21. előadás

Parciális és helyettesítéses integrál határozott esetben

Horváth Márton

BME, Matematika Intézet, Geometria Tanszék

2021. december 6.

Parciális integrálás

$$\int_a^b f(x)g'(x)\ \mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\ \mathrm{d}x$$

Példa:

$$\int_1^e x\ln x\ \mathrm{d}x = \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{1}{x}\frac{x^2}{2}\ \mathrm{d}x = \frac{e^2}{2} - 0 - \int_1^e \frac{x}{2}\ \mathrm{d}x =$$

$f(x) = \ln x \quad f'(x) = \frac{1}{x}$

$g'(x) = x \qquad g(x) = \frac{x^2}{2}$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right) = \frac{e^2}{4} + \frac{1}{4}$$

Feladat

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos(2x)\ \mathrm{d}x =$$

Feladat

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos(2x)\ \mathrm{d}x = \left[x\frac{\sin(2x)}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(2x)}{2}\ \mathrm{d}x = 0 - \frac{\pi}{8} - \left[-\frac{\cos(2x)}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$\begin{array}{ll} f(x) = x & f'(x) = 1 \\ g'(x) = \cos(2x) & g(x) = \frac{\sin(2x)}{2} \end{array}$$

$$= -\frac{\pi}{8} - \left(-\frac{-1}{4} - 0\right) = -\frac{\pi}{8} - \frac{1}{4}$$

Helyettesítéses integrál

$$\int_a^b f(x)\ \mathrm{d}x = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\ \mathrm{d}u,$$

ahol $x = g(u)$ monoton függvény.

Példa:

$$\int_0^1 \frac{e^x}{e^{2x}+1}\ \mathrm{d}x = \int_1^e \frac{t}{t^2+1}\frac{1}{t}\ \mathrm{d}t = \int_1^e \frac{1}{t^2+1}\ \mathrm{d}t = [\operatorname{arctg} t]_1^e = \operatorname{arctg} e - \frac{\pi}{4}$$

$t = e^x \rightsquigarrow x = \ln t$, így $g(t) = \ln t$, melynek deriváltja: $g'(t) = \frac{1}{t}$
(Vagy: $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t} \rightsquigarrow \mathrm{d}x = \frac{1}{t}\mathrm{d}t$)
Határok:
ha $x = 0$, akkor $t = e^0 = 1$
ha $x = 1$, akkor $t = e^1 = e$

Feladat

$$\int_0^{\pi^2} \sin\sqrt{x}\;\mathrm{d}x =$$

Feladat

$$\int_0^{\pi^2} \sin\sqrt{x}\ \mathrm{d}x = \int_0^{\pi} \sin t \cdot 2t\ \mathrm{d}t = 2\int_0^{\pi} t\sin t\ \mathrm{d}t =$$

$$\begin{aligned} t &= \sqrt{x} \\ x &= t^2 \\ \tfrac{\mathrm{d}x}{\mathrm{d}t} &= 2t \\ \mathrm{d}x &= 2t\,\mathrm{d}t \end{aligned}$$

határok:

$x = \pi^2 \rightsquigarrow t = \pi$

$x = 0 \rightsquigarrow t = 0$

$$= 2\left([-t\cos t]_0^{\pi} - \int_0^{\pi} -\cos t\ \mathrm{d}t\right) = 2\left(-\pi\cdot(-1) + [\sin t]_0^{\pi}\right) =$$

$$\begin{aligned} f(t) &= t & f'(t) &= 1 \\ g'(t) &= \sin t & g(t) &= -\cos t \end{aligned}$$

$$= 2\pi$$