The right for doubting - and the necessity of doubt
Thoughts concerning the analysis of Erdély's Spidron System[1]

LAJOS SZILASSI
Department of Mathematics
Juhász Gyula Teacher's Training College, University of Szeged
Boldogasszony sgt. 6, H-6701 Szeged, Hungary
E-mail: szilassi@jgytf.u-szeged.hu

When the author of this article begun to examine the paper model of the geometric construction called the "Spidron-nest", first he was surprised and then doubts were overcoming his first astonishment. The author's doubts were stimulated by the disbeliefs in the remarkable geometric properties of the construction, discovered and named by Dániel Erdély, that can be observed during the deformation of the model. In this paper, we will describe the history and the result of these initial doubts regarding the Spidron System. Particularly, we analyze what are the admissible motions of the Spidron System without describing all its interesting mathematical features. We show that the motions of the "abstract" Spidron-nest can be described with rigorous mathematical tools. For this examination, we used the computer algebra system Maple that supplied us with both precise symbolic and numerical computations as well as the software provided valuable visual images for our work. Prior to our detailed analysis of the Spidron System we illustrate why mathematicians must be careful while investigating an unknown geometric construction.

First, let us begin with a seemingly irrelevant problem:

How many real solutions does the equation $\left(\frac{1}{16}\right)^x = \log_{\frac{1}{16}}(x)$ have? This problem does not seem to be difficult at first sight.

The functions $y = \left(\frac{1}{16}\right)^x$ and $y = \log_{\frac{1}{16}}(x)$ are inverses of each other, therefore, their graphs are mirror images with respect to the $y = x$ line. Thus, the graphs intersect on the $y = x$ line and because of the monotony of the functions the equation has **only one** solution that can be found only by approximation.

[1] http://www.szinhaz.hu/edan/spidronh/

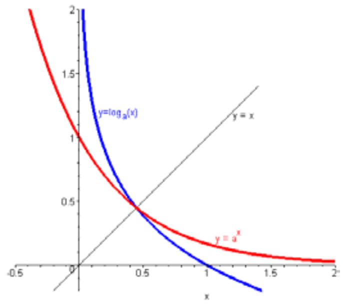

Figure 1

Without solving the original equation we can approximate the root of $x=\left(\frac{1}{16}\right)^x$ with Maple providing the (approximate) solution $x \approx 0.3642498896...$

Those readers who find the explanations above appropriate we recommend to try to solve the same equation using $x=\frac{1}{2}$ and $x=\frac{1}{4}$. What is the error in the solution[2]?

It is accurate to state that the functions on both sides of the equations are inverses and their graphs have an intersection on the line $y=x$. However, monotony of the graphs does not imply that there is no additional intersection. We illustrate this with another example that is more convincing.

Let us graph the function $y=a^x$ and its inverse $y=\log_a x$ in the same coordinate system together with their tangent lines at the point of intersection $(y_0;x_0)$ with the y=x line. In Figure 2 $a=\frac{1}{8}$ and in Figure 3 $a=\frac{1}{32}$.

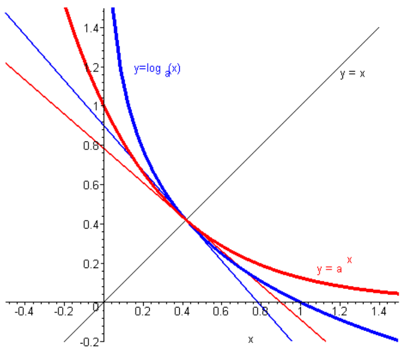

Figure 2

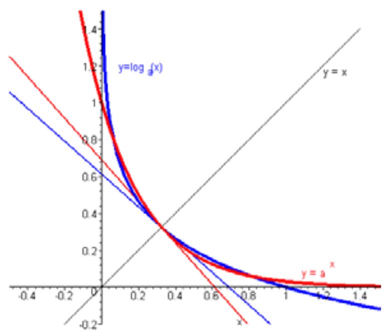

Figure 3

[2] This very interesting question is due to the Japanese mathematician SHARAGIN [3], who warned his readers to caution by nice exercises like this.

It can be observed that in the case of the exponential function with $a=\frac{1}{8}$ the slope of the tangent line is less than 45° while it is greater than 45° for its inverse. Thus, for example, when $x < x_0$ the tangent line of the logarithm function lies above the tangent line of the exponential function, and the logarithm function itself lies above this tangent line. Hence it cannot intersect the exponential function on this interval.

On the other hand, if $a=\frac{1}{32}$, then the logarithm function lies below the exponential function in the neighborhood of x_0 for $x < x_0$, since the slope of the tangent line of the logarithm function is less than (the absolute value of) 45°. At the same time, there must be an intersection, since the exponential function does and the logarithm function does not intersect the y axis. In addition, the mirror image to the $y=x$ line of this point of intersection must be a point of intersection as well. Thus we see that, when $0<a<1$, there are exactly three solutions if a is close enough to 0 and there is exactly one solution if a is close enough to 1. This result shows that a monotone function and its inverse may intersect not only on the $y = x$ line, but also at other locations, contradicting our naïve expectation.

