Stochastic analysis and its applications, Exam 1, May 21, 2010

Please give sufficiently detailed arguments. Each question is worth 10 points.

1. Let $(B(t))_{t\geq 0}$ be Brownian motion, $B(0)=0$. Evaluate the following conditional expectations.

   (a) $\mathbb{E}\left(B(s) \mid B(t)\right)$, when $0 \leq s < t$. *Hint:* What is the joint distribution of $(B(s), B(t))$?

   (b) $\mathbb{E}\left(\int_0^t B(s)\, ds \mid B(t)\right)$. *Hint:* You may approximate the integral by sums.

   (c) $\mathbb{E}\left(\int_0^t s\, dB(s) \mid B(t)\right)$. *Hint:* You may approximate the integral by sums.

2. Let $(B(t))_{t\geq 0}$ be Brownian motion, $B(0) = x \in (a,b)$, where (a,b) is a bounded interval. Define $\tau = \inf\{t \geq 0 : B(t) \notin (a,b)\}$, the first exit time from the interval. Show that $\mathbb{E}^x(\tau) = (x-a)(b-x)$. *Hint:* You may consider the martingale $B^2(t) - t$ and apply Doob's optional stopping theorem: When $(M(t))_{t\geq 0}$ is a martingale and τ is a stopping time, $(M(t\wedge\tau))_{t\geq 0}$ is a martingale as well.

3. Let $\mathbf{B}(t)$ be three-dimensional Brownian motion, $\mathbf{B}(0) = \mathbf{x} \neq \mathbf{0}$. Define $\tau_r = \inf\{t \geq 0 : |\mathbf{B}(t)| \leq \frac{1}{r}\}$, where r is a positive integer, $\frac{1}{r} < |\mathbf{x}|$.

   (a) Show that $X_r(t) = 1/|\mathbf{B}(t\wedge\tau_r)|$ is a martingale for any fixed r. (This shows that $Y(t) = 1/|\mathbf{B}(t)|$ is a so-called *local martingale*.) *Hint:* You may apply the multidimensional Itô's formula. Check that $1/|\mathbf{x}|$ is a harmonic function for $\mathbf{x} \neq \mathbf{0}$.

   (b) Show that $Y(t)$ is notwithstanding not a martingale (Why?), because $\mathbb{E}^x(1/|\mathbf{B}(t)|) \neq 1/|\mathbf{x}|$, when t is large enough. *Hint:* Write down the expectation $\mathbb{E}^x(1/|\mathbf{B}(t)|)$ by an integral, and by estimating it, show that it tends to 0 as $t \to \infty$. $\Rightarrow$ no martingale regularization

4. Let $(X(t))_{t\geq 0}$ be time-homogeneous Itô diffusion, $dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t)$, where $\mu(x) = \frac{1}{2}\sin x \cos x$, $\sigma(x) = \cos x$, $X(0) = x \in \left(0, \frac{\pi}{4}\right)$. Define $T_a = \inf\{t > 0 : X(t) = a\}$, the first hitting time of the point a. First solve the ODE

$$(Af)(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) = 0$$

   in the interval $\left(0, \frac{\pi}{4}\right)$. Show that then

$$\mathbb{P}^x(T_{\frac{\pi}{4}} < T_0) = \frac{f(x) - f(0)}{f(\frac{\pi}{4}) - f(0)}.$$

   (Evaluate it as well.) *Hint:* You may use Itô's formula, Doob's optional stopping theorem with $t\wedge\tau$, where $\tau = T_a \wedge T_b$, and then you can take expectation. Check that you can have a limit in the expectation as $t \to \infty$.

5. Let $\mathbf{B}(t)$ be d-dimensional Brownian motion, $d \geq 2$, $\mathbf{B}(0) = \mathbf{x}_0 \neq \mathbf{0}$. Define the scalar process (*d-dimensional Bessel process*) $R(t) = |\mathbf{B}(t)| = \left(\sum_{i=1}^d B_i^2(t)\right)^{1/2}$.

   Using the fact that $\mathbb{P}(\mathbf{B}(t) = \mathbf{0}$ for some $t \geq 0) = 0$, show that $R(t)$ is an Itô diffusion, driven by $\mathbf{B}(t)$, with drift coefficient $\mu(t) = \frac{d-1}{2}|\mathbf{B}(t)|^{-1}$ and diffusion coefficient $\sigma(t) = \mathbf{B}(t)\,|\mathbf{B}(t)|^{-1}$, which is a $1 \times d$ matrix. Argue that then $R(t)$ is a time-homogeneous diffusion (Markov) process with infinitesimal generator

$$A = \tilde{\mu}(x)\frac{d}{dx} + \frac{1}{2}\tilde{\sigma}(x)\tilde{\sigma}^T(x)\frac{d^2}{dx^2} = \frac{d-1}{2}\,\frac{1}{x}\,\frac{d}{dx} + \frac{1}{2}\,\frac{d^2}{dx^2}.$$

   So $R(t)$ has the same generator as the solution of the following scalar SDE:

$$dX(t) = \frac{d-1}{2}\,\frac{1}{X(t)}\,dt + dW(t), \qquad X(0) = |\mathbf{x}_0|,$$

   where $W(t)$ is one-dimensional Brownian motion. (Hence $R(t)$ is a so-called *weak solution* of this SDE.)