Improprius integrálok

$$\int_1^2 \frac{1}{x^2-3x+2}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{1}{x^2-3x+2} &= \log|x-2| - \log|x-1| = \log\left|\frac{x-2}{x-1}\right| \\
\left[\log\left|\frac{x-2}{x-1}\right|\right]_a^b &= \log\frac{2-b}{b-1} - \log\frac{2-a}{a-1} \\
\lim_{b\to 2^-} \log\frac{2-b}{b-1} &= -\infty \\
\lim_{a\to 1^+} \log\frac{2-a}{a-1} &= +\infty \\
\int_1^2 \frac{1}{x^2-3x+2}\ dx &= -\infty
\end{aligned}$$

$$\int_3^{+\infty} \frac{1}{x^2-3x+2}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{1}{x^2-3x+2} &= \log|x-2| - \log|x-1| = \log\left|\frac{x-2}{x-1}\right| \\
\left[\log\left|\frac{x-2}{x-1}\right|\right]_3^b &= \log\frac{b-2}{b-1} - \log\frac{3-2}{3-1} = \log\frac{b-2}{b-1} + \log 2 \\
\lim_{b\to +\infty} \log\frac{b-2}{b-1} &= \lim_{b\to +\infty} \log\left(1-\frac{1}{b-1}\right) = 0 \\
\int_3^{+\infty} \frac{1}{x^2-3x+2}\ dx &= \log 2
\end{aligned}$$

$$\int_{\pi/3}^{\pi/2} \tan^2 x \quad dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \tan^2 x &= \tan x - x \\
[\tan x - x]_{\pi/3}^b &= \tan b - b - \sqrt{3} + \frac{\pi}{3} \\
\lim_{b\to(\pi/2)^-} (\tan b - b) &= +\infty \\
\int_{\pi/3}^{\pi/2} \tan^2 x \quad dx &= +\infty
\end{aligned}$$

$$\int_{-1}^{0} \log(1+x)\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \log(1+x) &= (1+x)\log(1+x) - x \\
[(1+x)\log(1+x) - x]_a^0 &= a - (1+a)\log(1+a) \\
\lim_{a\to -1^+} (1+a)\log(1+a) &= \lim_{a\to -1^+} \frac{\log(1+a)}{(1+a)^{-1}} \\
&= \lim_{a\to -1^+} \frac{(1+a)^{-1}}{-(1+a)^{-2}} = 0 \\
\int_{-1}^{0} \log(1+x)\ dx &= \lim_{a\to -1^+} a = -1
\end{aligned}$$

$$\int_{-1}^{0} x\log(-x)\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int x\log(-x) &= \frac{x^2(2\log(-x)-1)}{4} \\
\left[\frac{x^2(2\log(-x)-1)}{4}\right]_{-1}^{b} &= \frac{b^2(2\log(-b)-1)}{4} + \frac{1}{4} \\
\lim_{b\to 0^-} b^2(2\log(-b)-1) &= \lim_{b\to 0^-} b^2\log(b^2/e) \\
&= e\lim_{z\to 0^+} z\log z \qquad \text{ahol} \quad z = b^2/e \\
\lim_{z\to 0^+} z\log z &= \lim_{z\to 0^+} \frac{\log z}{1/z} \\
&= \lim_{z\to 0^+} \frac{1/z}{-1/z^2} = \lim_{z\to 0^+}(-z) = 0 \\
\int_{-1}^{0} x\log(-x)\ dx &= \frac{1}{4}
\end{aligned}$$

$$\int_{-1}^{0} \sqrt[3]{\frac{5}{x^2}}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \sqrt[3]{\frac{5}{x^2}} &= \sqrt[3]{5} \cdot \int x^{-2/3} = 3 \cdot \sqrt[3]{5x} \\
\left[3 \cdot \sqrt[3]{5x}\right]_{-1}^{b} &= 3\sqrt[3]{5b} + 3\sqrt[3]{5} \\
\lim_{b \to 0^-} \sqrt[3]{5b} &= 0 \\
\int_{-1}^{0} \sqrt[3]{\frac{5}{x^2}}\, dx &= 3\sqrt[3]{5} = \sqrt[3]{135}
\end{aligned}$$

$$\int_{-e}^{+\infty} e^{1-x}\, dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int e^{1-x} &= -e^{1-x} \\
\left[-e^{1-x}\right]_{-e}^{b} &= -e^{1-b} + e^{1+e} \\
\lim_{b \to +\infty} e^{1-b} &= 0 \\
\int_{-e}^{+\infty} e^{1-x}\, dx &= e^{1+e}
\end{aligned}$$