Local Regularity of Affine Zipper Fractal Curves

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Construction of affine zippers

Affine zippers satisfying dominated splitting of index-1

A system $\mathcal{S} = \{f_0, \dots, f_{N-1}\}$ of contracting affine mappings of $\mathbb{R}^d$ to itself of the form $f_i(x) = A_i x + t_i$ is called an affine zipper with vertices $Z = \{z_0, \dots, z_N\}$ and signature $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{N-1})$, $\varepsilon_i \in \{0,1\}$, if the cross condition is satisfied, i.e. ,

$$f_i(z_0) = z_{i+\varepsilon_i} \text{ and } f_i(z_N) = z_{i+1-\varepsilon_i} \text{ for any } i = 0, \dots, N-1.$$

An affine fractal curve is the unique non-empty compact set Γ, for which

$$\Gamma = f_0(\Gamma) \cup f_1(\Gamma) \cup \dots \cup f_{N-1}(\Gamma).$$

Subdivide the $[0,1]$ interval according to a probability vector $(\lambda_0, \dots, \lambda_{N-1})$. A linear parametrization of Γ is the unique continuous function $v : [0,1] \mapsto \Gamma$ defined by

$$v(x) = f_i\left(v\left(\frac{x-\gamma_i}{(-1)^{\varepsilon_i}\lambda_i}\right)\right) \text{ if } x \in [\Sigma_{j=0}^{i-1}\lambda_j, \Sigma_{j=0}^{i}\lambda_j).$$

We study the local regularity of $v(x)$, provided the matrices A_i satisfy the following:

We assume that the matrices $\{A_0, \dots, A_{N-1}\}$ have dominated splitting of index-1, i.e. there exists a non-empty subset $M \subset \mathbb{PR}^{d-1}$ with a finite number of connected components, whose closures are pairwise disjoint such that

$$\bigcup_{i=0}^{N-1} A_i M \subset M^o,$$

and there is a $d-1$-plane that is transverse to all elements of M.

Assumption A: If M can be chosen to be a convex, simply connected cone C such that

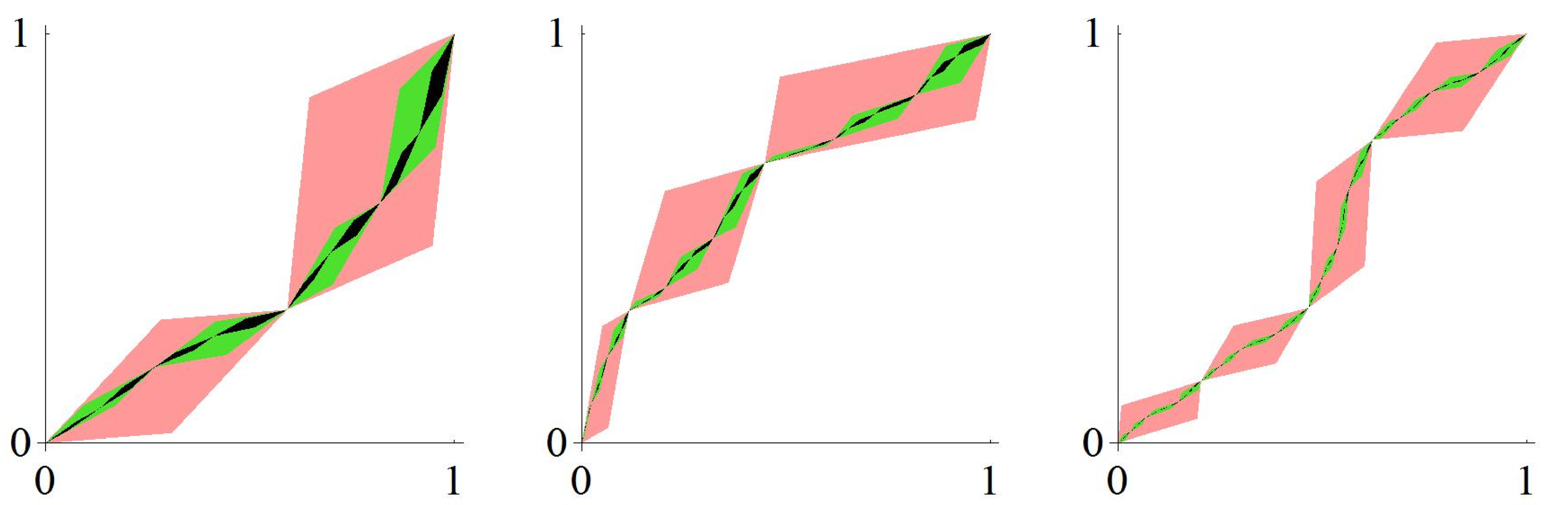
- $< z_N - z_0 > \in C$ and for every $0 \neq v \in C$, $\langle A_i v, v \rangle > 0$,
- $\mathcal{S}$ satisfies the SOSC w.r.t. the bounded component of $C^o(z_0) \cap C^o(z_N) =: U$. That is, $f_i(U) \subseteq U$ for every $i = 0, \dots, N-1$

