Overlapping triangular Gatzouras–Lalley-type carpets

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Definition

A triangular Gatzouras–Lalley-type (TGL) carpet Λ is the attractor of a self-affine iterated function system (IFS)

$$\mathcal{F}=\{f_i(\underline{x})=A_i\underline{x}+t_i\}_{i=1}^N,\text{ where } A_i=\begin{pmatrix} b_i & 0\\ d_i & a_i\end{pmatrix},\ t_i=\begin{pmatrix} t_{i,1}\\ t_{i,2}\end{pmatrix},$$

such that $f_i(R)\subset R$ (throughout $R=[0,1]^2$),

$0<a_i<b_i<1$, we say that direction-x dominates and the parallelograms $f_i(R)$ are aligned in M non-overlapping columns $\mathcal{I}_1,\dots,\mathcal{I}_M$ so that $\sum_{i\in\mathcal{I}_j}a_i\leq 1$ for $j=1,\dots,M$. A shifted TGL carpet may have overlapping columns.

Λ is diagonally homogeneous if $a_i\equiv a$ and $b_i\equiv b$.
Λ has uniform vertical fibres if $\sum_{i\in\mathcal{I}_j}a_i^{s_B-s_{\mathcal{H}}}=1$ for all j.

A TGL carpet Λ is a Gatzouras–Lalley (GL) carpet if $d_i\equiv 0$ and Λ satisfies the ROSC.
A GL carpet $\tilde{\mathcal{F}}=\{\tilde{A}_i\underline{x}+\tilde{t}_i\}_{i=1}^N$ is a GL brother of $\mathcal{F}$ if $\tilde{A}_i=\operatorname{diag}A_i$ and $\tilde{\mathcal{F}}$ has the same column structure as $\mathcal{F}$.

Separation conditions

The rectangular open set condition (ROSC) holds if the strong open set condition holds with $U=(0,1)^2$, i.e. $U\cap\Lambda\neq\emptyset$,

$$\bigcup_{i=1}^N f_i(U)\subseteq U\ \text{ and }\ f_i(U)\cap f_j(U)=\emptyset\text{ for every } i\neq j.$$

The TGL IFS $\mathcal{F}$ satisfies the transversality condition if there exists a $K_1>0$ such that for any two nth level cylinders $R_{i_1\dots i_n}$ and $R_{j_1\dots j_n}$ in the same column with $i_1\neq j_1$ we have

$$\operatorname{proj}_x(R_{i_1\dots i_n}\cap R_{j_1\dots j_n})<K_1\cdot\max\{a_{i_1}\cdot\ldots\cdot a_{i_n},\ a_{j_1}\cdot\ldots\cdot a_{j_n}\}.$$

The IFS $\mathcal{H}$ projected to the x-axis (one map per column) has Exponential Separation Condition (ESC) if there exist an $\varepsilon>0$ and $n_k\uparrow\infty$ such that for

$$\Delta_n:=\min_{\substack{\bar{\imath},\bar{\jmath}\in\{1\dots M\}^n\\ \bar{\imath}\neq\bar{\jmath}}}\begin{cases}|h_{\bar{\imath}}(0)-h_{\bar{\jmath}}(0)|, & \text{if } h'_{\bar{\imath}}(0)=h'_{\bar{\jmath}}(0);\\ \infty, & \text{otherwise.}\end{cases}$$

we have $\Delta_{n_k}>e^{-\varepsilon\cdot n_k}$. Hochman: ESC $\Rightarrow$ the dimension of self-similar measures does not drop. We show that this is equivalent to another condition we call Weak Almost Unique Coding.

