Gyakorló feladatok a 2. zéhára

Primitív függvények:

$$\int \frac{\sqrt{3}-2x}{x-\sqrt{2}} = ?$$

Megoldás vázlata:

$$\begin{aligned}
\frac{\sqrt{3}-2x}{x-\sqrt{2}} &= \frac{\sqrt{3}-2\sqrt{2}+2\sqrt{2}-2x}{x-\sqrt{2}} = \frac{\sqrt{3}-\sqrt{8}}{x-\sqrt{2}} - 2 \\
\int \frac{\sqrt{3}-2x}{x-\sqrt{2}} &= \left(\sqrt{3}-\sqrt{8}\right)\log\left|x-\sqrt{2}\right| - 2x
\end{aligned}$$

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$$\int \frac{12}{2-x^2} = ?$$

Megoldás vázlata:

$$\frac{-12}{x^2-2} = \frac{A}{x+\sqrt{2}} + \frac{B}{x-\sqrt{2}} = \frac{\sqrt{2}(B-A)+(A+B)x}{x^2-2}$$

$$\begin{aligned}
-12 &= \sqrt{2}(B-A) \\
0 &= A+B \\
A &= \sqrt{18}, \quad B = -\sqrt{18} \\
\int \frac{12}{2-x^2} &= \sqrt{18}\left(\log\left|x+\sqrt{2}\right| - \log\left|x-\sqrt{2}\right|\right)
\end{aligned}$$

Néhány határozott integrál közelítő végeredménnyel:

$$\begin{aligned}
\int_1^{\sqrt{2}} \frac{x-\sqrt{2}}{x-2}\,dx &\approx 0.1 \\
\int_1^{e} xe^x\,dx &\approx 26 \\
\int_{\pi/3}^{\pi/2} x\sin x\,dx &\approx 0.66 \\
\int_4^{5} 2^{3x}\,dx &\approx 14000 \\
\int_{\pi/8}^{\pi/6} \tan 2x\,dx &\approx 0.17 \\
\int_2^{5} \log_7 8x\,dx &\approx 5
\end{aligned}$$

Kiszámítási ötletek:

$$\int \frac{x-\sqrt{2}}{x-2} = \int \frac{x-\sqrt{2}-2+2}{x-2} = \int \left(1+\frac{2-\sqrt{2}}{x-2}\right)$$
$$= x+\left(2-\sqrt{2}\right)\log|x-2|$$

$$\int xe^x = xe^x - \int e^x$$

$$\int x\sin x = x(-\cos x) - \int(-\cos x) = \sin x - x\cos x$$

$$\int 2^{3x} = \int 8^x = \int e^{x\log 8} = \frac{e^{\log 8}}{\log 8} = \frac{2^{3x}}{3\log 2}$$

$$\int \tan 2x = \int \frac{\sin 2x}{\cos 2x} = \frac{-1}{2}\int \frac{(\cos 2x)'}{\cos 2x} = \frac{-\log|\cos 2x|}{2}$$

$$\int \log_7 8x = \int \frac{\log 8x}{\log 7} = \frac{1}{\log 7}\int(\log 8 + \log x)$$
$$= \frac{\log 8}{\log 7}x + \frac{x\log x - x}{\log 7} = \frac{x\log\frac{8x}{e}}{\log 7}$$

Pontos végeredmények:

$$\left[x+\left(2-\sqrt{2}\right)\log|x-2|\right]_1^{\sqrt{2}} = \sqrt{2}-1+\left(2-\sqrt{2}\right)\log\left(2-\sqrt{2}\right)$$
$$[(x-1)e^x]_1^e = (e-1)e^e$$
$$[\sin x - x\cos x]_{\pi/3}^{\pi/2} = 1-\frac{\sqrt{3}}{2}+\frac{\pi}{6}$$
$$\left[\frac{2^{3x}}{3\log 2}\right]_4^5 = \frac{28\,672}{3\ln 2}$$
$$\left[\frac{-\log|\cos 2x|}{2}\right]_{\pi/8}^{\pi/6} = \frac{\log 2}{4}$$
$$\left[\frac{x\log\frac{8x}{e}}{\log 7}\right]_2^5 = \frac{5\log 40 - 2\log 16 - 3}{\log 7}$$

Közrefogott területek:

$$y=\sqrt{x} \quad \text{és} \quad y=\frac{x}{\sqrt{2}}$$

Megoldás vázlata:

$$\begin{aligned}\sqrt{x} &= \frac{x}{\sqrt{2}} \\ x &= 0 \text{ vagy } x = 2 \\ \int_0^2 \left(\sqrt{x} - \frac{x}{\sqrt{2}}\right) dx &= \frac{\sqrt{2}}{3}\end{aligned}$$

