2}$ is similar). Let

$$V(x, t) = \begin{cases} t & \text{if } x \in (0, a) \\ 0 & \text{if } x \in (a, \pi), \end{cases}$$

then for $t = 0$

$$\dot{(\lambda_2 - \lambda_1)} = \int_0^a \left(y_2^2(x, 0) - y_1^2(x, 0)\right) dx.$$

Here $y_1(x,0) = \sqrt{\frac{2}{\pi}}\sin x,\ y_2(x,0) = \sqrt{\frac{2}{\pi}}\sin 2x$, so $\int_0^{\frac{\pi}{2}}(y_2^2 - y_1^2) = 0$. This means by (20) that

$$(\lambda_2 - \lambda_1)^{\cdot} = \int_0^a \left(y_2^2 - y_1^2\right) < 0$$

so for small values $t > 0$ the single-well potential $V(x,t)$ gives an eigenvalue gap $\lambda_2 - \lambda_1 < 3$. The same potential is a good counterexample for $a = \pi$ while for $a = 0$ we can choose e. g.

$$V = \begin{cases} 0 \text{ on } \left(0, \frac{2\pi}{3}\right) \\ t \text{ on } \left(\frac{2\pi}{3}, \pi\right) \end{cases}$$

if $t > 0$ is small enough. Theorem 1.1 is proved. $\square$

Proof of Theorem 1.3 It is similar to the above proof. The eigenfunctions of (4)–(5) are normalized by the rule

$$\int_0^\pi u_n^2(x)\varrho(x)dx = 1 \tag{25}$$

and such that $u_n(x) > 0$ for small $x > 0$. Then $u_1 > 0$ on $(0,\pi)$ and there exists $0 < x_0 < \pi$ such that $u_2 > 0$ on $(0, x_0)$ and < 0 on (x_0, π). Further there exist values $0 \leqslant x_- < x_0 < x_+ \leqslant \pi$ such that

$$u_1^2 - u_2^2 \begin{cases} < 0 & \text{on} \quad (0, x_-) \cup (x_+, \pi) \\ > 0 & \text{on} \quad (x_-, x_+), \end{cases} \tag{26}$$

both sets being nonempty; see Huang [7]. On the other hand if ϱ_0, ϱ_1 are single-barrier densities and

$$\varrho(x,t) = t\varrho_1(x) + (1-t)\varrho_0(x)$$

then the derivative of λ_n in t is

$$\dot{\lambda}_n = -\lambda_n \int_0^\pi \left(\varrho_1(x) - \varrho_0(x)\right) u_n^2(x,t)dx \tag{27}$$

see Keller [9] or Huang [7]. Consequently

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{\cdot} = \frac{\lambda_2}{\lambda_1}\int_0^\pi \left(\varrho_1(x) - \varrho_0(x)\right)\left(u_1^2(x,t) - u_2^2(x,t)\right) dx. \tag{28}$$

For $M > 1$ denote

$$A_M = \left\{ \varrho \,:\, \frac{1}{M} \leqslant \varrho \leqslant M,\ \varrho \text{ is single-barrier with transition point at } \frac{\pi}{2} \right\}.$$

Since the minimum of λ_1 over A_M must be positive, there exists an optimal $\varrho_0 \in A_M$ giving the minimal ratio $\frac{\lambda_2}{\lambda_1}$ over A_M. We analyze some cases like above. If $x_- \leqslant \frac{\pi}{2} < x_+$, then for

$$\varrho_1 = \begin{cases} \varrho_0(x_-) \text{ on } \left(0, \frac{\pi}{2}\right) \\ \varrho_0(x_+) \text{ on } \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

we get $(\varrho_1 - \varrho_0)(u_1^2 - u_2^2) \leqslant 0$ on $(0, \pi)$, hence (for $t = 0$)

$$0 \leqslant \left(\frac{\lambda_2}{\lambda_1}\right)^{\cdot} = \frac{\lambda_2}{\lambda_1} \int_0^\pi (\varrho_1 - \varrho_0)(u_1^2 - u_2^2)$$

which is possible only when $\varrho_1 = \varrho_0$ (except for $x = 0, \frac{\pi}{2}$ and π). The case $x_+ = \frac{\pi}{2}$ is similar. If $\frac{\pi}{2} < x_-$ then analogously we get

$$\varrho_0 = \varrho_2 = \begin{cases} M & \text{on } (0, x_-) \\ \frac{1}{M} & \text{on } (x_-, \pi). \end{cases}$$

