Discrete Mathematics 310 (2010) 3398–3403

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Maximal cliques in $\{P_2 \cup P_3, C_4\}$-free graphs

S.A. Choudum, T. Karthick *

Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India

ARTICLE INFO

Article history:
Received 28 January 2010
Received in revised form 19 July 2010
Accepted 12 August 2010
Available online 9 September 2010

Keywords:
Chordal graphs
C_4-free graphs
Maximal cliques
Chromatic number

ABSTRACT

We prove a decomposition theorem for the class $\mathcal{G}$ of $\{P_2 \cup P_3, C_4\}$-free graphs. This theorem enables us to show that (i) every graph G in $\mathcal{G}$ has at most $n+5$ maximal cliques where n is the number of vertices in G, and (ii) for every G in $\mathcal{G}$, $\chi(G) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$, where $\chi(G)$ ($\omega(G)$) is the chromatic (clique) number of G.

1. Introduction

All our graphs are simple, finite and undirected. For a graph G, let n and m, respectively denote the number of vertices and number of edges in G. A *clique* in a graph G is a set of pairwise adjacent vertices in G, and is *maximal* if it is not contained in a larger clique. We denote the number of maximal cliques in a graph G by $\mu(G)$. Moon and Moser [12] proved that $\mu(G) \leq 3^{\lceil \frac{n}{3} \rceil}$, for any graph G, and observed that this bound is attained by the graph $(\lceil \frac{t}{3} \rceil K_3)^c$, where t is a multiple of 3. A class of graphs $\mathcal{G}$ is said to have *few cliques* [14] if $\mu(G) \leq p(n)$, for every $G \in \mathcal{G}$, where $p(n)$ denotes a polynomial in n.

In 1965, Fulkerson and Gross [7] obtained an interesting characterization of interval graphs. During the course of their proof, they observed that any chordal graph has at most n maximal cliques. Since then several more classes of graphs with few cliques have been identified. These results are summarized in Table 1. It follows that the Maximum Clique Problem and the Maximum Weight Clique Problem for these classes of graphs are solvable in polynomial time, where as for a general class of graphs these problems are well known to be NP-complete. Various complexity issues on graphs with few cliques are investigated in [15].

For $(P_2 \cup P_3)$-free graphs, Maximum Independent Set Problem and Independent Dominating Set Problem are solvable in polynomial time [11] but Minimum Dominating Set Problem [1] and Maximum Clique Problem [13] are known to be NP-complete.

In this paper, we are concerned with the class of $\{P_2 \cup P_3, C_4\}$-free graphs, which includes interesting subclasses such as split graphs, pseudo-split graphs, $\{2K_2, C_4\}$-free graphs and $\{3K_1, C_4\}$-free graphs. See Fig. 1 for examples of graphs which are (i) $\{P_2 \cup P_3, C_4\}$-free, (ii) $(P_2 \cup P_3)$-free but contains an induced C_4, and (iii) C_4-free but contains an induced $(P_2 \cup P_3)$.

We prove a decomposition theorem for the class $\mathcal{G}$ of $\{P_2 \cup P_3, C_4\}$-free graphs. This theorem enables us to show that (i) every graph G in $\mathcal{G}$ has at most $n+5$ maximal cliques, where n is the number of vertices in G, and (ii) for every G in $\mathcal{G}$, $\chi(G) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$, where $\chi(G)$ ($\omega(G)$) is the chromatic (clique) number of G. These extend the results proved in [2,3].

Throughout the paper, P_t, C_t, K_t, respectively denote the induced path, induced cycle, complete graphs on t vertices. For notations and terminology not defined here, we follow West [17]. If H is an induced subgraph of G, we write $H \sqsubseteq G$. If H is

* Corresponding author.
E-mail addresses: sac@iitm.ac.in (S.A. Choudum), karthick@smail.iitm.ac.in (T. Karthick).

0012-365X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2010.08.005

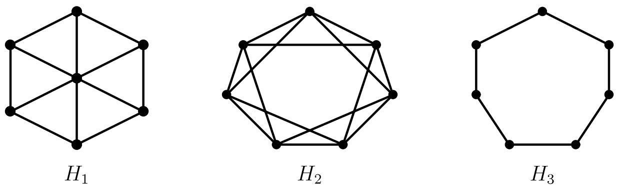

Fig. 1. H_1 is a $\{P_2 \cup P_3, C_4\}$-free graph, H_2 is a $(P_2 \cup P_3)$-free graph but contains an induced C_4, and H_3 is a C_4-free graph but contains an induced $(P_2 \cup P_3)$.

Table 1
Classes of graphs with few cliques.

