$$\int_0^{\infty} \frac{1}{1-\sqrt{x}}\ dx \quad = \ ?$$

$$\int_0^{\infty} \frac{1}{1-\sqrt{x}}\ dx = \int_0^{1} \frac{1}{1-\sqrt{x}}\ dx + \int_1^{\infty} \frac{1}{1-\sqrt{x}}\ dx$$

$$y = \sqrt{x}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$$
$$dx = 2y\ dy$$

$$\begin{aligned}\int \frac{1}{1-\sqrt{x}}\ dx &= \int \frac{1}{1-y}\ 2y\ dy = 2\int \frac{y}{1-y}\ dy \\ &= 2\int \frac{y-1+1}{1-y}\ dy = 2\int -1 + \frac{1}{1-y}\ dy \\ &= 2\left(-y - \ln|1-y|\right) = 2(-\sqrt{x} - \ln|1-\sqrt{x}|)\end{aligned}$$

$$\int_0^{1} \frac{1}{1-\sqrt{x}}\ dx = \lim_{x\to 1^-} \big(2(-\sqrt{x} - \ln(1-\sqrt{x}))\big) - 2(-\sqrt{0} - \ln(1-\sqrt{0})) = +\infty$$
$$\int_1^{\infty} \frac{1}{1-\sqrt{x}}\ dx = \lim_{x\to\infty} \big(2(-\sqrt{x} - \ln(\sqrt{x}-1))\big) - \lim_{x\to 1^+} \big(2(-\sqrt{x} - \ln(\sqrt{x}-1))\big) = -\infty$$

Az eredeti improprius integrál tehát nem értelmes!