Precoloring Extension on Grid Graphs

extended abstract

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1 Introduction

Precoloring extension is a vertex coloring problem under the situation that colors are already assigned to some of the vertices [1, 2, 3, 5]. Formally, precoloring extension is defined as follows.

Precoloring Extension
Input: A graph $G = (V, E)$, a subset $W \subseteq V$, an integer k and a k-coloring c_p of W.
Question: Is there a k-coloring of G which satisfy $c(v) = c_p(v)$ for all $v \in W$?

In this paper, we consider precoloring extension on grid graphs. A rectangular grid is a graph (V, E), where $V = \{v_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ for some n, m and $E = \{(v_{i,j}, v_{i+1,j}) \mid 1 \leq i < n, 1 \leq j \leq m\} \cup \{(v_{i,j}, v_{i,j+1}) \mid 1 \leq i \leq n, 1 \leq j < m\}$. A grid graph is a subgraph of a rectangular grid. In the following of this paper, we assume that vertices of a grid graph is given as the form of $v_{i,j}$.

We show that precoloring extension on grid graphs is NP-complete for $k = 3$ and solved in polynomial time for $k \neq 3$.

2 Precoloring Extension on Grid Graphs

2.1 Polynomial Time Solvability

Precoloring extension problem on gird graphs has obvious solutions for $k = 2$ and $k \geq 5$. For $k = 2$, it is known that precoloring extension is solved in polynomial time for general graphs [2]. For $k \geq 5$, as each vertex has degree at most 4, any precoloring can be extended to a k-coloring of the whole graph. We show the following result for $k = 4$.

Theorem 1. *Precoloring extension of grid graphs can be solved in polynomial time for* $k = 4$.

In the rest of this section, we give the sketch of the proof of this theorem. From the precoloring c_p of W, colors of some vertices may be uniquely decided. If three different colors are used in the neighboring vertices, the color of the vertex is uniquely decided. Let *obvious extension* be the process to repeatedly assign colors to such vertices until there exists no vertex whose color is uniquely decided. We call that coloring c of some vertices *causes a conflict* if a vertex whose four neighbors has different colors are found in the obvious extension.

Lemma 2. *If precoloring c_p does not cause a conflict, it can be extended to a 4-coloring of G*

Proof. Let W' be the set of vertices that are colored after obvious extension of c_p, and c'_p be the coloring of W'. A coloring of the vertices in $V - W'$ can be obtained in the following algorithm. In the algorithm, the priority of vertices is determined so that vertex $v_{a,b}$ has higher priority than $v_{c,d}$ if either $a < c$ or $(a = c) \wedge (b < d)$ is satisfied.

Extension Algorithm
1. Repeat steps 2 to 4 until all the vertices are colored.
2. Arbitrarily assign a color to the uncolored vertex v with the highest priority.
3. Execute obvious extension from the present coloring.
4. If the coloring causes a conflict, change the color of v and execute obvious extension.

It remains to prove that a 4-coloring of G is obtained by the above algorithm. As vertex v is chosen according to the priority, at most two neighbors of v are not colored. Let Y be the set of vertices that are colored in one execution of Step 3. The graph induced by Y forms a path because a vertex whose color is uniquely decided has at most one neighbor which is not colored yet. Especially, the graph induced by $Y \cup \{v\}$ forms a cycle when a conflict is caused. As no vertex has three different colors in its neighbors before executing Step 2, a conflict is caused when two of the neighbors are newly colored in the obvious extension. It is the case that can occur only if $Y \cup \{v\}$ forms a circuit.

An example is shown in Fig.1. Numbers in the vertices represent the colors of the vertices that are already assigned. In this figure, v can be colored with 3 or 4. A conflict occurs if v is colored with 3, and there exists an extension if v is colored with 4.

Proposition 3. *Assume that a conflict is caused in extending a coloring to vertices in $Y \cup \{v\}$ when we assigned a color to vertex v in Step 2. Then we*

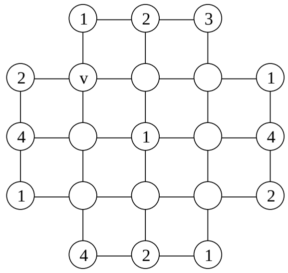

Figure 1: A cycle induced by $Y \cup \{v\}$.

can extend the coloring to $Y \cup \{v\}$ when we assign another color to v.

Proof of Proposition. Let $\{1, 2, 3, 4\}$ be the set of colors and $v = v_{i,j}$ can be colored with 1 and 2. Assume that a conflict is caused in $Y \cup \{v_{i,j}\}$ both when v is colored with 1 and when v is colored with 2. Due to the priority of vertices, $v_{i+1,j} \in Y$. As a conflict is caused by coloring $v_{i,j}$ with 1, no vertex adjacent to $v_{i+1,j}$ is colored with 1 before Step 2. Otherwise, at most three colors can be used in the neighbors of $v_{i+1,j}$, which means that no conflict is caused. Similarly, no vertex adjacent to $v_{i+1,j}$ is colored with 2. That is, the colors used in the vertices adjacent to $v_{i+1,j}$ are 3 and 4, and so $v_{i+1,j}$ can be colored with 2 if $v_{i,j}$ is colored with 1, and can be colored with 1 if $v_{i,j}$ is colored with 2.

By repeating this argument to adjacent vertices in Y, all the vertices in Y must be colored with 1 or 2 alternately. As the length of the cycle induced by $Y \cup \{v\}$ is even, no conflict can be caused, which is a contradiction. □

The coloring of the cycle does not affect the coloring of the other vertices because all the vertices adjacent to the cycle are already colored before Step 2. Therefore, there exists a 4-coloring of G unless the precoloring causes a conflict. □

From Lemma 2, whether the precoloring has an extension can be checked by executing obvious extension once. Thus Theorem 1 is proved.

2.2 NP-Completeness for $k = 3$

Theorem 4. *Precoloring extension on grid graphs is NP-complete for $k = 3$.*

Proof. We prove the NP-completeness by reduction from PLANER-3SAT[4]. The variable and clause gadgets are shown in Fig.2 and Fig.3. The value of a variable corresponds to the color of the vertex represented by a bold circle in Fig.2. Color 2 corresponds to value 1 and color 3 corresponds to value 0. Copies of the variable are generated in two vertices represented by dotted circles.

In the clause gadget, three bold vertices are connected to variable gadgets and colored with 2 or 3. The precoloring in the dotted rectangles depends on whether the literal is positive or negative. There exists an extension of the coloring in the gadget iff at least one of the literals in the clause is true. □

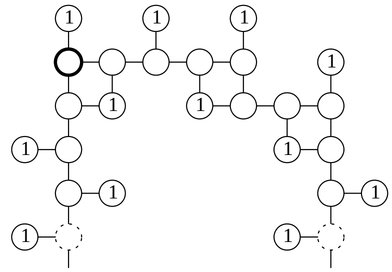

Figure 2: A variable gadget.

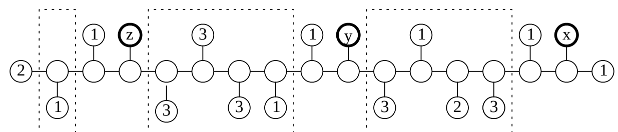

Figure 3: A clause gadget for $\overline{x} + y + z$.

Acknowledgement

This research is partially supported by by the Grant-in-Aid for Scientific Research(C) (21500007) of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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