G	Upper bound for $\mu(G)$	
Chordal	n	[7]
C_4-free	cn^2, c is a constant	[6]
$\{2K_2, C_4\}$-free	n	[2]
$\{P_5, C_4\}$-free	n	[8]
Planar	$\frac{7n}{3} - 6$	[14]
K_t-free, $t \geq 2$	$\max\{n, \frac{n\Delta^{t-2}}{2^{t-2}}\}$	[14]
$\{K_{1,3}, K_1 + C_4\}$-free	$2m$	[9]
Graph with boxicity k	$(2n)^k$	[16]

a subgraph of a graph G and if $v \in V(G)$, then $N_H(v) = N_G(v) \cap V(H)$. If $\mathcal{F}$ is a family of graphs, G is said to be *$\mathcal{F}$-free* if it contains no induced subgraph isomorphic to any graph in $\mathcal{F}$. If S and T are two vertex disjoint subsets, then $[S, T]$ denotes the set of edges with one end in S and the other in T. $[S, T]$ is said to be *complete* if every vertex in S is adjacent with every vertex in T. $[S]$ denotes the subgraph induced by S. The length of a shortest path between two vertices x, y is denoted by $d(x, y)$, and $d(x, S) := \min\{d(x, y) : y \in S\}$. For any integer $i \geq 0$, $N_i(S) := \{y \in V(G) : d(x, y) = i, \text{ for some } x \in S\}$, and we simply denote it by N_i. $N_1(\{x\})$ is denoted by $N(x)$, and $N[x] = N(x) \cup \{x\}$. For a graph G, we denote a partition of $V(G)$ into $k\,(\geq 1)$ sets by $(V_1, V_2, \ldots, V_k)$.

For a graph G, G^c denotes the complement of G. If G_1 and G_2 are two vertex disjoint graphs, then their *union* $G_1 \cup G_2$ is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Similarly, their *join* $G_1 + G_2$ is the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(x, y) : x \in V(G_1),\ y \in V(G_2)\}$. For any positive integer k, kG denotes the union of k disjoint graphs each isomorphic with G. For a graph G, let $\chi(G)$ ($\omega(G)$) denote its chromatic (clique) number.

Next, we define an operation to combine various graphs. Let G be a graph on n vertices $v_1, v_2, \ldots, v_n$, and let $H_1, H_2, \ldots, H_n$ be n vertex disjoint graphs. Then an *expansion* $G(H_1, H_2, \ldots, H_n)$ of G is the graph obtained from G by

(i) replacing the vertex v_i of G by H_i, $i = 1, 2, \ldots, n$, and
(ii) joining the vertices $x \in H_i$, $y \in H_j$ iff v_i and v_j are adjacent in G.

An expansion is also called a *composition*; see [17]. If H_i's are complete, it is called a *complete expansion* of G, and is denoted by $\mathbb{K}[G]$ or $\mathbb{K}[G](m_1, m_2, \ldots, m_n)$ if $H_i = K_{m_i}$. It is shown by Lovász [10] that if $G, H_1, H_2, \ldots, H_n$ are perfect, then $G(H_1, H_2, \ldots, H_n)$ is perfect. So, if G is perfect, then $\mathbb{K}[G]$ is perfect. It is easily verified that if G is chordal, then $\mathbb{K}[G]$ is chordal, so in particular $G + K_t$ $(t \geq 0)$ is chordal.

2. The class of $\{P_2 \cup P_3, C_4\}$-free graphs

2.1. Structure of $\{P_2 \cup P_3, C_4\}$-free graphs

In this section, we prove a decomposition theorem for a non-chordal $\{P_2 \cup P_3, C_4\}$-free graph G which says that its vertex set $V(G)$ can be partitioned into three subsets V_1, V_2 and V_3 such that $[V_3]$ can be obtained as a complete expansion of one of the seventeen basic graphs shown in Fig. 2, and V_1 and V_2, respectively induce an edgeless graph and a complete graph.

Theorem 1. *A connected graph G that contains an induced C_6 is $\{P_2 \cup P_3, C_4, C_5\}$-free if and only if there exists a partition (V_1, V_2, V_3) of $V(G)$ such that* (1) $[V_1] \cong K_m^c$, *for some* $m \geq 0$, (2) $[V_2] \cong K_t$, *for some* $t \geq 0$, (3) $[V_3] \cong C_6$, (4) $[V_1, V_3] = \emptyset$, *and* (5) $[V_2, V_3]$ *is complete.*

Proof. Observe that if such a partition exists, then G is a $\{P_2 \cup P_3, C_4, C_5\}$-free graph that contains an induced C_6. To see the reverse implication, let us assume that $[N_0] := [\{v_1, v_2, v_3, v_4, v_5, v_6\}] \cong C_6 \sqsubseteq G$. By assumptions on G, any $x \in N_1$ is adjacent to all the vertices of $[N_0]$. Then

(i) $[N_1]$ is complete (else, $C_4 \sqsubseteq G$).
(ii) N_2 is independent (else, $P_2 \cup P_3 \sqsubseteq G$).
(iii) $N_r = \emptyset$, if $r \geq 3$ (else, $P_2 \cup P_3 \sqsubseteq G$).