Let us find the value of the parameter a separating the two solutions. Obviously, this must occur where the tangent line of the two functions at $(y_0\,;\,x_0)$ coincides. In other words, where the tangent line is perpendicular to the line $y = x$ (its slope is –1). From this condition we obtain a system of equations in variables a and x:

$$\left.\begin{aligned} x &= a^x \\ a^x \ln(a) &= -1 \end{aligned}\right\}$$

(On the left side of the equation the derivative of the exp function occurs.)

From this, a bit unusual, system of equations x and a – with multiple substitutions – can be easily determined:

$$x\ln a=-1 \qquad \ln a^x=-1 \qquad a^x=\frac{1}{e} \qquad x=\frac{1}{e} \qquad a^{\frac{1}{e}}=\frac{1}{e} \qquad a=\left(\frac{1}{e}\right)^e=e^{-e}\approx 0.065988$$

Thus, we obtain that if $a \le \left(\frac{1}{e}\right)^e = e^{-e} < 1$, then the equation $a^x = \log_a x$ has one real root, while in the case $0 < a < \left(\frac{1}{e}\right)^e = e^{-e}$, including $a=\frac{1}{16}=0.0625$, the equation has three real roots.

We note, in addition, that the solution $a=\frac{1}{16}=0.0625$ is so close to the limiting case considered above, that graphing the functions does not yield any hint for solving our problem. For example, according to the description above, the equation $\left(\frac{1}{15}\right)^x = \log_{\frac{1}{15}}(x)$ has only one root, since $e^{-e} \approx 0.06598 < \frac{1}{15} \approx 0.0666$.

We can only get a reasonable conjecture concerning the number of roots if we plot the difference of the two functions by some suitable graphical software. Figure 4 shows the graph of the function $y = 100\left(\left(\frac{1}{16}\right)^x - \log_{\frac{1}{16}} x\right)$. It can be seen that on the given interval (0,2; 0,52) the value of the functions in the two sides of our original equation differs only by 0,001.

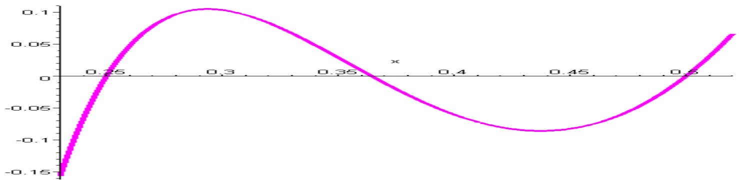

Figure 4

Mathematics can easily be mystified if mathematicians pretend to be magicians by presenting a statement (whether true or false) like the magicians take a rabbit from a stove-pipe-hat to the great amazement of the audience. Here is a "magician style" statement which was spread in algebraists circles[3]:

The value of $e^{\pi\sqrt{163}}$ is an integer, which equals exactly 262537412640768744. This statement is fairly surprising since both the base and the exponent contains transcendental number.

Is this statement true? How can anyone suspect such a relation? How can we prove the validity of this statement?

It would be a difficult task to check some complicated "proof". To disprove, it is enough to calculate its value to a certain precision. It would be difficult to do this calculation "by hand" (as one can easily check by Maple), since the fractional part is zero to 10-digit precision and only after this we get digits differing from zero. Writing the statement in the form $\left(\frac{\ln(262537412640768744)}{\pi}\right)^2 = 163$ would also result in difficulties, since the right side of the equation is *163* to 20-digit precision (!!). However, Maple can provide an even higher precision, and in this way it turns out that the number in question is not an integer.

[3] http://www.shef.ac.uk/puremath/theorems/nearint.html

It is even easier to find examples in geometry showing that solutions that “look right” differ from that are *proven* to be “right”. Let us construct an “open” bipyramid from ten equilateral triangles with unit edge length. The “openness” of this bipyramid only depends on the distance between the opposite apexes. We can choose this distance equal to the distance between the free vertices of the base of the pyramids. Then we can glue two copies of these bipyramids to a polyhedron with 20 equilateral triangles (one may perceive it as an irregular icosahedron.).

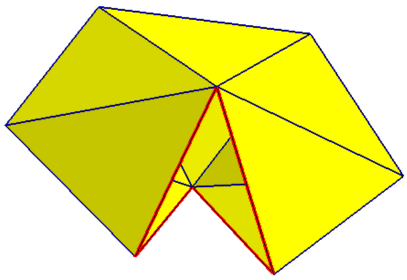

Figure 5

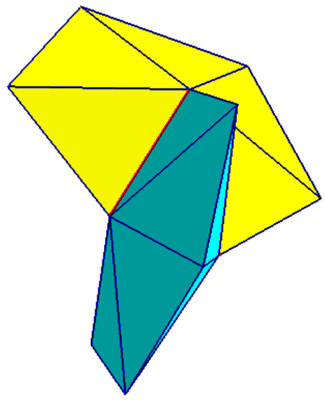

Figure 6

Using a paper model of this polyhedron we can find that it is “movable”. Is it really a “movable” polyhedron? It seems to be so. But is this really true?

To decide the question let us place this construction suitably into a three-dimensional coordinate system. Denote by x half the distance between vertex 1 and 2 (vertex 2 happens to be hidden in Figure 7 and by y half the distance between vertex 3 and 4 (that are free). Then write y as a function of x.

