Improprius integrálok

$$\int_1^\infty \frac{1}{x}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned} \int \frac{1}{x} &= \log|x| \\ [\log|x|]_1^b &= \log b \\ \lim_{b\to+\infty} \log b &= +\infty \\ \int_1^\infty \frac{1}{x}\ dx &= +\infty \end{aligned}$$

$$\int_{-\infty}^0 xe^x\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned} \int_a^0 xe^x &= e^x\,(x-1) \\ [e^x\,(x-1)]_a^1 &= (1-a)e^a - 1 \\ \lim_{a\to-\infty} (1-a)e^a &= 0 \\ \int_{-\infty}^0 xe^x\ dx &= -1 \end{aligned}$$

$$\int_{-\infty}^0 \frac{1}{(2x-1)^2}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned} \int (2x-1)^{-2} &= \frac{1}{2-4x} \\ \left[\frac{1}{2-4x}\right]_a^0 &= \frac{1}{2} - \frac{1}{2-4a} \\ \lim_{a\to-\infty} \frac{1}{2-4a} &= 0 \\ \int_{-\infty}^0 \frac{1}{(2x-1)^2}\ dx &= \frac{1}{2} \end{aligned}$$

$$\int_0^{+\infty} \frac{x}{\sqrt{1+x^2}}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{x}{\sqrt{1+x^2}} &= \sqrt{x^2+1} \\
\left[\sqrt{x^2+1}\right]_0^b &= \sqrt{b^2+1} - 1 \\
\lim_{b\to+\infty} \sqrt{b^2+1} &= +\infty \\
\int_0^{+\infty} \frac{x}{\sqrt{1+x^2}} &= +\infty
\end{aligned}$$

$$\int_3^{+\infty} \frac{1}{\sqrt{(x-1)^3}}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{1}{\sqrt{(x-1)^3}} &= \int (x-1)^{-1.5} = \frac{-2}{\sqrt{x-1}} \\
\left[\frac{-2}{\sqrt{x-1}}\right]_3^b &= \sqrt{2} - \frac{2}{\sqrt{b-1}} \\
\lim_{b\to+\infty} \frac{2}{\sqrt{b-1}} &= 0 \\
\int_3^{+\infty} \frac{1}{\sqrt{(x-1)^3}}\ dx &= \sqrt{2}
\end{aligned}$$

$$\int_1^{2} \frac{1}{\sqrt{(x-1)^3}}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{1}{\sqrt{(x-1)^3}} &= \int (x-1)^{-1.5} = \frac{-2}{\sqrt{x-1}} \\
\left[\frac{-2}{\sqrt{x-1}}\right]_a^2 &= \frac{2}{\sqrt{a-1}} - 2 \\
\lim_{a\to+1^+} \frac{2}{\sqrt{a-1}} &= +\infty \\
\int_1^{2} \frac{1}{\sqrt{(x-1)^3}}\ dx &= +\infty
\end{aligned}$$

$$\int_1^{+\infty} \frac{2}{x^3}\, dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{2}{x^2} &= 2\int x^{-2} = \frac{-2}{x} \\
\left[\frac{-2}{x}\right]_1^b &= \frac{2(b-1)}{b} \\
\lim_{b\to+\infty} \frac{2(b-1)}{b} &= 2 \\
\int_1^{+\infty} \frac{2}{x^3}\, dx &= 2
\end{aligned}$$

$$\int_{-\infty}^{-1} \frac{3}{x^4}\, dx = ?$$

Megoldás:

$$\begin{aligned}
\int \frac{3}{x^4} &= 3\int x^{-4} = \frac{-1}{x^3} \\
\left[\frac{-1}{x^3}\right]_a^{-1} &= \frac{1}{a^3} + 1 \\
\lim_{a\to-\infty} \frac{1}{a^3} &= 0 \\
\int_{-\infty}^{-1} \frac{3}{x^4}\, dx &= 1
\end{aligned}$$

$$\int_1^{+\infty} \frac{2}{x(x+2)}\, dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\frac{2}{x(x+2)} &= \frac{1}{x} - \frac{1}{x+2} \\
\int \frac{2}{x(x+2)} &= \log|x| - \log|x+2| = \log\left|\frac{x}{x+2}\right| \\
\left[\log\left|\frac{x}{x+2}\right|\right]_1^b &= \log\frac{b}{b+2} + \log 3 \\
\lim_{b\to+\infty} \log\frac{b}{b+2} &= 0 \\
\int_1^{+\infty} \frac{2}{x(x+2)}\, dx &= \log 3
\end{aligned}$$

$$\int_0^{+\infty} x^2 e^{-x}\, dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int x^2 e^{-x} &= -(x^2+2x+2)e^{-x} \\
\left[-(x^2+2x+2)e^{-x}\right]_0^b &= 2-2(b^2+2b+2)e^{-b} \\
\lim_{b\to+\infty} \frac{(b^2+2b+2)}{e^b} &= 0 \\
\int_0^{+\infty} x^2 e^{-x}\, dx &= 2
\end{aligned}$$

$$\int_{-4}^{3} \frac{1}{\sqrt{3-x}}\, dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{1}{\sqrt{3-x}} &= -2\sqrt{3-x} \\
\left[-2\sqrt{3-x}\right]_{-4}^b &= \sqrt{28}-2\sqrt{3-b} \\
\lim_{b\to+3^-} \sqrt{3-b} &= 0 \\
\int_{-4}^{3} \frac{1}{\sqrt{3-x}}\, dx &= \sqrt{28}
\end{aligned}$$

$$\int_{-\infty}^{+\infty} \frac{1}{2+3x^2}\, dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{1}{2+3x^2} &= \frac{\arctan\sqrt{1.5}x}{\sqrt{6}} \\
& \left[\frac{\arctan\left(\sqrt{1.5}x\right)}{\sqrt{6}}\right]_{-b}^b \\
\left[\frac{\arctan\left(\sqrt{3/2}x\right)}{\sqrt{6}}\right]_{-b}^b &= \sqrt{\frac{2}{3}}\arctan\left(\sqrt{3/2}b\right) \\
\lim_{b\to\infty} \arctan\left(\sqrt{3/2}b\right) &= \frac{\pi}{2} \\
\int_{-\infty}^{+\infty} \frac{1}{2+3x^2}\, dx &= \frac{\pi}{\sqrt{6}}
\end{aligned}$$

$$\int_{-\infty}^{0} \frac{1}{1+e^{-2x}}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int \frac{1}{1+e^{-2x}} &= x + \log\sqrt{1+e^{-2x}} \\
\left[x + \log\sqrt{1+e^{-2x}}\right]_a^0 &= \log\sqrt{2} - a - \log\sqrt{1+e^{-2a}} \\
\lim_{a\to-\infty}\left(a + \log\sqrt{1+e^{-2a}}\right) &= \lim_{a\to-\infty}\left(-\log\sqrt{e^{-2a}} + \log\sqrt{1+e^{-2a}}\right) \\
&= \lim_{a\to-\infty}\log\sqrt{\frac{1+e^{-2a}}{e^{-2a}}} = 0 \\
\int_{-\infty}^{0} \frac{1}{1+e^{-2x}}\ dx &= \log\sqrt{2}
\end{aligned}$$