It follows that (N_2, N_1, N_0) is a required partition of $V(G)$. $\square$

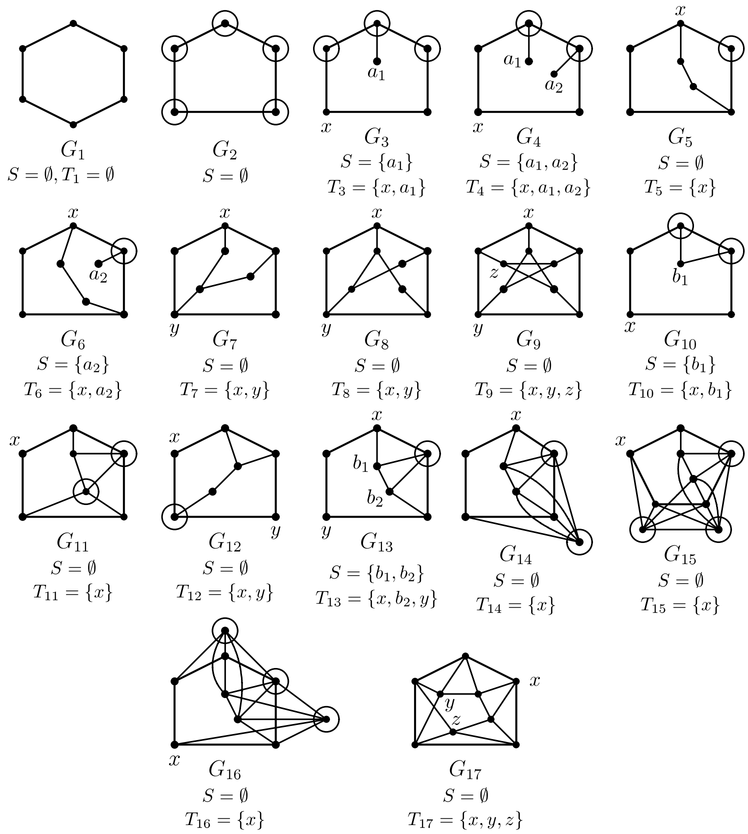

Fig. 2. Basic graphs used in Theorem 2.

Theorem 2. *If G is a connected $\{P_2 \cup P_3, C_4\}$-free graph, then G is either chordal or there exists a partition (V_1, V_2, V_3) of $V(G)$ such that (1) $[V_1] \cong K_m^c$, for some $m \geq 0$, (2) $[V_2] \cong K_t$, for some $t \geq 0$, (3) $[V_3]$ is isomorphic to a graph obtained from one of the basic graphs G_t ($1 \leq t \leq 17$) shown in Fig. 2 by expanding each vertex indicated in circle by a complete graph (of order ≥ 1), (4) $[V_1, V_3] = \emptyset$, and (5) $[V_2, V_3 \setminus S]$ is complete (see Fig. 2 for the set S).*

Proof. Let G be a connected $\{P_2 \cup P_3, C_4\}$-free graph and for $1 \leq t \leq 17$, let $\mathcal{G}_t$ denote the class of graphs obtained from G_t by the operations stated in the theorem.

If G is $\{C_5, C_6\}$-free, then G is chordal. If G is C_5-free and contains an induced C_6, then a required partition is given by Theorem 1 where $[V_3] \cong C_6 \in \mathcal{G}_1$ and $S = \emptyset$. Hence, assume that G contains an induced C_5. Among all the induced 5-cycles in G, choose one, say C such that the number of vertices in $V(G) \setminus V(C)$ that are adjacent to exactly two consecutive vertices of C is maximum. Let $C = [N_0] = [\{v_1, v_2, v_3, v_4, v_5\}] \cong C_5 \sqsubseteq G$ and let

$$A = \{x \in N_1 : [N(x) \cap N_0] \cong K_1\},$$
$$B = \{x \in N_1 : [N(x) \cap N_0] \cong K_2\},$$
$$D = \{x \in N_1 : [N(x) \cap N_0] \cong P_3\},$$
$$F = \{x \in N_1 : [N(x) \cap N_0] \cong C_5\}.$$