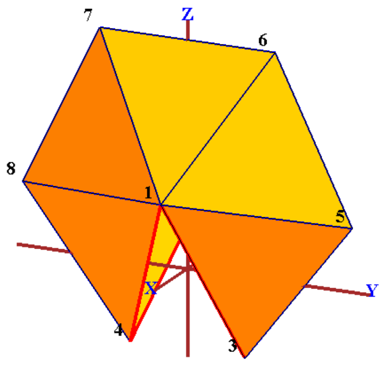

Figure 7

With the details omitted, we obtain $y = f(x) = \sqrt{1-x^2}\,\sin\left(5\arcsin\left(\frac{1}{2\sqrt{1-x^2}}\right)\right)$.

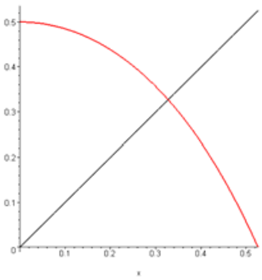

Figure 8

We have plotted the function on the interval $[0, x_0]$, where $f(x_0) = 0$, from which we obtain $x_0 = \frac{\sqrt{50-10\sqrt{5}}}{10} \approx 0.5257$

To find the coordinates for the vertices of the really existing polyhedron with regular triangular faces, we must solve the $f(x) = x$ equation. The approximate solution of this equation can be easily calculated with Maple, to a suitably high precision, yielding $x \approx 0.327267...$

If the polyhedron is movable, then the following condition must be fulfilled: the height of the polyhedron (the distance 1–2) must be equal to the distance between the free vertices (3 and 4) while continuously moving throughout a certain interval. In other words, if $y_1 = f(x_1)$ is satisfied then $x_1 = f(y_1)$ must also be satisfied. This amounts to saying that f(x) and its inverse must coincide on this interval.

The inverse of *f(x)* (on the interval in question) is the function $\bar{f}(x) = \sqrt{\frac{4x-5+\sqrt{8x+5}}{4x-10}}$.

It can be easily observed from the graphs (see Figure 9 that $f(x)$ and $\bar{f}(x)$ have exactly three points of intersection – similarly to our first example. Again, using Maple the roots of the equation $f(x) = \bar{f}(x)$ within the interval in question can also be traced.

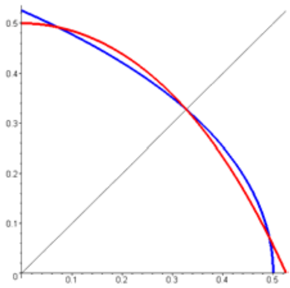

Figure 9

From the previous examples, it is not surprising that monotone functions and their inverses may have three points of intersection. Figure 10 shows that the polyhedron described above is not “movable”. However, there are two additional “stable” variants of this polyhedron (these are congruent with each other). This corresponds to the fact that the function *f(x)* and its inverse have in fact three points of intersection on the interval in question. This result has first been proved by M. Goldberg in 1978 [1]. The corresponding (x,y) pairs of coordinates are: (0.071185256… , 0.492373…) and its inverse, as well as (0.327267375, 0.327267375). The number 0.327267375 is at the same time the (approximated) root of the equation *x=f(x)*.

It is not necessary to find the inverse of f(x) to be able to calculate the numerical approximation of the equation *x=f(x)*. These values are equal to the roots of the function $g(x) = x - f\left(f\left(x\right)\right)$:

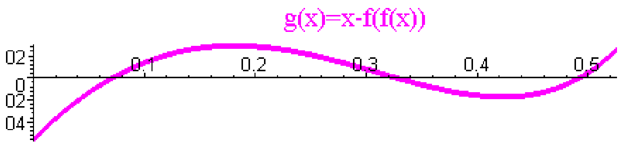

Figure 10

If we construct our polyhedron, instead of equilateral triangles, from isosceles triangles whose base is greater or less than their lateral sides by 3%, we only get one single “stable” polyhedron. This polyhedron would only be movable if the function *g(x)* took – on at least one certain interval – constantly zero value. This condition, however, is impossible.

Paper models of another kind of polyhedron, whose vertices coincide with the vertices of a regular icosahedron, are easily movable as well. Eight faces of this polyhedron are regular triangles and the other twelve faces are obtuse “golden” triangles with lateral sides of unit length and base of length $\tau = \dfrac{1+\sqrt{5}}{2}$ (where τ is the golden ratio).

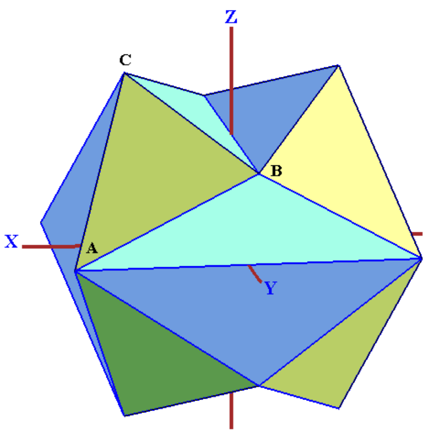

Figure 11

Let *A, B* and *C* three vertices that determine a regular triangular face of the polyhedron, as in Figure 11. To simplify computation we choose the side length of the *ABC* triangle 2. We place this polyhedron into a 3-dimensional coordinate system in a position such that each of the three points is lying on a coordinate-plane; thus their coordinates are:

$$A=[y,x,0]$$
$$B=[0,y,x]$$
$$C=[x,0,y]$$

Expressing the distance of the points in coordinates, we obtain a single relation between *x* and *y*, namely, the function $y=f(x)=\dfrac{x+\sqrt{8-3x^2}}{2}$. This function – as expected – satisfies the condition $f(1)=\dfrac{1+\sqrt{5}}{2}=\tau$ (giving the coordinates of a regular icosahedron). In addition, $f(\tau-1)=f\left(\dfrac{-1+\sqrt{5}}{2}\right)=\dfrac{1+\sqrt{5}}{2}=\tau$ is also satisfied, moreover, between the two stable states, the value of $f(x)-\tau$ differs from 0 with no more than 0.015. That is, this difference is less than 1%. Thus, due to this difference, the polyhedron given with these faces is not movable in mathematical sense, but has two stable states.

