21. előadás

Parciális és helyettesítéses integrál határozott esetben

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Parciális integrálás

$$\int_a^b f(x)g'(x)\ \mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\ \mathrm{d}x$$

Példa:

$$\int_1^e x\ln x\ \mathrm{d}x = \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{1}{x}\frac{x^2}{2}\ \mathrm{d}x = \frac{e^2}{2} - 0 - \int_1^e \frac{x}{2}\ \mathrm{d}x =$$

$f(x) = \ln x \quad f'(x) = \frac{1}{x}$

$g'(x) = x \qquad g(x) = \frac{x^2}{2}$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right) = \frac{e^2}{4} + \frac{1}{4}$$

Feladat

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos(2x)\ \mathrm{d}x =$$

Feladat

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos(2x)\ \mathrm{d}x = \left[x\frac{\sin(2x)}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(2x)}{2}\ \mathrm{d}x = 0 - \frac{\pi}{8} - \left[-\frac{\cos(2x)}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$\begin{array}{ll} f(x) = x & f'(x) = 1 \\ g'(x) = \cos(2x) & g(x) = \frac{\sin(2x)}{2} \end{array}$$

$$= -\frac{\pi}{8} - \left(-\frac{-1}{4} - 0\right) = -\frac{\pi}{8} - \frac{1}{4}$$

Helyettesítéses integrál

$$\int_a^b f(x)\ \mathrm{d}x = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\ \mathrm{d}u,$$

ahol $x = g(u)$ monoton függvény.

Példa:

$$\int_0^1 \frac{e^x}{e^{2x}+1}\ \mathrm{d}x = \int_1^e \frac{u}{u^2+1}\frac{1}{u}\ \mathrm{d}u = \int_1^e \frac{1}{u^2+1}\ \mathrm{d}u = [\operatorname{arctg} u]_1^e = \operatorname{arctg} e - \frac{\pi}{4}$$

$u = e^x \rightsquigarrow x = \ln u$, így $g(u) = \ln u$, melynek deriváltja: $g'(u) = \frac{1}{u}$
(Vagy: $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{u} \rightsquigarrow \mathrm{d}x = \frac{1}{u}\mathrm{d}u$)

Határok:
ha $x = 0$, akkor $u = e^0 = 1$
ha $x = 1$, akkor $u = e^1 = e$

Feladat

$$\int_0^{\pi^2} \sin\sqrt{x}\ \mathrm{d}x =$$

Feladat

$$\int_0^{\pi^2} \sin\sqrt{x}\ \mathrm{d}x = \int_0^{\pi} \sin u \cdot 2u\ \mathrm{d}u = 2\int_0^{\pi} u\sin u\ \mathrm{d}u =$$

$$u = \sqrt{x}$$
$$x = u^2$$
$$\frac{\mathrm{d}x}{\mathrm{d}u} = 2u$$
$$\mathrm{d}x = 2u\,\mathrm{d}u$$

határok:

$$x = \pi^2 \rightsquigarrow u = \pi$$
$$x = 0 \rightsquigarrow u = 0$$

$$= 2\left(\left[-u\cos u\right]_0^{\pi} - \int_0^{\pi} -\cos u\ \mathrm{d}u\right) = 2\left(-\pi\cdot(-1) + \left[\sin u\right]_0^{\pi}\right) =$$

$$f(u) = u \qquad f'(u) = 1$$
$$g'(u) = \sin u \qquad g(u) = -\cos u$$

$$= 2\pi$$