$$\int_0^\infty \frac{1}{(x+1)(x^2+1)}dx = ?$$

Először keresünk olyan A, B, C konstansokat, melyekre

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx}{x^2+1} + \frac{C}{x^2+1}$$

A jobboldalt átrendezve:

$$\frac{(A+B)\,x^2 + (B+C)\,x + A + C}{(x+1)(x^2+1)}$$

Tehát

$$\begin{aligned} A+B &= 0 \\ B+C &= 0 \\ A+C &= 1 \end{aligned}$$

azaz

$$A = \frac{1}{2}, \quad B = \frac{-1}{2}, \quad C = \frac{1}{2}$$

Tehát így folytathatjuk:

$$\begin{aligned} \int \frac{1}{(x+1)(x^2+1)}dx &= \frac{1}{2}\int \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}dx \\ &= \frac{1}{2}\left(\ln|x+1| - \frac{1}{2}\ln(x^2+1) + \arctan x\right) + \text{ konstans} \end{aligned}$$

Ebből:

$$\begin{aligned} \int_0^b \frac{1}{(x+1)(x^2+1)}dx &= \frac{1}{2}\left[\ln|x+1| - \frac{1}{2}\ln(x^2+1) + \arctan x\right]_0^b \\ &= \frac{1}{2}\left(\ln(b+1) - \frac{\ln(b^2+1)}{2} + \arctan b - 0 + 0 - 0\right) \\ &= \frac{\ln(b+1)}{2} - \frac{\ln(b^2+1)}{4} + \frac{\arctan b}{2} \\ &= \frac{1}{4}\cdot \ln\frac{(b+1)^2}{b^2+1} + \frac{\arctan b}{2} \end{aligned}$$

Mármost

$$\begin{aligned} \lim_{b\to\infty} \frac{(b+1)^2}{b^2+1} &= 1 \\ \lim_{b\to\infty} \arctan b &= \frac{\pi}{2} \end{aligned}$$

Ezért

$$\int_0^\infty \frac{1}{(x+1)(x^2+1)}dx = \frac{\pi}{4}$$