In this case we have

$$u_2 = \begin{cases} c \sin\left(\sqrt{\lambda_2 M} x\right) & \text{on } (0, x_-) \\ d \sin\left(\sqrt{\lambda_2 \frac{1}{M}} (\pi - x)\right) & \text{on } (x_-, \pi). \end{cases}$$

The only zero of u_2 lies between x_- and x_+. Since there is no zeros on $(0, x_-]$,

$$\sqrt{\lambda_2 M} x_- < \pi \quad \text{i.e.} \quad M < \frac{1}{\lambda_2} \cdot \left(\frac{\pi}{x_-}\right)^2.$$

Since there is a zero on (x_-, π) we get

$$\sqrt{\lambda_2 \frac{1}{M}} (\pi - x_-) > \pi \quad \text{i.e.} \quad M < \lambda_2 \left(\frac{\pi - x_-}{\pi}\right)^2.$$

Multiplying these inequalities leads to a contradiction

$$M^2 < \left(\frac{\pi - x_-}{x_-}\right)^2 < 1.$$

The case $x_+ < \frac{\pi}{2}$ is similar. This shows that (after multiplying by an appropriate constant) an optimal ϱ_0 must have the form

$$\varrho_0 = \begin{cases} 1 & \text{on } \left(0, \frac{\pi}{2}\right) \\ m & \text{on } \left(\frac{\pi}{2}, \pi\right) \end{cases} \qquad \text{or } \varrho_0 = \begin{cases} m & \text{on } \left(0, \frac{\pi}{2}\right) \\ 1 & \text{on } \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

with some $m \geqslant 1$. We analyze only the first case. An eigenfunction of ϱ_0 is of the form

$$u = \begin{cases} c\sin\left(\sqrt{\lambda}x\right) & \text{on } \left(0, \frac{\pi}{2}\right) \\ d\sin\left(\sqrt{\lambda m}(\pi - x)\right) & \text{on } \left(\frac{\pi}{2}, \pi\right). \end{cases}$$

The condition of being in C^1 at $\frac{\pi}{2}$ gives

$$\sqrt{\lambda}\cot\left(\sqrt{\lambda}\frac{\pi}{2}\right) = -\sqrt{\lambda m}\cot\left(\sqrt{\lambda m}\frac{\pi}{2}\right). \tag{29}$$

The eigenvalues are the positive solutions of (29), we allow that both sides be infinite. Using the notation $t = \sqrt{\lambda}\frac{\pi}{2} > 0, d = \sqrt{m} \geqslant 1$ this becomes

$$\tan t = -\frac{1}{d}\tan(td). \tag{30}$$

It is enough to prove that the first two positive solutions of (30) satisfy $t_2 > 2t_1$, unless $d = 1$. This is exactly what is stated in Lemma 2.2, so Theorem 1.2 is proved if $a = \frac{\pi}{2}$ is the transition point. Finally we can prove that $a \neq \frac{\pi}{2}$ doesn't yield $\lambda_2 \geqslant 4\lambda_1$ by just the same way as in Theorem 1.1: if

$$\varrho = \begin{cases} 1 & \text{on } (0, a) \\ c & \text{on } (a, \pi) \end{cases}$$

and $0 < a < \pi,\ a \neq \frac{\pi}{2}$ then for small $|c - 1| \neq 0$ we have $\lambda_2 < 4\lambda_1$ if $c > 1$ and $a > \frac{\pi}{2}$ or if $c < 1$ and $a < \frac{\pi}{2}$. These densities are good counterexamples also for $a = 0$ resp. $a = \pi$. Theorem 1.3 is proved. $\square$

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