Results for carpets with ROSC

Let $\mathbf{p}$ be a probability vector of length N. Then

$$s_H:=\sup_{\mathbf{p}}\frac{\log\langle\mathbf{p}\rangle_{\mathbf{p}}}{\log\langle\mathbf{a}\rangle_{\mathbf{p}}}+\left(1-\frac{\log\langle\mathbf{b}\rangle_{\mathbf{p}}}{\log\langle\mathbf{a}\rangle_{\mathbf{p}}}\right)\frac{\log\langle\mathbf{q}\rangle_{\mathbf{q}}}{\log\langle\mathbf{b}\rangle_{\mathbf{p}}},$$

where $\log\langle\mathbf{a}\rangle_{\mathbf{p}}=\sum_{i=1}^N p_i\log a_i$ and $q_j=\sum_{i\in\mathcal{I}_j}p_i$. The sup is attained for a $\mathbf{p}$ with strictly positive entries. Furthermore, let

s_B be the unique solution of $\sum_{i=1}^N b_i^{s_{\mathcal{H}}}a_i^{s_B-s_{\mathcal{H}}}=1$, s_{Aff} be the unique solution of $\sum_{i=1}^N b_i^{\min\{\tilde{s}_x,1\}}a_i^{s_{Aff}-\min\{\tilde{s}_x,1\}}=1$,

where $s_{\mathcal{H}}=\dim_B\operatorname{proj}_x(\Lambda)$. where $\tilde{s}_x$ solves $\sum_{i=1}^N b_i^{\tilde{s}_x}=1$.

Theorem (Gatzouras–Lalley; Barański)
If a TGL carpet Λ has non-overlapping columns and satisfies ROSC (first and second picture), then

$$\dim_H\Lambda=s_H,\quad \dim_B\Lambda=s_B,\quad \dim_{Aff}\Lambda=s_{Aff},$$

where $\dim_{Aff}$ denotes the affinity dimension. Moreover, $\dim_H\Lambda=\dim_B\Lambda$ if and only if Λ has uniform vertical fibres.

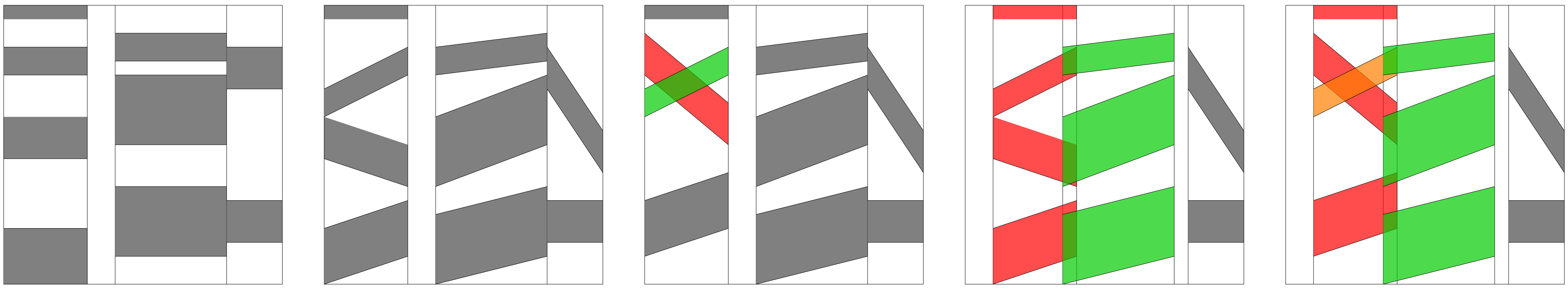


Figure: **The "evolution" of Gatzouras–Lalley carpets. First**: original construction of Gatzouras and Lalley. **Second**: construction of Barański using lower triangular matrices satisfying ROSC. **Our overlapping constructions: third** has non-overlapping columns and transversality condition; **fourth** has shifted columns satisfying ESC; **fifth** has a mixture of both. The first one is the GL brother of all others.

Our main results for overlapping constructions

Irrespective of overlaps, for any shifted TGL carpet Λ

$$\dim_H\Lambda\leq s_H,\ \dim_P\Lambda=\overline{\dim}_B\Lambda\leq\overline{s}_B,\ \dim_{Aff}\Lambda=s_{Aff}$$

In other words, the dimension of the GL brother dominates.
$\dim_H\Lambda=\dim_B\Lambda$ if and only if Λ has uniform vertical fibres.
$\dim_B\Lambda=\dim_{Aff}\Lambda$ if and only if $s_{\mathcal{H}}=\min\{\tilde{s}_x,1\}$.