Then it is easily verified that $N_1 = A \cup B \cup D \cup F$. For convenience, we further partition A, B and D as follows:

$$A_i = \{x \in A : N(x) \cap N_0 = \{v_i\}\},$$
$$B_i = \{x \in B : N(x) \cap N_0 = \{v_i, v_{i+1}\}\},$$
$$D_i = \{x \in D : N(x) \cap N_0 = \{v_{i-1}, v_i, v_{i+1}\}\},$$

where $1 \leq i \leq 5$, i mod 5. Then $A = \cup_{i=1}^5 A_i$, $B = \cup_{i=1}^5 B_i$, and $D = \cup_{i=1}^5 D_i$, and the following hold:

(R1) $N_j = \emptyset$, for all $j \geq 3$ (else, $P_2 \cup P_3 \sqsubseteq G$); so $V(G) = N_0 \cup N_1 \cup N_2$.
(R2) $[A \cup B \cup D, N_2] = \emptyset$ (else, $P_2 \cup P_3 \sqsubseteq G$).
For every i, $1 \leq i \leq 5$, i mod 5, we have:
(R3) (i) $|A_i| \leq 1$ (else, $P_2 \cup P_3 \sqsubseteq G$).
(ii) $[A_i, A_{i+1}] = \emptyset$ (else, $C_4 \sqsubseteq G$).

(iii) $[A_i, A_{i+2}]$ is complete (else, $P_2 \cup P_3 \sqsubseteq G$).
(iv) $[A]$ is an induced subgraph of C_5 (a consequence of (i), (ii) and (iii)).

(R4) (i) $|B_i| \leq 1$ (else, $P_2 \cup P_3 \sqsubseteq G$).
(ii) $[B_i, B_{i+1}]$ is complete (else, $P_2 \cup P_3 \sqsubseteq G$).
(iii) If $B_i \neq \emptyset$, then $B_{i+2} = \emptyset = B_{i-2}$ (else, $P_2 \cup P_3$ or $C_4 \sqsubseteq G$).
(iv) If $|B| = 2$, then $B_k, B_{k+1} \neq \emptyset$, for some k, $1 \leq k \leq 5$, k mod 5 (a consequence of (iii)).

(R5) (i) $[D_i]$ is complete (else, $C_4 \sqsubseteq G$).
(ii) $[D_i, D_{i+2}] = \emptyset$ (else, $C_4 \sqsubseteq G$).
(iii) $[D_i, D_{i+1}]$ is complete or there exist a unique $d_i \in D_i$ and a unique $d_{i+1} \in D_{i+1}$ such that $d_id_{i+1} \notin E(G)$ (else, $P_2 \cup P_3$ or $C_4 \sqsubseteq G$). In the latter case, $D_{i+3} = \emptyset$ (else, $P_2 \cup P_3 \sqsubseteq G$).
(iv) There exists a j, $1 \leq j \leq 5$, j mod 5, such that $[N_0 \cup D] \cong \mathbb{K}(C_5) - Z$, where $Z \subseteq \{(d_j, d_{j+1}), (d'_{j+1}, d_{j+2})\}$ with $d_j \in D_j$, $d_{j+1}, d'_{j+1} \in D_{j+1}$ ($d_{j+1} \neq d'_{j+1}$) and $d_{j+2} \in D_{j+2}$ (—a consequence of (i), (ii) and (iii)).

(R6) (i) If $A_i \neq \emptyset$, then $B_j = \emptyset$, for all $j \neq i + 2$ (else, $P_2 \cup P_3$ or $C_4 \sqsubseteq G$).
(ii) $[A_i, B_{i+2}]$ is complete (else, $P_2 \cup P_3 \sqsubseteq G$).

(R7) (i) $[A_i, D_i]$ is complete (else, $P_2 \cup P_3 \sqsubseteq G$).
(ii) $[A_i, D_{i-1}] = \emptyset = [A_i, D_{i+1}]$ (else, $P_2 \cup P_3 \sqsubseteq G$).
(iii) If $A_i \neq \emptyset$, then $D_{i-2} = \emptyset = D_{i+2}$ (else, $P_2 \cup P_3$ or $C_4 \sqsubseteq G$).
(iv) If $A_i \neq \emptyset$, then $[D_i, D_{i+1}]$ and $[D_i, D_{i-1}]$ are complete (else, $P_2 \cup P_3 \sqsubseteq G$, by (i) and (ii)).