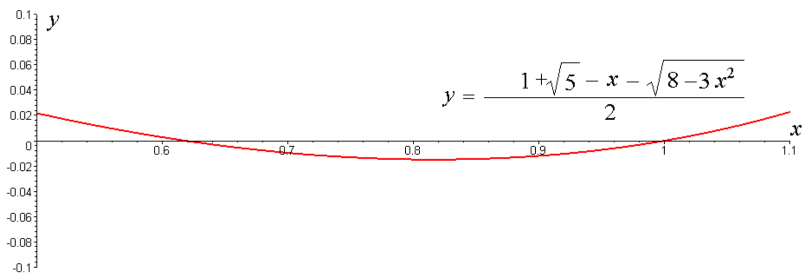

Figure 12

We note that several seemingly movable polyhedra are known ([2], p. 224). However, it is very difficult to construct a polyhedron such that is really movable and this can be proved in fact by mathematical tools [2], p. 246. In fact, the author does not know the existence of any polyhedron whose movability is proved other than the concave polyhedron described in [2], p. 246; it has 9 vertices, 14 faces and 21 edges. László Csirmaz [4] has given a proof of the movability of this polyhedron; the discovery of the polyhedron is due to Klaus Steffen.

It was Cauchy who first proved in 1813 that every convex polyhedron is rigid. Cauchy's theorem is formulated as follows: *"If two convex polyhedron structures are the same and their corresponding faces are congruent, then they are either congruent in an orientation- preserving way or mirror congruent (with respect to a plane)".* This theorem states more than just the immovability of polyhedra; convex polyhedra with the same (combinatorial) structure and congruent faces cannot even behave like our examples above, namely, cannot exist in several not congruent variants. The proof of Cauchy's theorem is rather long ([2], p. 228; [5], Exercise 58).

We hope that our argumentation above helped to show why it is important to doubt or at least to be cautious when investigating the movement of the "spidron-nest".

The "spidron-*nest*" is a regular hexagon folded up with a special procedure. Such a nest consists of six congruent *arms*, where each arm is a connected sequence of *base-figures*. This sequence may consist of arbitrary number of elements, however, it is sufficient to compute (or draw) a few of its starting elements. Define the *depth* of the spidron (nest, or arm) to be n if the spidron-arm consists of n connected base-figures.

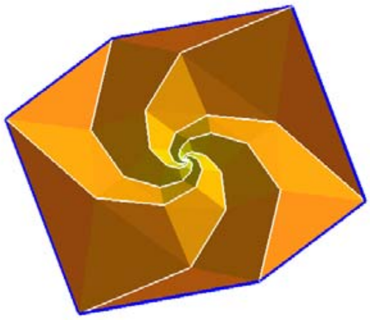

Figure 13

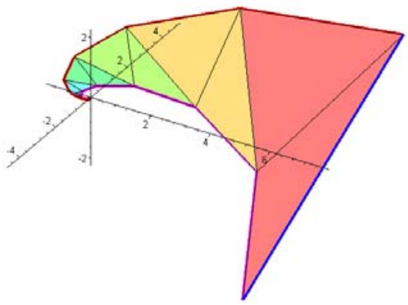

Figure 14

The base-figure contains an isosceles triangle with vertex angle 120° (ABD triangle) and a matching equilateral triangle (ACD triangle) such that they together form a right triangle, provided they are in the same plane.

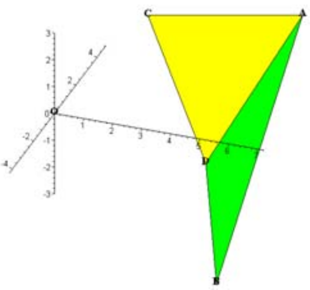

Figure 15

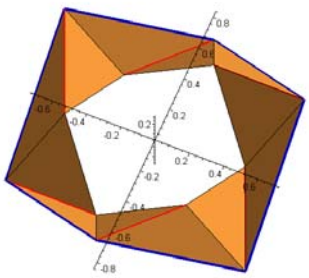

Figure 16

Later on we shall use the term spidron *ring*, which is, so to say, a “spidron-nest of depth 1”: it consists of six congruent base-forms.

Observe that performing a suitable deformation (i.e. that is possible along the common sides of the adjacent triangles) the “edge” of the nest forms a skew hexagon such that all of its (equal) sides incline at the same angle to a certain plane – in fact, to the plane determined by their midpoints. We call this plane the *plane of the spidron-nest* (in the limiting case the spidron-nest is a two-dimensional object – it is the planar spidron-nest). In addition, we observe that the orthogonal projection of the regular skew hexagon onto this plane is a regular hexagon as well.