$$f_i(U) \cap f_j(U) = \emptyset \text{ if } i \neq j \text{ and } f_i(U) \cap f_j(U) = \begin{cases} \emptyset & \text{if } |i-j| > 1 \\ \{z_{i+1}\} & \text{if } j = i+1. \end{cases}$$

Special examples

- All entries of A_i are strictly positive, then the positive quadrant is mapped into itself.
- de Rham's curve. Construction goes as follows:
  - Start from a square and trisect each side with ratios $\omega : (1-2\omega) : \omega$ $(\omega \in (0, 1/2))$,
  - "Cut the corners" by connecting adjacent partitioning points and repeat.

In the figure below $d = 2$ and $N = 2, 3, 4$, respectively. It shows the first (red), second (green) and third (black) level cylinders of the image of $[0,1]^2$.



Pointwise Hölder exponent

Let $g : [0,1] \to \mathbb{R}^d$ be a continuous function. Then for every $x \in (0,1)$ the following definitions are equivalent

$$\alpha_1(x) = \liminf_{y \to x} \frac{\log |g(x) - g(y)|}{\log |x-y|},$$

$$\alpha_2(x) = \sup\left\{\alpha : \forall \rho > 0, \sup_{y \in B_\rho(x)} \frac{|g(x)-g(y)|}{|x-y|^\alpha} < \infty\right\},$$

$$\alpha_3(x) = \sup\left\{\alpha : \exists C > 0, \quad |g(x) - g(y)| \leq C \cdot |x-y|^\alpha \quad \forall y \in [0,1]\right\}.$$

We call the common value the pointwise Hölder exponent and denote it by $\alpha(x)$.
If the $\liminf$ in $\alpha_1(x)$ coincides with the $\limsup$ for a function $g : [0,1] \to \mathbb{R}^d$ at x, we say it has regular pointwise Hölder exponent and denote this common limit by $\alpha_r(x)$.

References

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Main results

Denote by $P(t)$ the pressure function which is defined as the unique root of the equation

$$0 = \lim_{n\to\infty} \frac{1}{n} \log \Sigma_{i_1,\dots,i_n=0}^{N-1} \|A_{i_1} \cdots A_{i_n}\|^t (\lambda_{i_1} \cdots \lambda_{i_n})^{-P(t)}.$$

Let $d_0 > 0$ be the unique real number such that

$$0 = \lim_{n\to\infty} \frac{1}{n} \log \Sigma_{i_1,\dots,i_n=0}^{N-1} \|A_{i_1} \cdots A_{i_n}\|^{d_0}.$$

Since $\mathcal{S}$ defines a curve, $d_0 \geq 1$. Let

$$\alpha_{\min} = \lim_{t\to\infty} \frac{P(t)}{t}, \quad \alpha_{\max} = \lim_{t\to-\infty} \frac{P(t)}{t} \text{ and } \bar{\alpha} = P'(0).$$