(R8) (i) $[B_i, D_j]$ is complete, for all $j \neq i - 2$ (else, $P_2 \cup P_3 \sqsubseteq G$).
(ii) $[B_i, D_{i-2}] = \emptyset$ (else, $C_4 \sqsubseteq G$).
(iii) If $B_i \neq \emptyset$, then either $D_{i-1} = \emptyset$ or $D_{i+2} = \emptyset$ (else, $P_2 \cup P_3$ or $C_4 \sqsubseteq G$).
(iv) If $B_i \neq \emptyset$, then $[D_{i+1}, D_{i+2}]$ and $[D_i, D_{i-1}]$ are complete (else, $C_4 \sqsubseteq G$).

(R9) $[F]$ is complete (else, $C_4 \sqsubseteq G$).

(R10) $[D_i, F]$ is complete (else, $C_4 \sqsubseteq G$). Hence, $[N_0 \cup D, F]$ is complete.

(R11) N_2 is independent (else, $P_2 \cup P_3 \sqsubseteq G$).

(R12) (i) If $x, y \in V(G) \setminus V(C)$ are such that $xy \in E(G)$, and if there exists an index j ($1 \leq j \leq 5$, j mod 5) such that $xv_j, yv_{j-2} \in E(G)$ and $xv_{j-2}, yv_j, xv_{j-1}, yv_{j-1} \notin E(G)$, then $[\{x, y\}, F]$ is complete (else, $P_2 \cup P_3$ or $C_4 \sqsubseteq G$).
(ii) If $x, y \in A \cup B$ and $xy \in E(G)$, then $N(x) \cap F = N(y) \cap F$ (else, $C_4 \sqsubseteq G$).

If we define $V_1 = N_2$, $V_2 = F$ and $V_3 = N_0 \cup A \cup B \cup D$, then the partition (V_1, V_2, V_3) of $V(G)$ satisfies the following requirements of the theorem: (1) by (R11), (2) by (R9), (4) by (R2), and (5) by (R10) and (R12). So, the theorem is proved if we show that $[V_3]$ satisfies (3). We will consider three cases depending on $|B|$. Note that $|B| \leq 2$, by (R4).

Case 1: $|B| = 0$.

If $A = \emptyset$, we claim that the set Z defined in (R5(iv)) is empty. If not, (w. l. o. g.) assume that $(d_1, d_2) \in Z$. Then $C' := [\{v_1, d_2, v_3, v_4, v_5\}]$ is an induced 5-cycle in G such that d_1 is adjacent to exactly two consecutive vertices v_1 and v_5 of C', a contradiction to the choice of C.

So, by properties (R3), (R5) and (R7), we conclude that $[V_3] \cong \mathbb{K}(C_5) \in \mathcal{G}_2$, if $A = \emptyset$, and $[V_3] \in \cup_{i=3}^{9} \mathcal{G}_i$, if $A \neq \emptyset$.

Case 2: $|B| = 1$.

Assume (w.l.o.g.) that $B = \{b_1\}$. Then by (R6(i)), $A_i = \emptyset$, for all $i \in \{1, 2, 3, 5\}$. So, $|A| = |A_4| \leq 1$. Also, by (R8(iii)), one of D_3 or D_5 is empty. Assume (w.l.o.g.) that $D_5 = \emptyset$. Next, we claim that $Z = \emptyset$. Assume to the contrary that $Z \neq \emptyset$. By (R8(iv)), $Z = \{(d_1, d_2)\}$ or $Z = \{(d_3, d_4)\}$. If $Z = \{(d_1, d_2)\}$, then $C' := [\{v_1, d_2, v_3, v_4, v_5\}]$ is an induced 5-cycle in G such that d_1 is adjacent to v_1 and v_5 of C', and b_1 is adjacent to v_1 and d_2 of C', a contradiction to the choice of C, and if $Z = \{(d_3, d_4)\}$, we get a similar contradiction.

So, if $|A| = 0$, then by properties (R4) and (R8), we see that $[V_3] \in \mathcal{G}_{10} \cup \mathcal{G}_{11}$.

If $|A| = 1$, let $A = A_4 = \{a_4\}$. By (R7(iii)), $D_1 = \emptyset = D_2$. By (R6(ii)), $[A_4, B_1]$ is complete. Then by using (R7(ii)) and (R8(i)), we see that $D_3 = \emptyset$ (else, $C_4 \sqsubseteq G$). Since $[A_4, D_4]$ is complete (by R7(i)), these observations imply that $[V_3] \in \mathcal{G}_{12}$.

Case 3: $|B| = 2$.

Assume (w.l.o.g.) that $B = \{b_1, b_2\}$. Then by (R6(i)), $A = \emptyset$. By (R8(iii)), we deduce that $D_5 = \emptyset$ or $D_3 = \emptyset$, since $B_1 \neq \emptyset$, and $D_1 = \emptyset$ or $D_4 = \emptyset$, since $B_2 \neq \emptyset$. Hence one of the four sets $D_1 \cup D_5$, $D_5 \cup D_4$, $D_3 \cup D_1$ or $D_4 \cup D_3$ is empty.

