## Problems for the first week

1. Find the general solution of the differential equation
$$y' = \frac{y}{x}.$$

2. Find the general solution of the differential equation
$$y' = \mathrm{e}^{x+y}.$$

3. Consider a sample of radioactive material which has mass $y(t)$ kg at time $t$. It has been observed that a constant factor of those radioactive atoms will spontaneously decay (into atoms of another element or into another isotope of the same element) during each time unit.

   (a) Find the differential equation which describes this process if the half-life of the material is $T = 100sec$.

   (b) Assume that at the beginning we had $1kg$ from this material. Find $y(t)$.

   The **half-life** of a radioactive material is the time for an amount of this material to decay to one-half of its original value.

4. Find the solution of the following initial value problem:
$$y' = \frac{\mathrm{e}^x}{y+1} \quad ; \quad y(0) = -4.$$

5. Assume that as a result of the drag force the decay of the speed of a moving object is proportional to the square of the speed of the object. Let $v(t)$ be the velocity as a function of the time. Write a differential equation for $v(t)$ and solve this differential equation.

6. Find the general solution of the differential equation
$$y' = \frac{1+2\mathrm{e}^y}{\mathrm{e}^y x \ln(x)}.$$

7. Find the general solution of the differential equation
$$(\mathrm{e}^{-2y} - \mathrm{e}^{-y})y' = \frac{\mathrm{e}^{x-y} + \mathrm{e}^{-x-y}}{\mathrm{e}^y + 1}.$$

8. Find the solution of the following initial value problem:
$$x + y - xy' = 0 \quad ; \quad y(1) = 1.$$

9. Find the general solution of the differential equation
$$x\mathrm{e}^{y/x} + y = xy'.$$

10. Find the general solution of the differential equation
$$xy' = y(\ln y - \ln x).$$

## Results and some Solutions

1.) $y = C \cdot x$, where $C \neq 0$.
2.) $y = -\ln(-e^x - C)$, where $C < 0$.
3.)

**(a)** The differential equation is:

$$\frac{dy}{dt} = Ay.$$

Thus $y = Ce^{At}$. Since the half life is $T = 100$ we can write

$$y(100) = \frac{1}{2} \cdot y(0).$$

This yields $A = -\frac{\ln 2}{100}$. So the general solution is

$$y = Ce^{-\frac{\ln 2}{100} \cdot t}$$

**(b)** Using that $y(0) = 1$ we obtain that $C = 1$. Substitute this into the previous formulae to get

$$y(t) = e^{-\frac{\ln 2}{100} \cdot t}$$

4.) We write the equation in the form

$$\int (y+1) dy = \int e^x dx.$$

That is

$$\frac{y^2}{2} + y = e^x + C.$$

Solving this for $y$ yields

$$y(x) = -1 \pm \sqrt{1 + 2(e^x + C)}.$$

Using that $y(0) = -4$ we obtain that $C = 3$. Thus the solution of the initial value problem is:

$$y(x) = -1 - \sqrt{7 + 2e^x}.$$

5.) The differential equation is:

$$-\frac{dv}{dt} = Av^2$$

for a constant $A < 0$. The solution of this equation is

$$v(t) = \frac{1}{At + C}.$$

6.) We write the separable equation in the form

$$\int \frac{e^y}{1 + 2e^y} dy = \int \frac{1}{x \ln x} dx.$$

Apply the substitution $u = e^y, v = \ln x$ to get

$$\int \frac{du}{1 + 2u} = \int \frac{dv}{v}.$$

After integrating both sides we obtain

$$\frac{1}{2}\ln(1+2u) = \ln(|v|) + C.$$

This yields

$$u = \frac{1}{2}\left(\mathrm{e}^{2C}v^2 - 1\right).$$

That is

$$y = \ln u = \ln\left(\mathrm{e}^{2C}\left(\ln x\right)^2 - 1\right) - \ln 2.$$

7.) $y = \cosh^{-1}\left(\sinh(x) + C\right)$, where $\sinh(x) + C \geq 1$.
8.) After the substitution $u = \frac{y}{x}$ we get a separable equation. The general solution is : $y = x\ln(|x|) + Cx$. Using that $y(1) = 1$ we obtain that $C = 1$. Thus the solution of the initial value problem is: $y = x\ln(|x|) + x$.
9.) After the substitution $u = \frac{y}{x}$ we get a separable equation. This leads to the general solution: $y = x\ln\left(C + \ln|x|\right)$.
10.) First we use the substitution $u = \frac{y}{x}$. We obtain that

$$\int \frac{du}{u\ln u - u} = \int \frac{dx}{x}.$$

Then assuming that $x, y > 0$ we write $z = \ln u$. This leads to

$$\int \frac{dz}{z-1} = \int \frac{dx}{x}.$$

After integration: $\ln(|z-1|) = \ln(x) + C$. In this way we get

$$|z-1| = \mathrm{e}^{C}x.$$

If $z > 1$ (that is $y > \mathrm{e}\cdot x$) then

$$y = ux = \mathrm{e}^{z}x = \mathrm{e}^{\mathrm{e}^{C}x+1}x.$$

If $y < \mathrm{e}\cdot x$ then

$$y = ux = \mathrm{e}^{z}x = \mathrm{e}^{-\mathrm{e}^{C}x+1}x.$$

During our calculations we excluded the case $y = \mathrm{e}\cdot x$ which is also a solution of the equation.