Theorem: Multifractal analysis of $\alpha(x)$ and $\alpha_r(x)$

Assume dominated splitting of index-1. Then there exists a constant $\bar{\alpha}$ such that for $\mathcal{L}$-a.e. $x \in [0,1]$, $\alpha(x) = \bar{\alpha} \geq 1/d_0$. Moreover, $\exists \varepsilon > 0$ s. t. for every $\beta \in [\bar{\alpha}, \bar{\alpha} + \varepsilon]$

$$\dim_H \{x \in [0,1] : \alpha(x) = \beta\} = \inf_{t \in \mathbb{R}} \{t\beta - P(t)\}.$$

Assumption A holds if and only if for $\mathcal{L}$-a.e. x, $\alpha_r(x)$ exists.
In this case, for $\mathcal{L}$-a.e. $x \in [0,1]$, $\alpha_r(x) = \alpha(x) = \bar{\alpha} \geq 1$ and the multifractal analysis holds for the full spectrum, i.e. for every $\beta \in [\alpha_{\min}, \alpha_{\max}]$

$$\dim_H \{x \in [0,1] : \alpha(x) = \beta\} = \dim_H \{x \in [0,1] : \alpha_r(x) = \beta\} = \inf_t \{t\beta - P(t)\}.$$

In each case, the functions $\beta \mapsto \dim_H E(\beta)$ and $\beta \mapsto \dim_H E_r(\beta)$ are continuous and concave on their respective domains, where $E_r(\beta) = \{x \in [0,1] : \alpha_r(x) = \beta\}$.

Properties of the matrix pressure function

Extension of results of Feng-Lau for matrices with strictly positive entries to family of matrices with dominated splitting of index-1 using work of Bochi-Gourmelon.

- The map $t \mapsto P(t)$ exists and it is monotone increasing, concave and continuously differentiable for every $t \in \mathbb{R}$.
- Existence of Gibbs-measure: for every $t \in \mathbb{R}$ there exists a unique ergodic, left-shift invariant Gibbs measure μ_t on $\Sigma = \{0, \dots, N-1\}^{\mathbb{N}}$ s. t. $\exists C > 0$ that for any $(i_1, \dots, i_n) \in \{0, \dots, N-1\}^*$

  $$C^{-1} \leq \frac{\mu_t([i_1, \dots, i_n])}{\|A_{\mathbf{i}|n} E(\sigma^n \mathbf{i})\|^t \cdot \lambda_{\mathbf{i}|n}^{-P(t)}} \leq C.$$

  Furthermore, for every $t \in \mathbb{R}$

  $$\dim_H \mu_t = tP'(t) - P(t),$$

  and

  $$\lim_{n\to\infty} \frac{\log \|A_{i_1} \cdots A_{i_n}\|_1}{-n \log N} = P'(t) \text{ for } \mu_t\text{-almost every } \mathbf{i} \in \Sigma.$$
- Pressure for $t = 0$ and $t = 1$. $P(0) = -1$, $P(d_0) = 0$, $P'(0) \geq 1/d_0$, $P'(d_0) \leq 1/d_0$.