So, if $Z = \emptyset$, then since $b_1b_2 \in E(G)$ (by R4(ii)), we see that one of $D_4 = \emptyset$ or $D_5 = \emptyset$ (else, $C_4 \sqsubseteq G$). Hence, by properties (R4), (R5) and (R8), we conclude that $[V_3] \in \mathcal{G}_{13} \cup \mathcal{G}_{14} \cup \mathcal{G}_{15}$, if $D_1 \cup D_5 = \emptyset$ or $D_4 \cup D_3 = \emptyset$ or $D_3 \cup D_1 = \emptyset$, and $[V_3] \in \mathcal{G}_{13} \cup \mathcal{G}_{14} \cup \mathcal{G}_{16}$, if $D_4 \cup D_5 = \emptyset$.

If $Z \neq \emptyset$, then by (R8(iv)), $Z = \{(d_4, d_5)\}$. Now, we claim that $D_4 \setminus \{d_4\} = \emptyset = D_5 \setminus \{d_5\}$. Assume to the contrary that $d'_4 \in D_4 \setminus \{d_4\}$. Then $b_1b_2, d'_4d_5, b_1d_5, b_2d'_4 \in E(G)$ (by R4(ii), R5(iii), and R8(i)) and $b_1d'_4, b_2d_5 \notin E(G)$ (by R8(ii)). But then $[b_3, b_4, d'_1, d_2] \cong C_4 \sqsubseteq G$, a contradiction. So, $D_4 \setminus \{d_4\} = \emptyset$. Similarly, $D_5 \setminus \{d_5\} = \emptyset$. Thus, by properties (R4), (R5) and (R8), and by the nature of Z, we conclude that $[V_3] \in \mathcal{G}_{17}$. □

Corollary 1. *If G is a connected $\{P_2 \cup P_3, C_4\}$-free graph, then there exists a vertex v in G such that $[N[v]]$ is chordal.*

Proof. If G is chordal, then $[N[v]]$ is chordal, for all $v \in V(G)$. If G is not chordal, then consider the partition (V_1, V_2, V_3) of $V(G)$ described in Theorem 2. Let $v \in V_3$. It can be directly verified that $[N_{G_j}[v]]$ is chordal, for all j, $1 \leq j \leq 17$. Moreover, $[N_G(v) \cap V_2]$ is a complete graph. Thus, $[N_G(v)]$ is a complete expansion of $[N_{G_j}[v]] + K_1$, where $1 \leq j \leq 17$, and hence it is chordal. □

It follows that if G is a $\{P_2 \cup P_3, C_4\}$-free graph (connected or disconnected), then there exists a vertex v such that $[N[v]]$ is chordal.

2.2. Number of maximal cliques

A *universal vertex* of a graph G is a vertex which is adjacent to all other vertices in G. For later use, we make the following simple observations and state a result due to Fulkerson and Gross [7].

(M1) $\mu(G_1 \cup G_2) = \mu(G_1) + \mu(G_2)$. So, $\mu(G) = \mu(G_1) + \mu(G_2) + \cdots + \mu(G_k)$, if $G_1, G_2, \ldots, G_k$ are the components of G.
(M2) $\mu(G_1 + G_2) = \mu(G_1) \cdot \mu(G_2)$. So, if T is a set of universal vertices in a graph G, then $\mu(G - T) = \mu(G)$.
(M3) $\mu(\mathbb{K}[G]) = \mu(G)$.
(M4) If each vertex in G belongs to at most k maximal cliques, then $\mu(G) \leq \frac{kn}{2}$.

Theorem A ([7]). *If G is a chordal graph, then $\mu(G) \leq n$. Equality holds if and only if G has no edges.* □

Theorem 3. *If G is a $\{P_2 \cup P_3, C_4\}$-free graph, then $\mu(G) \leq n + 5$.*

Proof. If G is chordal, then $\mu(G) \leq n$, by Theorem A. So, assume that G is not chordal. If G is connected, then $V(G)$ admits a partition (V_1, V_2, V_3) as described in Theorem 2. Let T denote the set of all universal vertices in $[V_2 \cup V_3]$. Note that $T \subseteq V_2$. Then:

$$\begin{aligned}
\mu(G) &\leq \mu(G - V_1) + |V_1|, \quad \text{since } [V_1] \text{ is edgeless and every } v \in V_1 \text{ is simplicial} \\
&= \mu(G - V_1 - T) + |V_1|, \quad \text{by (M2)} \\
&= \mu(G_i) + |V_1|, \quad \text{for some } i,\ 1 \leq i \leq 17, \text{ by (M3)} \\
&\leq n + 5, \quad \text{since } \mu(G_i) \leq |V(G_i)| + 5, \text{ for all } i,\ 1 \leq i \leq 17.
\end{aligned}$$