<u>We obtain a *regular skew hexagon*</u> if we only require of a closed broken line consisting of six line segments that its sides as well as the angles of its adjacent sides be equal. Obviously, this spatial hexagon does not form a regular planar hexagon. We define a *spatial regular hexagonal surface* to be a figure consisting of six isosceles triangles such that their bases form a regular skew hexagon and they have their vertices in common forming the center of the whole figure. This spatial hexagonal surface may be movable along its adjacent edges. However, if we require that it remain to be a spatial *regular* hexagonal surface, then it will be rigid. For, if its edges incline at a greater angle to the plane of the hexagon, then they get closer to the center, and hence the lateral sides of the constituent isosceles triangles must also decrease. This spatial regular hexagonal surface is a centrally symmetric figure, and so is the spidron-nest, as we shall see later.

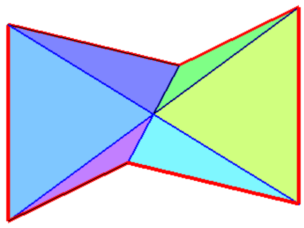

Figure 17

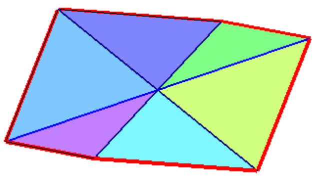

Figure 18

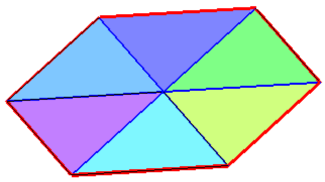

Figure 19

<u>We demand</u> that the edge of the spidron-nest must be a regular (either skew or planar) hexagon.

We shall prove that if the (outer) edge of the spidron-nest (which at the same time is the edge of the first spidron ring as well) is a regular skew hexagon, then its interior edge is also a regular skew hexagon. This provides the possibility to form the spidron-nest as a sequence of nested spidron rings. It should be mentioned that the interior edges of the spidron ring form smaller angles with the plane of the spidron-nest than its outer edges. Therefore, we cannot state that, in general, the spidron-nest of infinite depth is a self-similar object. However, we can state this for its planar variant.

For the description of the motions of a spidron-nest, it is sufficient to investigate only the motions of one of its base elements, since by using appropriate parameters we can build a spidron-arm of such elements, and by applying consecutive congruent transformations, the whole spidron-nest as well.

Let us place the spidron-nest into a 3-dimensional coordinate system in a position such that its plane coincides with the *xy*-plane and so does its center with the origin.

A line segment in the edge of the spidron-nest can be described by three parameters:

- d : the length of the line segment;
- α : the direction angle of the position vector pointing to the midpoint of the edge;
- β : the angle of the edge with the *xy*-plane.

The corresponding commands in a Maple procedure are as follows:

```
V:= proc(d,alpha,beta)
    local F,A,c,V;
       c:=d*cos(beta)*sqrt(3)/2:
       F:=[c*cos(alpha),c*sin(alpha),0]:
       A:=[d*sin(alpha)*cos(beta)/2,
           d*cos(alpha)*cos(beta)/2,
           d*sin(beta)/2]:
       V:=[F+A,F-A]:
    end proc:
```

The orthogonal projection of the edge onto the *xy*-plane is $\mathrm{d}\cos(\beta)$, thus the distance of its midpoint from the origin is $\mathrm{c} = \mathrm{d}\cos(\beta)\frac{\sqrt{3}}{2}$. Using these expressions we can determine the coordinates of the midpoint and of one endpoint of the edge. These coordinates can be obtained as the output of the Maple procedure above.

The spidron-nest is uniquely determined by an outer spidron ring which, in turn, is uniquely determined by the skew hexagon described above (the outer edge of this ring) . This skew hexagon can be described by two parameters: *d,* the length of its edge, and *f*, the angle formed by the *xy*-plane and by the edge.

Let us place the edge *AB* of the base-figure into a 3-dimensional coordinate system with its midpoint located on the *x*-axis. Then determine the segment *CD* using the procedure above and temporarily taken *a*, *b* as indeterminate parameters. Then the vertices of the base-figure can be given as follows:

$$A := \left[\frac{1}{2}\, d \cos(f) \sqrt{3}, \frac{1}{2}\, d \cos(f), \frac{1}{2}\, d \sin(f)\right]$$

$$B := \left[\frac{1}{2}\, d \cos(f) \sqrt{3}, -\frac{1}{2}\, d \cos(f), -\frac{1}{2}\, d \sin(f)\right]$$

$$C := \left[-\frac{1}{6}\, d \sqrt{3} \sin(a) \cos(b) + \frac{1}{2}\, d \cos(b) \cos(a), \right.$$
$$\left. \frac{1}{6}\, d \sqrt{3} \cos(a) \cos(b) + \frac{1}{2}\, d \cos(b) \sin(a), \frac{1}{6}\, d \sqrt{3} \sin(b)\right]$$

$$\mathrm{D} := \left[\frac{1}{6} d\sqrt{3}\,\sin(a)\cos(b) + \frac{1}{2} d\cos(b)\cos(a),\right.$$
$$\left. -\frac{1}{6} d\sqrt{3}\,\cos(a)\cos(b) + \frac{1}{2} d\cos(b)\sin(a), -\frac{1}{6} d\sqrt{3}\,\sin(b)\right]$$