  $$P'(0) > 1/d_0 \iff P'(d_0) < 1/d_0 \iff \mu_{d_0} \neq \mu_0.$$

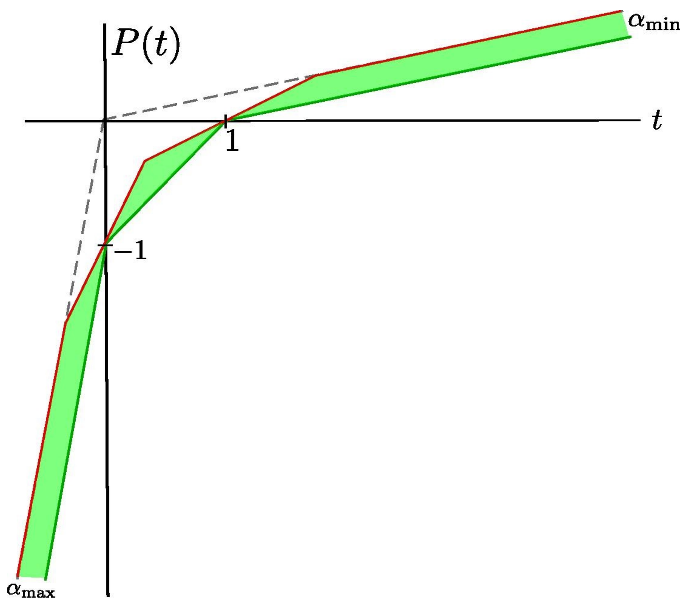
Matrices with strictly positive entries

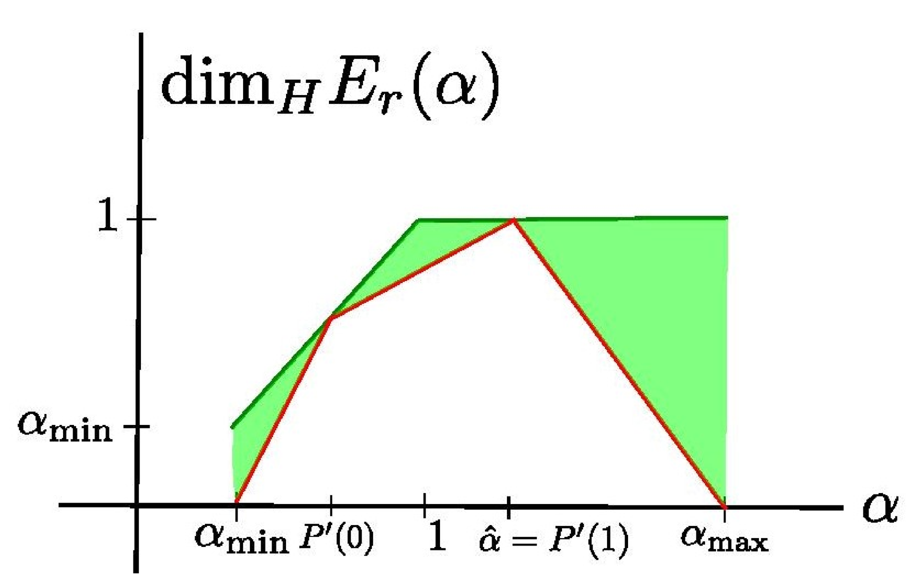
- A measure on Σ: Define ν on the cylinder sets $[\mathbf{i}|n] = \cup_{\mathbf{j}\in\Sigma}(i_1, \dots, i_n, \mathbf{j})$ of Σ as follows

  $$\nu[\mathbf{i}|n] = p^T A_{\mathbf{i}|n} e = p^T A_{i_1} \dots A_{i_n} e \text{ for every } \mathbf{i} \in \Sigma,$$

  where p is the left normalized eigenvector of $\Sigma_{i=1}^{N-1} A_i$ corresponding to largest eigenvalue and $e = (1, \dots, 1)^T$.
  ν uniquely extends to a σ-invariant, ergodic and mixing probability measure on Σ.
- In particular, $d_0 = 1$. Hence, from the Gibbs property it follows that $\mu_0 = \mathbb{P}$ is the equidistributed measure and $\mu_{d_0} = \mu_1 = \nu$. Therefore,

  $$\nu = \mathbb{P} \iff P(t) = t - 1 \text{ for } t \in [0,1].$$
- These give lower and upper bounds on $P(t)$ (shaded green below left) which in turn, after taking the Legendre transform, give bounds for $\dim_H E_r(\alpha)$ (shaded green on right).





Corollaries

- If $\nu \neq \mathbb{P}$, then the set $\mathcal{N} \subseteq [0,1]$ where the curve v is not differentiable has positive Hausdorff dimension. Moreover, for $\mathcal{L}$-a.e. $x \in [0,1]$: $\alpha_r(x) = \bar{\alpha} > 1$. In particular, v is differentiable at Lebesgue-almost every point with derivative zero.
- Push forward measures: If $\nu[\mathbf{i}|n] = N^{-n}$ for every $\mathbf{i} \in \Sigma$, then $\dim_H \pi_* \mathbb{P} = 1/P'(0) = 1$. Otherwise, if $\nu \neq \mathbb{P}$, then $\dim_H \pi_* \mathbb{P} < 1$.
- On the other hand $\dim_H \pi_* \nu = 1$ always holds. Furthermore, $\dim_H v([0,1]) = 1$.