Next, assume that G is disconnected. Let $F_1, F_2, \ldots, F_k$ denote the components of G. Since G is not chordal, at least one component, say F_1 admits a decomposition (V_1, V_2, V_3) of $V(F_1)$ as given in Theorem 2 such that $C_5 \sqsubseteq [V_3]$ or $C_6 \cong [V_3]$. Since G is $(P_2 \cup P_3)$-free, for every i, $2 \leq i \leq k$, F_i is an edgeless graph and hence it has exactly one vertex. So,

$$\begin{aligned}
\mu(G) &= \mu(F_1) + n - |V(F_1)|, \quad \text{by (M1)} \\
&\leq |V(F_1)| + 5 + n - |V(F_1)|, \quad \text{by the above argument} \\
&= n + 5. \quad \square
\end{aligned}$$

The upper bound given in Theorem 3 is attained for the graphs $P \cup K_t^c$, where $t \geq 0$ and P is the Petersen graph.

Corollary 2. *If G is a $\{K_1 + (P_2 \cup P_3), K_1 + C_4\}$-free graph, then $\mu(G) \leq \frac{n^2+5n}{2}$.*

Proof. Note that $[N(x)]$ is $\{P_2 \cup P_3, C_4\}$-free, for any $x \in V(G)$. So, by Theorem 3, any $x \in V(G)$ belong to at most $n + 5$ maximal cliques. Hence the result follows by (M4). □

If G is a $\{K_1 + (P_2 \cup P_3), K_1 + C_4\}$-free graph, then the upper bound for $\mu(G)$ in Corollary 2 has to be quadratic in n. For example, consider the graph $H = C_5(K_t^c, K_t^c, K_t^c, K_t^c, K_t^c)$, which is $\{K_1 + (P_2 \cup P_3), K_1 + C_4\}$-free and has $5t$ vertices. Since maximal cliques in H are edges, we have $\mu(H) = 5t^2$.

An $O(n^2m)$-time algorithm to generate all the maximal cliques in a graph which repeatedly contains a vertex whose neighborhood is chordal is described in Section 2 of [5]. Therefore, by Corollary 1, we have the following result.

Theorem 4. *All the maximal cliques in a $\{P_2 \cup P_3, C_4\}$-free graph can be generated in $O(n^2m)$-time.* □

2.3. Chromatic bound

Blászik et al. [2] proved that for every $\{2K_2, C_4\}$-free graph G, $\chi(G) \leq \omega(G) + 1$. It is shown in [3] that if G is a $\{3K_1, C_4\}$-free graph, then $\chi(G) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$. We prove that this latter bound holds for the class of $\{P_2 \cup P_3, C_4\}$-free graphs, which is a superclass of both $\{2K_2, C_4\}$-free graphs and $\{3K_1, C_4\}$-free graphs. This bound is optimal; see [3] for optimal graphs.

A graph G is said to be *perfect* if $\chi(H) = \omega(H)$, for every induced subgraph H of G. The strong perfect graph theorem (SPGT) [4] states that G is perfect if and only if G is $\{C_{2k+1}, C_{2k+1}^c\}$-free, for every $k \geq 2$.

The following assertions are easy consequences of SPGT [4].

Lemma 1. *If G is a $\{P_2 \cup P_3, C_4, C_5\}$-free graph, then G is perfect.* □

Lemma 2. *If G is a graph which admits a partition (V_1, V_2, V_3) of $V(G)$ such that (1) $[V_1]$ is an edgeless graph, (2) $[V_2]$ is complete, (3) $[V_3]$ is perfect, (4) $[V_1, V_3] = \emptyset$, and (5) $[V_2, V_3]$ is complete, then G is perfect.* □

Theorem 5. *If $G(V, E)$ is a $\{P_2 \cup P_3, C_4\}$-free graph, then $\chi(G) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$.*

Proof. Assume that G is connected. We use Theorem 2 to deduce the upper bound. If G is chordal, then G is perfect and so $\chi(G) = \omega(G)$. Else, there exists a partition (V_1, V_2, V_3) of $V(G)$ as described in Theorem 2.

If $[V_3] \in \mathcal{G}_j$, where $1 \le j \le 17, j \ne 2$, let $T_j \subseteq G_j$ be the independent set defined in Fig. 2. It is easily verified that $G_j - T_j$ is a perfect graph. So, $[V_3 \setminus T_j]$ is a complete expansion of a perfect graph. Hence, by Lemmas 1 and 2, $G - T_j$ is perfect. Therefore, $\chi(G) \le \chi(G - T_j) + \chi([T_j]) \le \omega(G) + 1$.