**Problems for the second week**

1. Consider the following differential equation

$$y' - 2y = 3e^t.$$

   (a) Draw a direction field for this equation.

   (b) Based on the inspection of the direction field, describe how solutions behave for large $t$.

   (c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$!

2. Find the solutions of the given initial value problems:

   (a)
$$y' - y = 2te^{2t}, \quad y\,(0) = 1$$

   (b)
$$ty' + 2y = t^2 - t + 1, \quad y\,(1) = 1/2, \quad t > 0$$

   (c)
$$ty' + (t+1)\,y = t, \quad y\,(\ln 2) = 1, \quad t > 0$$

3. The following problems involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $y \to f(y)$, determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable. Draw the phase line and sketch some graphs of solutions in the $ty$-plane.

   (a)
$$dy/dt = y(y-1)(y-2), \quad y(0) \geq 0$$

   (b)
$$dy/dt = \mathrm{e}^y - 1, \quad y_0 \in \mathbb{R}$$

   (c)
$$dy/dt = -2(\arctan y)/(1+y^2), \quad y_0 \in \mathbb{R}$$

   We learned that

**Theorem 1** *Let the function* $f$ *and* $\partial f/\partial y$ *be continuous in some rectangle* $\alpha < t < \beta, \gamma < y < \delta$ *containing the point* $(t_0, y_0)$. *Then in some interval* $t_0 - h < t < t_0 + h$ *contained in the interval* $(\alpha, \beta)$, *there is a unique solution of the initial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{1}$$

4. In each of the following problems state where in the $ty$ plane the hypotheses the Theorem above are satisfied.

   (a) $y' = \frac{t-y}{2t+5y}$

   (b) $y' = \frac{\ln|ty|}{1-t^2+y^2}$

   (c) $y' = (t^2 + y^2)^{3/2}$

   The use of mathematical methods to study the spread of contagious diseases goes back at least to some work by Daniel Bernoulli in 1760 on smallpox. The following two problems deals with some simpler models. Similar models have also been used to describe the spread of rumors and of consumer products.

5. Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let $x$ be the proportion of susceptible individuals and $y$ the proportion of infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread $dy/dt$ is proportional to the number of such contacts. Further, assume that members of both groups move about freely each other, so the number of contacts is proportional to the product of $x$ and $y$. Since $x = 1-y$,we obtain the initial value problem

   $$dy/dt = \alpha y(1-y), \quad y(0) = y_0,$$

   where $\alpha > 0$ is a proportional factor, and $y_0$ is the initial proportion of infectious individuals.

   (a) Find the equilibrium points for this differential equation and determine whether each is asymptotically stable, or unstable.

(b) Solve the initial value problem above and verify that the conclusions you reached in part (a) are correct. Show that

$$\lim_{t\to\infty} y(t) = 1$$

which means that ultimately the disease spreads through the entire population.

6. Some diseases (such as typhoid fever) are spread largely by *carriers*, individuals who can transmit the disease but who can transmit the disease but who exhibit no overt symptoms. Let $x$ and $y$, respectively, denote the proportion of susceptibles and carriers in the population. Suppose that carriers are identified and removed from the population at a rate $\beta$, so

$$dy/dt = -\beta \cdot y. \tag{2}$$

Suppose also that the disease spreads at a rate proportional to the product of $x$ and $y$; thus

$$dx/dt = -\alpha xy. \tag{3}$$

(a) Determine $y$ at any time $t$ by solving equation (2) subject to the initial condition $y(0) = y_0$.

(b) Use the result of part (a) to find $x$ at any time $t$ by solving equation (3) subject to the initial condition $x(0) = x_0$.

(c) Find the proportion of the population that escapes the epidemic by finding the limiting value

$$\lim_{t\to\infty} x(t).$$

**Results and some solutions**

1.
(b) $\lim_{t\to\infty} y(t) = \infty$.
(c) $y(t) = -3e^t + C \cdot e^{2t}$.

2. (a)

$$y(t) = \left(2te^t - 2e^t + 3\right) e^t$$

(b)

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

(c)

$$y(t) = 1 - \frac{1}{t} + 2\frac{e^{-t}}{t}$$

(b) $y \equiv 0$: unstable
(c) $y \equiv 0$: stable

4. (a) $2t + 5y > 0$ or $2t + 5y > 0$.
(b) $1 - t^2 - y^2 > 0$ or $1 - t^2 - y^2 < 0$
(c) Everywhere.
5.
(b) $y = y_0 / \left[y_0 + (1 - y_0)\mathrm{e}^{-\alpha t}\right]$

6. (a) $y = y_0 \mathrm{e}^{-\beta t}$
(b) $x = x_0 \exp\left[-\alpha y_0(1 - \mathrm{e}^{-\beta t})/\beta\right]$
(c) $x_0 \exp\left[-\alpha y_0/\beta\right]$