We want to express a and b as a function of f under the condition that the length of the line segments BD, DC, CA and DA be equal to $\mathrm{d}\frac{\sqrt{3}}{3}$ alike. The condition $\mathrm{CD} = \mathrm{d}\frac{\sqrt{3}}{3}$ is an identity that is valid for all (a, b) values, as a consequence of our equations above. The transformation that transforms the line segment AB to CD takes A to C and B to D; accordingly, $AC=BD$ must be satisfied. It can be easily seen from the Maple results. Thus we obtained a system of equations in two variables, where a and b are the unknowns and f is taken as an independent parameter:

$$e1 := \frac{3}{4}\cos(b)^2 - 2\cos(b)\cos(a)\cos(f)\sqrt{3} + \frac{9}{4}\cos(f)^2 - \frac{1}{2}\sqrt{3}\,\sin(b)\sin(f)$$

$$e2 := \frac{3}{4}\cos(b)^2 - 3\sin(a)\cos(b)\cos(f) - \cos(b)\cos(a)\cos(f)\sqrt{3} + \frac{9}{4}\cos(f)^2 + \frac{1}{2}\sqrt{3}\,\sin(b)\sin(f)$$

Nevertheless, we had to pick the solutions provided by Maple that satisfied the desired geometric properties. Moreover, in some cases we had to “help” Maple in performing some trigonometric simplifications that the geometric content made possible, but that are not permitted otherwise. Thus we obtained the following expressions for a and b, as functions of f:

$$\mathrm{a(f)} = \arccos\left(\frac{\sqrt{3}\left(1-2\cos(\mathrm{f})+3\cos^2(\mathrm{f})\right)-\sin(\mathrm{f})\sqrt{-1+4\cos(\mathrm{f})-3\cos^2(\mathrm{f})}}{4\cos(\mathrm{f})\sqrt{2-4\cos(\mathrm{f})+3\cos^2(\mathrm{f})}}\right)$$

$$\mathrm{b(f)} = \arctan\left(\sqrt{\frac{-1+4\cos(\mathrm{f})-3\cos^2(\mathrm{f})}{2-4\cos(\mathrm{f})+3\cos^2(\mathrm{f})}}\right)$$

Since we applied computer and manual computations alike, we could not be sure that our results described in fact the motions of the spidron. Thus we had to check that the equalities e_1=0 and e_2=0 were fulfilled in all cases, independently of f. This investigation yielded a certain trigonometric expression for which we had to check that it was identically zero, for all values of f; we present it here, to scare the reader:

$$-3\cos(f)^2 - 1 + 2\cos(f) + 2\sqrt{2}\left(\left(\left(2\cos(f)^2 - 12\cos(f)^3 + 9\cos(f)^4 - 1 + 4\cos(f) + \right.\right.\right.$$
$$\sqrt{2 - 4\cos(f) + 3\cos(f)^2}\,\sqrt{\frac{1}{2 - 4\cos(f) + 3\cos(f)^2}}\,\sqrt{3}$$

$$\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)-2\sqrt{2-4\cos(f)+3\cos(f)^2}$$
$$\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\,\cos(f)\sqrt{3}\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)+3$$
$$\cos(f)^2\sqrt{2-4\cos(f)+3\cos(f)^2}\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\sqrt{3}$$
$$\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)\bigg)\bigg/\left(\left(2-4\cos(f)+3\cos(f)^2\right)\cos(f)^2\right)\bigg)^{(1/2)}$$
$$\cos(f)\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}-4\sqrt{2}\bigg(\bigg(2\cos(f)^2-12\cos(f)^3+9\cos(f)^4-1$$
$$+4\cos(f)+\sqrt{2-4\cos(f)+3\cos(f)^2}\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\sqrt{3}$$
$$\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)-2\sqrt{2-4\cos(f)+3\cos(f)^2}$$
$$\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\,\cos(f)\sqrt{3}\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)+3$$
$$\cos(f)^2\sqrt{2-4\cos(f)+3\cos(f)^2}\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\sqrt{3}$$
$$\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)\bigg)\bigg/\left(\left(2-4\cos(f)+3\cos(f)^2\right)\cos(f)^2\right)\bigg)^{(1/2)}$$
$$\cos(f)^2\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}+3\sqrt{2}\bigg(\bigg(2\cos(f)^2-12\cos(f)^3+9\cos(f)^4-1$$
$$+4\cos(f)+\sqrt{2-4\cos(f)+3\cos(f)^2}\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\sqrt{3}$$
$$\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)-2\sqrt{2-4\cos(f)+3\cos(f)^2}$$
$$\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\,\cos(f)\sqrt{3}\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)+3$$
$$\cos(f)^2\sqrt{2-4\cos(f)+3\cos(f)^2}\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\sqrt{3}$$
$$\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)\bigg)\bigg/\left(\left(2-4\cos(f)+3\cos(f)^2\right)\cos(f)^2\right)\bigg)^{(1/2)}$$
$$\cos(f)^3\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}-\sqrt{2-4\cos(f)+3\cos(f)^2}$$
$$\sqrt{\frac{1}{2-4\cos(f)+3\cos(f)^2}}\sqrt{3}\sqrt{-1+4\cos(f)-3\cos(f)^2}\,\sin(f)$$

We successfully arrived at a result that also required the cooperative efforts of the software and of the doubting user.