Next suppose that $[V_3] \in \mathcal{G}_2$. Then, $[V_3] \cong \mathbb{K}[C_5](m_1, m_2, m_3, m_4, m_5)$, where $m_i \ge 1$, for all i, $1 \le i \le 5$; let $m_1 = \min\{m_i : 1 \le i \le 5\}$. Let $V_3' = V(\mathbb{K}[C_5](m_1, m_1, m_1, m_1, m_1))$. Then $[V_3 - V_3']$ is perfect. So, $(V_1, V_2, V_3 - V_3')$ is a partition of $V(G - V_3')$ as described in Lemma 2. Hence, $G - V_3'$ is perfect. Since $\omega([V_3']) = 2m_1$ and $\chi([V_1 \cup V_3']) = \left\lceil \frac{5m_1}{2} \right\rceil$ (see [3]), we deduce that

$$\begin{aligned}
\chi(G) &\le \chi([V - (V_1 \cup V_3')]) + \chi([V_1 \cup V_3']) \\
&\le \omega([V - (V_1 \cup V_3')]) + \left\lceil \frac{5m_1}{2} \right\rceil, \quad \text{since } [V - (V_1 \cup V_3')] \sqsubseteq G - V_3', \text{ and } G - V_3' \text{ is perfect} \\
&\le (\omega(G) - 2m_1) + \left\lceil \frac{5m_1}{2} \right\rceil \\
&\le \left\lceil \frac{5\omega(G)}{4} \right\rceil, \quad \text{since } 2m_1 \le \omega(G). \quad \square
\end{aligned}$$

Acknowledgements

The authors thank the referees for their valuable suggestions and comments. The second author acknowledges the CSIR, India for the financial support to carry out this research work.

References

[1] A.A. Bertossi, Dominating sets for split and bipartite graphs, Information Processing Letters 19 (1984) 37–40.
[2] Z. Blázsik, M. Hujter, A. Pluhár, Z. Tuza, Graphs with no induced C_4 and $2K_2$, Discrete Mathematics 115 (1993) 51–55.
[3] S.A. Choudum, M.A. Shalu, The class of $\{3K_1, C_4\}$-free graphs, Australasian Journal of Combinatorics 32 (2005) 111–116.
[4] M. Chudnovsky, P. Seymour, N. Robertson, R. Thomas, The strong perfect graph theorem, Annals of Mathematics 164 (1) (2006) 51–229.
[5] M.V.G. da Silva, K. Vušković, Triangulated neighborhoods in even-hole-free graphs, Discrete Mathematics 307 (2007) 1065–1073.
[6] M. Farber, On diameters and radii of bridged graphs, Discrete Mathematics 73 (1989) 249–260.
[7] D.R. Fulkerson, O. Gross, Incidence matrices and interval graphs, Pacific Journal of Mathematics 15 (1965) 835–855.
[8] J.L. Fouquet, V. Giakoumakis, F. Maire, H. Thuillier, On graphs without P_5 and $\overline{P_5}$, Discrete Mathematics 146 (1995) 33–44.
[9] T. Kloks, $K_{1,3}$-free and W_4-free graphs, Information Processing Letters 60 (1996) 221–223.
[10] L. Lovász, A characterization of perfect graphs, Journal of Combinatorial Theory Series B 13 (1972) 95–98.
[11] V.V. Lozin, R. Mosca, Independent sets in extensions of $2K_2$-free graphs, Discrete Applied Mathematics 146 (2005) 74–80.
[12] J.W. Moon, L. Moser, On cliques in graphs, Israel Journal of Mathematics 3 (1965) 23–28.
[13] S. Poljak, A note on stable sets and colorings of graphs, Commentationes Mathematicae Universitatis Carolinae 15 (1974) 307–309.
[14] E. Prisner, Graphs with few cliques, in: Proceedings of the 7th Quadrennial International Conference on the Theory and Applications of Graphs, 1992, in: Graph Theory, Combinatorics, and Applications, 1995, pp. 945–956.
[15] B. Rosgen, L. Stewart, Complexity results on graphs with few cliques, Discrete Mathematics and Theoretical Computer Science 9 (2007) 127–136.
[16] J.P. Spinrad, Efficient graph representations, in: Fields Institute Monographs, vol. 19, American Mathematical Society, 2003.
[17] D.B. West, Introduction to Graph Theory, 2nd Edition, Prentice-Hall, Englewood Cliffs, New Jersey, 2000.