$\dim_H\Lambda=s_H$ if the IFS $\mathcal{H}$ satisfies ESC and

- each column independently satisfies the ROSC or
- Λ satisfies transversality and technical condition (fifth pic).

Transversality and technical conditions hold if a_i is small enough compared to b_i, see below for examples.

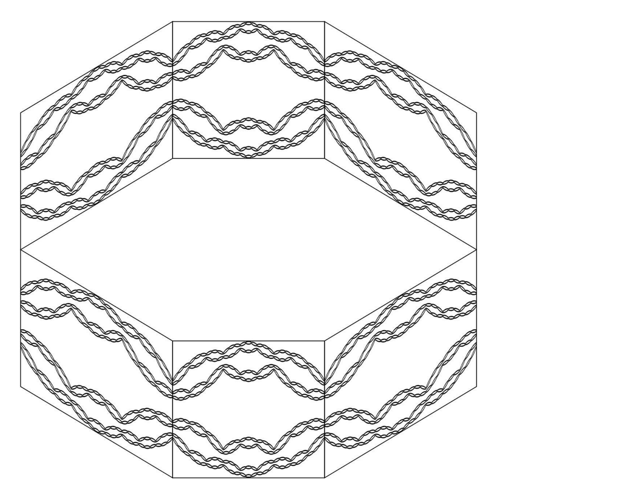
$\dim_B\Lambda=s_B$ if either of the following hold

- $\mathcal{H}$ satisfies ESC and each column independently satisfies the ROSC (fourth picture) or
- Λ is a TGL carpet, satisfies transversality and a technical condition (third picture).

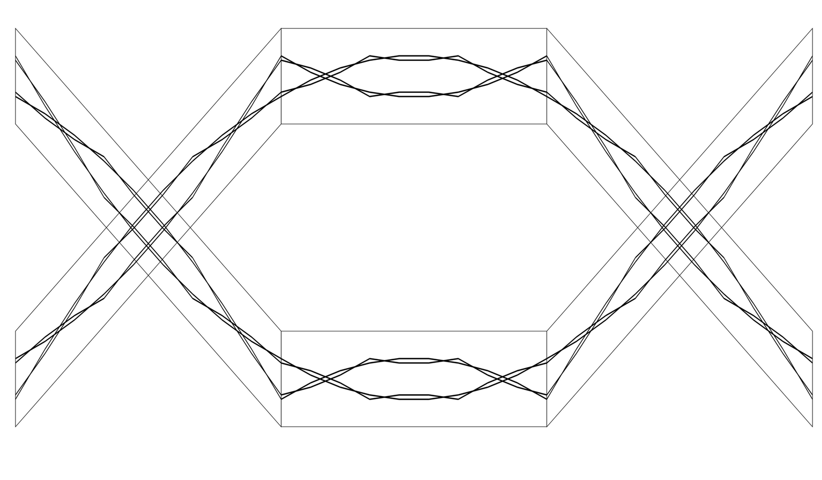
Breaking equality between dimensions

top left: diag. homo. & uniform vertical fibres & $\operatorname{proj}_x\Lambda=[0,1]\Rightarrow\dim_H=\dim_B=\dim_{Aff}$.
top right: overlaps, for every $a<1/6$ still have $\dim_H=\dim_B=\dim_{Aff}=1-\log 2/\log a$.
bottom left: uniform vertical fibres $\Rightarrow\dim_H=\dim_B$ but $s_{\mathcal{H}}<1<\tilde{s}_x\Rightarrow\dim_B<\dim_{Aff}$
bottom right: non uniform vertical fibres $\Rightarrow\dim_H<\dim_B<\dim_{Aff}$ for $a<0.10254\ldots$