Now, we can be assured that the spidron-nest is movable and its motion can be described by the functions $a(f)$ and $b(f)$ given above.

Although the domain of these two functions is wider than described, the values with actual geometric meaning occur within the interval $\beta_1 = f\left[0, \frac{\pi}{3}\right]$. For, if $f = \frac{\pi}{3}$, then the triangles *ABD* and *ACD* are both perpendicular to the *xy*-plane; thus, higher values would provide a self-intersecting surface, which cannot be realized physically.

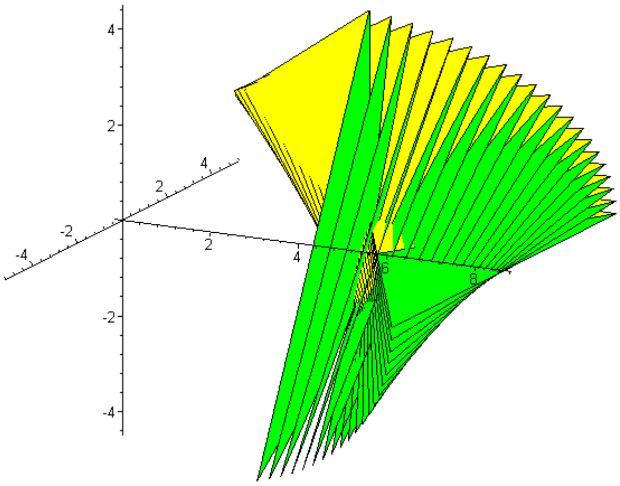

Figure 20

To realize a spidron-arm or a spidron-nest, knowing the values of the key parameters, is an easy task now.

We obtain the first base-figure starting with the $d_1 = d$, $\alpha_1 = 0$ $\beta_1 = f$ values. From the input data of the *k*-th base-figure of a spidron-arm one obtains the input data for the (*k+1*)-th base-figure using the following recursive formulas:

$$d_{k+1} = \frac{d_k\sqrt{3}}{3}$$
$$\alpha_{k+1} = \alpha_k + a(\beta_k)$$
$$\beta_{k+1} = b(\beta_k)$$

However, the graph of the functions $a(f)$ and $b(f)$ does not offer opportunity for a more thorough investigation of the properties of the spidron-nest.

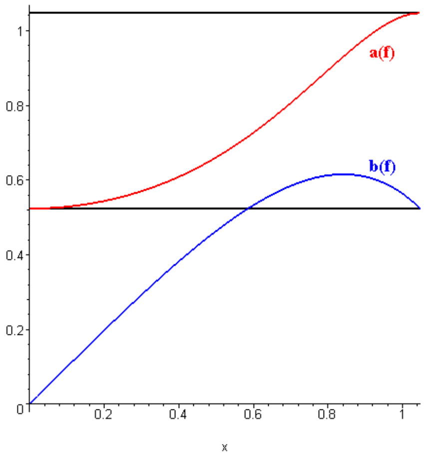

Figure 21

At most, we can observe that as the angle of the edges of the outermost ring with the *xy*-plane increases, the projection of the interior hexagon of the first spidron ring onto the *xy*-plane turns with respect to that of the outer hexagon to a greater extent as well; in the case of f = 60° the degree of this turning is just 60°. The degree of turning is continuously decreases to 30° corresponding to the planar case. Similarly, the angle of edges to the *xy*- plane is gradually decreasing as well, although this decrease is slow. Accordingly, the interior – and, smaller and smaller – rings of the spidron-nest are approaching the *xy*-plane.

Will actually the rings reach the *xy*-plane?

To answer this question we must prove that the sequence given by the following recursive formulas

$$\beta_1 = f \quad \left(0 \le f \le \frac{\pi}{3}\right)$$

$$\beta_{k+1} = b(\beta_k) \quad (k = 2, 3, \ldots)$$

tends to zero, that is, $\lim \beta_n = 0$, that means that the vertical lines in our Figure 22 below are accumulating at the origin.

This problem is not obvious, for if we would define this sequence through the function $y = f$ instead of $y = b(f)$, then it were a constant sequence. However, our function $b(f)$ converges just to the function $y = f$ around the origin: $\lim\limits_{f \to 0^+} \left(b'(f)\right) = 1$

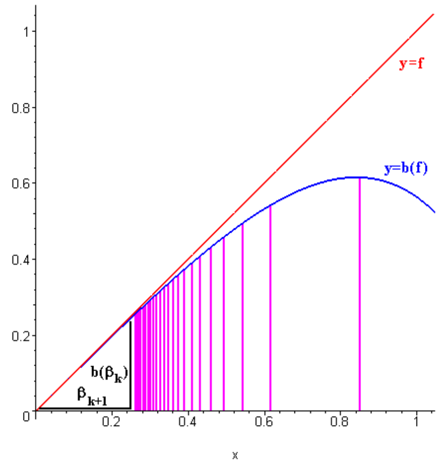

Figure 22

It is easily proven either by hand or by Maple that the function $b(f)$ is continuous on the $\left(0,\frac{\pi}{3}\right]$ interval in question. Moreover, on this interval the inequality $b(f)<f$ is also satisfied.

