$$\int \frac{x^3}{x+1} = ?$$

$$\int \frac{x^3}{x-2} = ?$$

$$\int \frac{x+1}{x^3+1} = ?$$

$$\int \frac{x-\sqrt{2}}{x^3-2\sqrt{2}} = ?$$

$$\int \frac{x^2}{x^3+1} = ?$$

$$\int \frac{x}{x^3-1} = ?$$

$$\int \frac{1}{x^3+2x^2-x-2} = ?$$

$$\int \frac{1+x^2}{1-x} = ?$$

$$\int \frac{1-x}{1+x^2} = ?$$

A megoldások vázlata (a konstans tagokat elhagyjuk):

$$\begin{aligned}
\int \frac{x^3}{x+1} &= \int \frac{x^3+1-1}{x+1} = \int \frac{x^3+1}{x+1} - \int \frac{1}{x+1} \\
&= \int \frac{(x+1)\left(x^2-x+1\right)}{x+1} - \int \frac{1}{x+1} \\
&= \int \left(x^2-x+1\right) - \log|x+1| \\
&= \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1|
\end{aligned}$$

$$\begin{aligned}
\int \frac{x^3}{x-2} &= \int \frac{x^3-8+8}{x-2} = \int \left(x^2+x+1+\frac{8}{x-2}\right) \\
&\frac{x^3}{3} + \frac{x^2}{2} + x - \log(x+1)^8
\end{aligned}$$

$$\begin{aligned}
\int \frac{x+1}{x^3+1} &= \int \frac{x+1}{(x+1)(x^2-x+1)} = \int \frac{1}{x^2-x+1} \\
&= \int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}} = \int \frac{4}{(2x-1)^2+3} = \frac{4}{3}\int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} \\
&= \frac{4}{3}\cdot\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\frac{2}{\sqrt{3}}} = \frac{2\sqrt{3}\arctan(\frac{2x-1}{\sqrt{3}})}{3}
\end{aligned}$$

$$\begin{aligned}
\int \frac{x-\sqrt{2}}{x^3-2\sqrt{2}} &= \int \frac{x-\sqrt{2}}{(x-\sqrt{2})(x^2+\sqrt{2}x+2)} = \int \frac{1}{x^2+\sqrt{2}x+2} = \int \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{3}{2}} \\
&= \frac{2}{3}\int \frac{1}{\left(\frac{x+\frac{\sqrt{2}}{2}}{\sqrt{\frac{3}{2}}}\right)+1} = \frac{2}{3}\cdot\frac{\arctan\frac{x+\frac{\sqrt{2}}{2}}{\sqrt{\frac{3}{2}}}}{\frac{1}{\sqrt{\frac{3}{2}}}} = \frac{\sqrt{6}}{3}\arctan\frac{\sqrt{2}x+1}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
y &= x^3 \text{ esetén} \\
\int \frac{x^2}{x^3+1} &= \frac{1}{3}\int \frac{(3x^3)'}{x^3+1} = \frac{1}{3}\int \frac{1}{y} = \frac{\log|y|}{3} = \log\sqrt[3]{|x^3+1|}
\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{x^3-1} &= \int \frac{x}{(x-1)(x^2+x+1)} \\
\frac{x}{(x-1)(x^2+x+1)} &= \frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} \\
&= \frac{A(x^2+x+1)+(x-1)(Bx+C)}{(x-1)(x^2+x+1)} \\
x &= A(x^2+x+1)+(x-1)(Bx+C) \\
&= (A+B)x^2+(A-B+C)x+A-C
\end{aligned}$$

$$\begin{aligned}
0 &= A+B \\
1 &= A-B+C \\
0 &= A-C \\
A &= \frac{1}{3},\quad B=-\frac{1}{3},\quad C=\frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\frac{x}{(x-1)(x^2+x+1)} &= \frac{1}{3(x-1)} + \frac{1-x}{3\left(x^2+x+1\right)} \\
&= \frac{1}{3(x-1)} + \frac{1}{2\left(x^2+x+1\right)} - \frac{2x+1}{6\left(x^2+x+1\right)} \\
\int \frac{1}{3(x-1)} &= \frac{\log|x+1|}{3} \\
\int \frac{1}{2\left(x^2+x+1\right)} &= \frac{\arctan\frac{2x+1}{\sqrt{3}}}{\sqrt{3}} \\
\int \frac{1+2x}{6\left(x^2+x+1\right)} &= \frac{\log(x^2+x+1)}{6} \\
\int \frac{x}{x^3-1} &= \frac{\log|x+1|}{3} - \frac{\log(x^2+x+1)}{6} + \frac{\arctan\frac{2x+1}{\sqrt{3}}}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
x^3+2x^2-x-2 &= (x-1)\,(x+1)\,(x+2) \\
\frac{1}{x^3+2x^2-x-2} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} \\
&= \frac{A(x+1)+B(x-1)}{x^2-1} + \frac{C}{x+2} \\
&= \frac{\left(A(x+1)+B(x-1)\right)(x+2)+C(x^2-1)}{(x^2-1)(x+2)} \\
1 &= \left(A(x+1)+B(x-1)\right)(x+2)+C(x^2-1) \\
&= 2A-2B-C+(3A+B)x+(A+B+C)x^2
\end{aligned}$$

$$\begin{aligned}
1 &= 2A-2B-C \\
0 &= 3A+B \\
0 &= A+B+C \\
A &= \frac{1}{6},\quad B=-\frac{1}{2},\quad C=\frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{x^3+2x^2-x-2} &= \frac{1}{6(x-1)} - \frac{1}{2(x+1)} + \frac{1}{3(x+2)} \\
\int \frac{1}{x^3+2x^2-x-2} &= \frac{\log|x-1|}{6} - \frac{\log|x+1|}{2} + \frac{\log|x+2|}{3}
\end{aligned}$$

$$\int \frac{1+x^2}{1-x} = \int \frac{2+x^2-1}{1-x} = \int \frac{2}{x} - \int (1+x) = \log x^2 - x - \frac{x^2}{2}$$

$$
\begin{aligned}
\int \frac{1-x}{1+x^2} &= \int \frac{1}{1+x^2} - \frac{1}{2}\int \frac{2x}{1+x^2} \\
&= \arctan x - \frac{1}{2}\int \frac{1}{y} \text{ ha } y = 1+x^2 \\
&= \arctan x - \frac{\log(1+x^2)}{2} \\
&= \arctan x - \log\sqrt{1+x^2}
\end{aligned}
$$