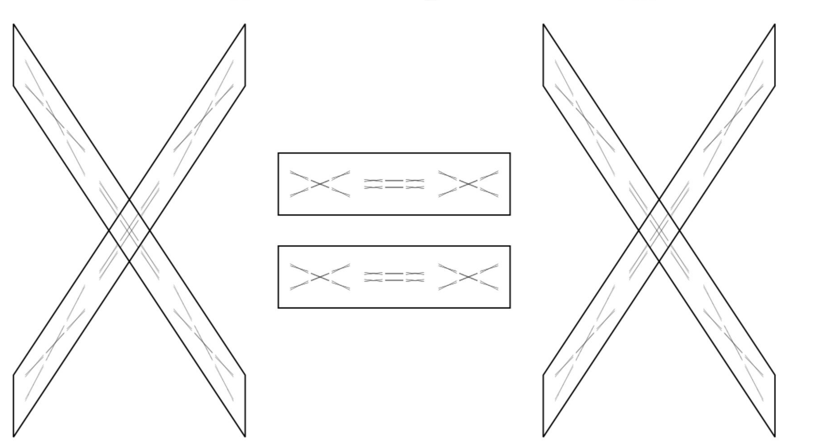
$\dim_H\Lambda=\dim_B\Lambda=\dim_{Aff}\Lambda$



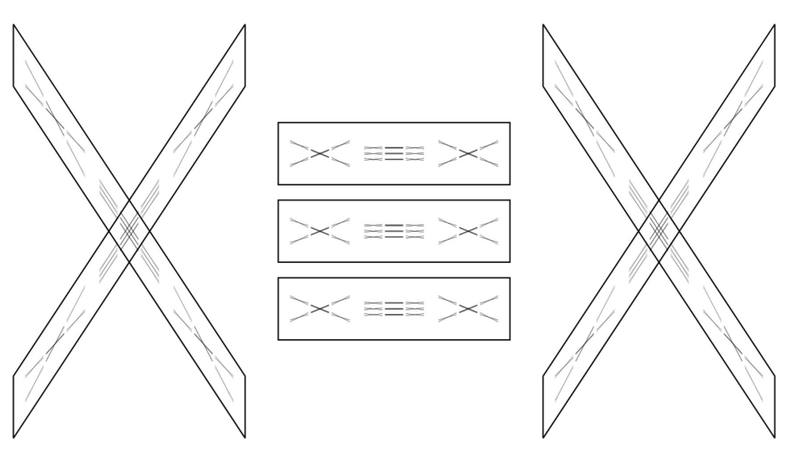
$\dim_H\Lambda=\dim_B\Lambda=\dim_{Aff}\Lambda$