In more generality: if the function $\mathrm{y}=\mathrm{b(f)}$ is continuous and everywhere on this interval its value satisfies the relation $0<\mathrm{b(f)}<\mathrm{f}$, then every similarly defined recursive sequence converges to zero. For, if the limit of the sequence would be a number *h* greater than zero: 0 < *h*, then, due to continuity, the function at *h* took just the value *h*, contradicting our conditions.

Of course, the tools of mathematics do not provide a good intuitive picture about the "speed" of convergence of this sequence. Thus, we don't know when the nest of the spatial spidron becomes more and more similar to a planar spidron-nest.

This remarkable website:
http://micro.magnet.fsu.edu/primer/java/scienceopticsu/powersof10/index.html illustrates the size of the Universe. To our present knowledge, the size of the Universe is 10^{23} meter and the size of the smallest known physical object, a quark, is 10^{-16} meter.

Let us imagine a "Universe-sized" spatial spidron-nest whose one outer edge is, say, 10^{23} meter. Since the interior edge of each spidron ring is $\frac{\sqrt{3}}{3}$ times smaller than the outer one, we get a quark-sized spidron ring in some 170 steps. Well, using Maple it is easy to show that the angle of its edge to the plane of the spidron is still more than 6°. Of course, this is merely playing with the forms (or with the notions) and is not to be taken too serious.

Practically, a model of the spidron-nest consists of merely 5-6 rings with holes in their center. It is only worthy even to draw a maximum of 8-9 rings.

If we want to have a spidron-nest with continuous surface we can patch the “hole” inside with a spatial regular hexagon. The resulting object is rigid in mathematical sense, as the regular hexagon is immovable in its center. This object consists of a finite number of triangles.

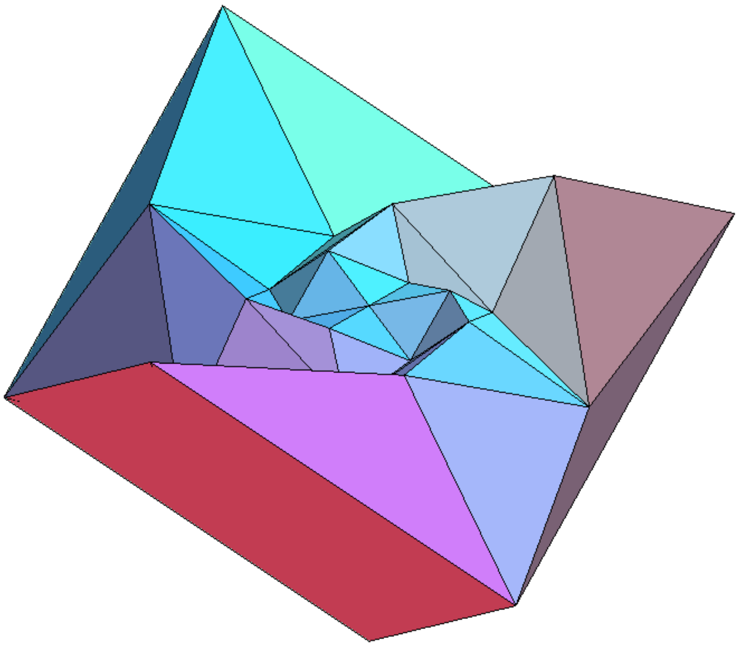

Figure 23

The object in Figure 23 is a polyhedron with 6*7+3=45 faces build from three spidron-nests and six triangles patching the hole in the center of the spidron. Moreover, it contains three mutually neighboring faces of a cube as well. One obtains a cube by joining up a “right-handed” and a “left-handed” variant of it.

We note, too, that it is senseless to ask whether a spidron-nest, containing a spiral-like broken line, twists left or right. At the most, we can only relate to each other two joining spidron-nests as they are of the same or opposite orientation.

Inventing further structures build from spidron rings and nests, delighting the eyes and stirring the fantasy is, nevertheless, the task of the discoverer of the Spidron, and not of a geometer who may sometimes seem to be too scrupulous.

In addition, it would be quite challenging, in mathematical sense, to understand the properties of an “infinitely deep” spidron-nest. The mathematical description of such a spidron requires the description of a spidron-ring which, except the first ring, has both inner and outer neighbors. Yet, this mathematical challenge is beyond the scope of this paper.

REFERENCES:

[1] CROMWELL, P. M.: *Polyhedra*, Cambridge University Press, Cambridge, 1997.

[2] GOLDBERG, M.: *Unstable Polyhedral Structures*, Math. Magazine, **51**, (1978) 165–170.

[3] SHARAGIN, I. F.: *So, what's wrong?* QUANTUM, **8** (6), (1998) 34–37; 53–54.

[4] CSIRMAZ, LÁSZLÓ: *Mozoghat-e valaki, akit síklapokkal határolt páncélba öltöztettünk?* (*Can one move who is dressed in an armor of polyhedral structure? – in Hungarian*), KÖMAL, 59(3–4), (1979) 119–123.

[5] SKLJARSZKIJ, D. O., CSENCOV, N. N., JAGLOM, I. N.: *Válogatott feladatok és tételek az elemi matematika köréből* (*Selected exercises – in Hungarian*) *3. Geometria II. (Sztereometria)*, Tankönyvkiadó, Budapest, 1968.