Megjegyzés: a végeredmény közelítőleg: 0.47

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$$x + y = 3 \qquad \text{és} \qquad xy = 1$$

Megoldás vázlata:

$$\begin{aligned}y &= 3 - x \quad \text{és} \quad y = \frac{1}{x} \\ 3 - x &= \frac{1}{x} \\ x &= \frac{3-\sqrt{5}}{2} \;\text{ és }\; x = \frac{3+\sqrt{5}}{2} \\ \int \left(3 - x - \frac{1}{x}\right) &= 3x - \frac{x^2}{2} - \log x \\ \int_{\frac{3-\sqrt{5}}{2}}^{\frac{3+\sqrt{5}}{2}} \left(3 - x - \frac{1}{x}\right) dx &= \left[3x - \frac{x^2}{2} - \log x\right]_{\frac{3-\sqrt{5}}{2}}^{\frac{3+\sqrt{5}}{2}} \\ &= \frac{3\sqrt{5}}{2} + \log \frac{3-\sqrt{5}}{3+\sqrt{5}}\end{aligned}$$

Megjegyzés: a végeredmény közelítőleg: 1. 43

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$$y = x^2 + x - 1 \quad \text{és} \quad y = 2 + x - x^2$$

Megoldás vázlata:

$$
\begin{aligned}
x^2+x-1 &= 2+x-x^2\\
x &= -\sqrt{3/2} \quad \text{és} \quad x=+\sqrt{3/2}\\
\left(2+x-x^2\right)-\left(x^2+x-1\right) &= 3-2x^2\\
\int\left(3-2x^2\right) &= 3x-\frac{2x^3}{3}\\
\int_{-\sqrt{3/2}}^{+\sqrt{3/2}}\left(3-2x^2\right)\,dx &= \left[3x-\frac{2x^3}{3}\right]_{-\sqrt{3/2}}^{+\sqrt{3/2}}\\
&= \left(3\sqrt{3/2}-\frac{2\cdot(3/2)\sqrt{3/2}}{3}\right)\\
&\quad -\left(-3\sqrt{3/2}+\frac{2\cdot(3/2)\sqrt{3/2}}{3}\right)\\
&= 2\sqrt{6}=\sqrt{24}
\end{aligned}
$$

Megjegyzés: a végeredmény közelítőleg: 4.9

Változatok egy témára:

$$\int\frac{1}{12-x-x^2}=?$$

Megoldás vázlata:

$$
\begin{aligned}
12-x-x^2 &= 0\\
x &= -4 \quad \text{és} \quad x=3\\
\frac{1}{12-x-x^2} &= \frac{A}{x+4}+\frac{B}{x-3}\\
\frac{1}{12-x-x^2} &= \frac{-1}{(x+4)\,(x-3)}\\
\frac{A}{x+4}+\frac{B}{x-3} &= \frac{-3A+4B+Ax+Bx}{(x+4)\,(x-3)}
\end{aligned}
$$

$$
\begin{aligned}
-1 &= -3A+4B\\
0 &= A+B\\
A &= \frac{1}{7},\quad B=\frac{-1}{7}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{12-x-x^2} &= \frac{1}{7}\left(\frac{1}{x+4}-\frac{1}{x-3}\right)\\
\int\frac{1}{12-x-x^2} &= \frac{\log|x+4|-\log|x-3|}{7}
\end{aligned}
$$

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$$\int_{-2}^{+2} \frac{1}{12-x-x^2}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int_{-2}^{+2} \frac{1}{12-x-x^2}\ dx &= \left[\frac{\log|x+4| - \log|x-3|}{7}\right]_{-2}^{+2} \\
&= \left(\frac{\log 6 - \log 1}{7}\right) - \left(\frac{\log 2 - \log 5}{7}\right) \\
&= \frac{\log 15}{7}
\end{aligned}$$

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$$\int_{2}^{3} \frac{1}{12-x-x^2}\ dx = ?$$

Megoldás vázlata:

$$\begin{aligned}
\int_{2}^{b} \frac{1}{12-x-x^2}\ dx &= \left[\frac{\log|x+4| - \log|x-3|}{7}\right]_{2}^{b} \\
&= \left(\frac{\log(b+4) - \log(3-b)}{7}\right) \\
&\quad - \left(\frac{\log 6 - \log 1}{7}\right) \\
\lim_{b\to 3^-} \left(\log(b+4) - \log(3-b)\right) &= \lim_{b\to 3^-} \log\frac{b+4}{3-b} = +\infty \\
\int_{2}^{3} \frac{1}{12-x-x^2}\ dx &= +\infty
\end{aligned}$$

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