

21. előadás

Parciális és helyettesítési integrál határozott esetben

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Parciális integrálás

$$\int_a^b f(x)g'(x) dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x) dx$$

Példa:

$$\int_1^e x \ln x dx = \left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{1}{x} \frac{x^2}{2} dx = \frac{e^2}{2} - 0 - \int_1^e \frac{x}{2} dx =$$

$$f(x) = \ln x \quad f'(x) = \frac{1}{x}$$

$$g'(x) = x \quad g(x) = \frac{x^2}{2}$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4} \right]_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4} \right) = \frac{e^2}{4} + \frac{1}{4}$$

Feladat

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \cos(2x) \, dx =$$

Feladat

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \cos(2x) \, dx = \left[x \frac{\sin(2x)}{2} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(2x)}{2} \, dx = 0 - \frac{\pi}{8} - \left[-\frac{\cos(2x)}{4} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$\begin{aligned} f(x) &= x & f'(x) &= 1 \\ g'(x) &= \cos(2x) & g(x) &= \frac{\sin(2x)}{2} \end{aligned}$$

$$= -\frac{\pi}{8} - \left(-\frac{1}{4} - 0 \right) = -\frac{\pi}{8} - \frac{1}{4}$$

Helyettesítéses integrál

$$\int_a^b f(x) dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u) du,$$

ahol $x = g(u)$ monoton függvény.

Példa:

$$\int_0^1 \frac{e^x}{e^{2x} + 1} dx = \int_1^e \frac{t}{t^2 + 1} \frac{1}{t} dt = \int_1^e \frac{1}{t^2 + 1} dt = [\operatorname{arctg} t]_1^e = \operatorname{arctg} e - \frac{\pi}{4}$$

$t = e^x \rightsquigarrow x = \ln t$, így $g(t) = \ln t$, melynek deriváltja: $g'(t) = \frac{1}{t}$

(Vagy: $\frac{dx}{dt} = \frac{1}{t} \rightsquigarrow dx = \frac{1}{t} dt$)

Határok:

ha $x = 0$, akkor $t = e^0 = 1$

ha $x = 1$, akkor $t = e^1 = e$

Feladat

$$\int_0^{\pi^2} \sin \sqrt{x} \, dx =$$

Feladat

$$\int_0^{\pi^2} \sin \sqrt{x} \, dx = \int_0^{\pi} \sin t \cdot 2t \, dt = 2 \int_0^{\pi} t \sin t \, dt =$$

$$t = \sqrt{x}$$

$$x = t^2$$

$$\frac{dx}{dt} = 2t$$

$$dx = 2t \, dt$$

határok:

$$x = \pi^2 \rightsquigarrow t = \pi$$

$$x = 0 \rightsquigarrow t = 0$$

$$= 2 \left([-t \cos t]_0^{\pi} - \int_0^{\pi} -\cos t \, dt \right) = 2(-\pi \cdot (-1) + [\sin t]_0^{\pi}) =$$

$$f(t) = t \quad f'(t) = 1$$

$$g'(t) = \sin t \quad g(t) = -\cos t$$

$$= 2\pi$$