PROBABILITY, Continuous Distributions Practice

1. A probability density function (*pdf*) of variable X is given as,

$$f(x) = \begin{cases} 2x, & \text{if } 0 < x < 1. \\ 0, & \text{otherwise.} \end{cases}$$

Then the cumulative density function(*cdf*)is defined as;

$$F(x) = \int_0^x f(t)dt = \begin{cases} 0, & \text{if } x \leq 0. \\ \int_0^x 2tdt = [t^2]_0^x = x^2, & \text{if } 0 < x < 1. \\ 1, & \text{if } x > 1. \end{cases}$$

Also finding the probability over a certain interval is computed by integrate the (*pdf*) over that interval. In this example,

$$\mathbb{P}\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} 2xdx = \left[x^2\right]_{\frac{1}{4}}^{\frac{3}{4}} = \frac{1}{2}$$

Memoryless (Markov) Property of the Exponential distribution

Theorem. If $X \sim Exp(\lambda)$, then

$$\mathbb{P}(X > t + s | X > t) = \mathbb{P}(X > s), \forall t, s > 0$$

Proof:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(X > t + s)}{\mathbb{P}(X > t)} = \frac{1 - F(t + s)}{1 - F(t)}$$
$$= \frac{1 - (1 - e^{-\lambda(t+s)})}{1 - e^{-\lambda t}} = \frac{e^{-\lambda t - \lambda s}}{e^{-\lambda t}} = e^{-\lambda s} = 1 - F(s) = \mathbb{P}(X > s).$$

Example: let the lifetime (year) of a radioactive isotope, starting in 1986, be X. X is exponentially distributed with parameter $\lambda = \frac{1}{140}$. Find the halving time of the isotope (median of the distribution): it is T such that $F(T) = \frac{1}{2}$. From here, $T = \frac{\ln 2}{\lambda} = 140 \ln 2 < 140$.

Which proportion of the isotopes is present now?

$$\mathbb{P}(X > 2017-1986) = \mathbb{P}(X > 31) = 1-F(310) = 1-(1-e^{\frac{1}{140}31}) = e^{-\frac{31}{140}}$$

Which proportion of the isotopes present at 2000, is also present now?

$\mathbb{P}(X > 31|X > 14) = \mathbb{P}(X > 17) = 1 - F(17) = 1-(1-e^{\frac{1}{140}17}) = e^{-\frac{17}{140}}$

Exercises to the Normal Distribution (Gaussian Distribution)

2. Let $X \sim \mathcal{N}(\mu, \sigma)$ be rv. Using *standardization* $Y = \frac{X-\mu}{\sigma}$, find the following probabilities:

$$\mathbb{P}(\mu - \sigma < X < \mu - \sigma) = \mathbb{P}(-\sigma < X - \mu < \sigma) = \mathbb{P}\left(-1 < \frac{X-\mu}{\sigma} < 1\right)$$
$$= \Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1)) = 2\Phi(1) - 1 \approx 0.68$$

The value of $\Phi(1) \approx 0.84$ is obtained from the standard normal table.
As well,

$$\mathbb{P}(\mu - 2\sigma < X < \mu - 2\sigma) = \mathbb{P}(-2 < \frac{X-\mu}{\sigma} < 2) = \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2))$$
$$= 2\Phi(2) - 1 \approx 0.95.$$

The value of $\Phi(2) \approx 0.9772$ is obtained from the standard normal table.

Exercises to the Central Limit theorem topic

3. A fair die is rolled 100 times. Find the exact and the approximate probability that the number of '6' outcomes is between 10 and 20.

Solution: let X be the number of the outcomes '6'. Since $X \sim Bin_{100}\left(\frac{1}{6}\right)$, the exact probability is

$$\mathbb{P}(10 \leq X \leq 20) = \sum_{k=10}^{20} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(1 - \frac{5}{6}\right)^{100-k}.$$

This formula is correct to be written in the exam. The approximate probability can be calculated by the De-Moivre–Laplace theorem, where $n = 100$ is "large".

$$X \sim \mathcal{N}\left(100\frac{1}{6}, \sqrt{100\frac{1}{6}\frac{5}{6}}\right)$$

which is,

$$\approx \mathcal{N}\left(\frac{50}{3}, \frac{\sqrt{5}10}{6}\right).$$

Therefore, the probability is

$$\mathbb{P}(10 \leq X \leq 20) = \mathbb{P}(\frac{10-\frac{50}{3}}{\sqrt{5}\frac{10}{6}} \leq \frac{X-\frac{50}{3}}{\sqrt{5}\frac{10}{6}} \leq \frac{20-\frac{50}{3}}{\sqrt{5}\frac{10}{6}})$$

By using the standard normal distribution function, this equals to

$$\Phi(\frac{2}{\sqrt{5}}) - \Phi(-\frac{4}{\sqrt{5}}) = \Phi(\frac{2}{\sqrt{5}}) + \Phi(\frac{4}{\sqrt{5}}) - 1 = 0.777$$

4. There are 300 parking permits and each permit holder comes to the university with probability 0.7, independently of the others. How many parking lots (l) to establish so that

$$\mathbb{P}(\text{someone with permit cannot find a place to park}) = 0.01.$$

Solution: Let X be the number of permit holders coming in. Where $n = 300$ and $p = 0.7$, therefore, $X \sim Bin_{300}(0.7)$. Then by the Central limit Theorem, $X \sim \mathcal{N}(210, \sqrt{63})$, and so,

$$\mathbb{P}(X > l) = 0.01$$

and

$$\mathbb{P}(X \leq l) = 0.99.$$

But

$$\mathbb{P}\left(\frac{X-210}{\sqrt{63}} \leq \frac{l-210}{\sqrt{63}}\right) = 0.99$$

$$\Phi\left(\frac{l-210}{\sqrt{62}}\right) = 0.99$$

From the standard normal table, the value that corresponds to the probability 0.99 is 2.33. Hence,

$$\frac{l-210}{\sqrt{63}} = 2.33$$

and $l = 2.33 \cdot \sqrt{63} + 210 \approx 229$.

5. 1000 persons arrive to the left or right entrance of a theater independently (they choose between the entrances with 0.5-0.5 probability). How many hangers (h)to place into the left and right cloak rooms, if they want to give only a 1 percent chance to the event that somebody cannot place his/her coat in the nearest cloakroom. (Each person has a coat.)

   Solution: let X be the number of people who arrive to the left. Since $X \sim B_{1000}(0.5)$, by using the Central Limit Theorem, $X \sim \mathcal{N}(500, \sqrt{250} = 5 \cdot \sqrt{10})$.

$$\mathbb{P}(X > h \text{ or } 1000 - X > h) = .05$$

   Using the complementary event,

$$\mathbb{P}(1000 - h \leq X \leq h) = \mathbb{P}(\frac{-h - 500}{5\sqrt{10}} \leq \frac{X - 500}{5\sqrt{10}} \leq \frac{h - 500}{5\sqrt{10}}) = 0.95$$

   and

$$\Phi\left(\frac{h - 500}{5\sqrt{10}}\right) - \left(1 - \Phi\left(\frac{h - 500}{5\sqrt{10}}\right)\right) = .95$$

   By solving this equation, $h = 531$.