$\dim_H\Lambda=\dim_B\Lambda<\dim_{Aff}\Lambda$



$\dim_H\Lambda<\dim_B\Lambda<\dim_{Aff}\Lambda$



Application to three-dimensional systems

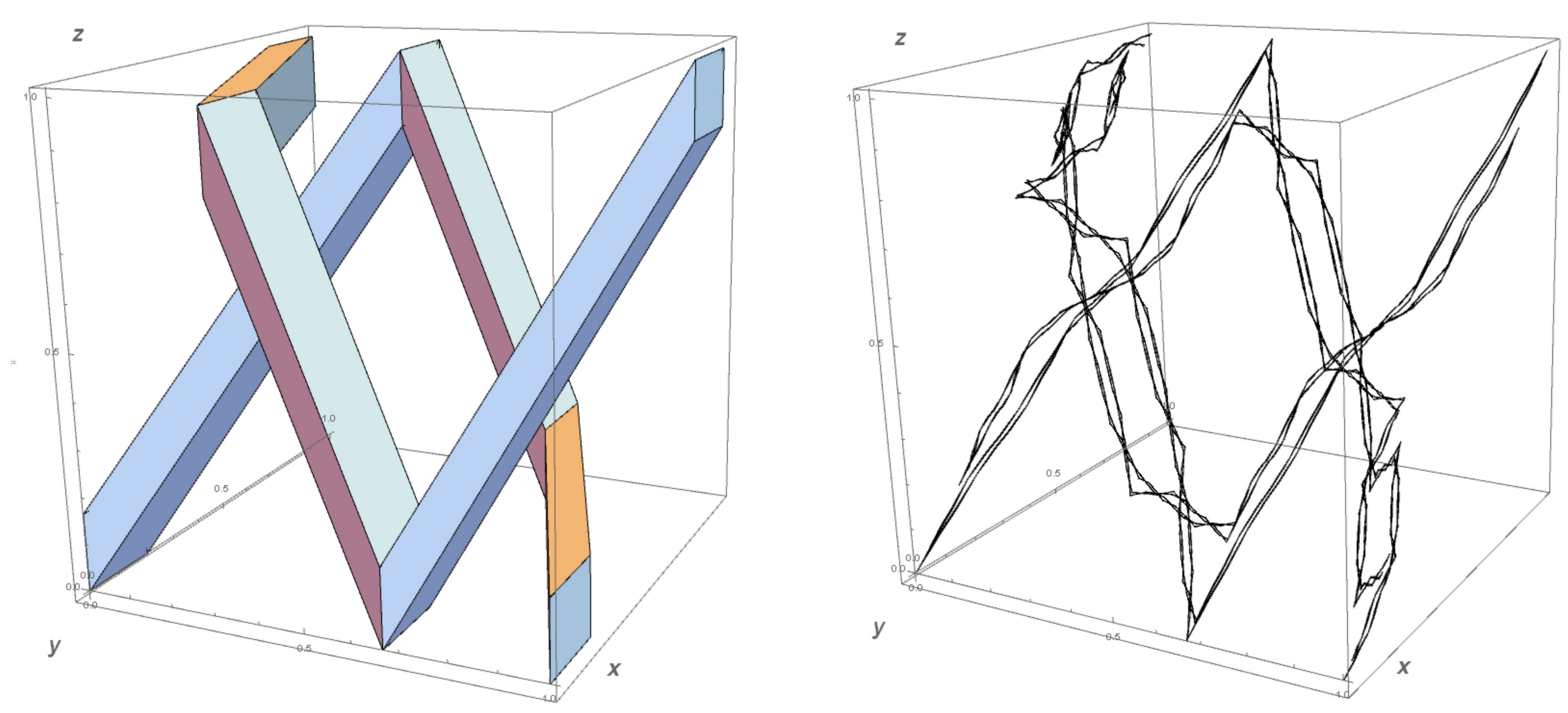


Figure: A three dimensional IFS (left) with its attractor (right), whose orthogonal projection to xy-plane is an overlapping system like the top right on the left-hand side.

Theorem

Start from a (possibly overlapping) diag. homo. TGL IFS $\mathcal{F}$ which has uniform vertical fibres, satisfies the conditions of our theorems and $\operatorname{proj}_x\Lambda=[0,1]$. We "lift" $\mathcal{F}$ to 3D space

$$\widehat{\mathcal{F}}:=\left\{F_i(\widehat{\underline{x}}):=\widehat{A}_i\cdot\widehat{\underline{x}}+\widehat{\mathbf{t}}_i\right\}_{i=1}^N,\text{ where }\widehat{A}_i=\begin{pmatrix} b & 0 & 0\\ d_i & a & 0\\ u_i & v_i & \lambda_i\end{pmatrix},\ \widehat{t}_i:=\begin{pmatrix} t_{i,1}\\ t_{i,2}\\ t_{i,3}\end{pmatrix},\ \widehat{\underline{x}}\in[0,1]^3$$

so that $0<|\lambda_i|<a<b<1$ (for small enough values of a) and $\widehat{\mathcal{F}}$ satisfies the SOSC. Then

$$\dim_H\widehat{\Lambda}=\dim_B\widehat{\Lambda}=\dim_{Aff}\widehat{\Lambda}=1+\frac{\log(Nb)}{-\log a}.$$

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