PROBABILITY, Problems to Lesson 4.

1. *Birthday holidays:* n workers are employed in a factory. Each working day, each worker manifactures a chair. When any of them has a birthday, there is a holiday (the factory is closed, they do not work at all). Under these conditions, how many workers have to be employed, if they want to maximize the number of chairs produced in a year in the factory? Equivalent problem: A worker's legal code specifies a holiday any day during which at least one worker in a certain factory has a birthday. All other days are working days. How many workers (n) must the factory employ so that the expected number of working man-days is maximized during the year?

2. Two people (A and B) play the following game: they cast a die, one after the other (alternately), until the first 6 comes out. What is the probability, that it is the starting player (A) who wins?

3. A secretary keeps 50-50 typing papers in her left and right drawer. Whenever she needs typing paper, she takes out one, randomly from one of the drawers. Let X be the following random variable: at which occasion it happens first that she cannot find more typing paper in the actual drawer (she chose). Give the distribution of X. What is the probability, that it happens at the 70^{th} occasion, or else, $\mathbb{P}(X = 70) =$?
   Equivalent problem: **Banach's match-box problem.** A certain mathematician always carries two mach-boxes; every time he wants a match he selects a box at random. Ineviatably a moment occurs when, for the first time, he finds a box empty. Give the distribution of this moment, if both match-boxes contain 50-50 matches at the beginning.

4. **Memoryless (Markov) property of the geometric distribution:** Prove that if X follows a geometric distribution then $\mathbb{P}(X > k + m \,|\, X > k) = \mathbb{P}(X > m)$ for any $k, m \in \mathbb{N}$.

5. **Memoryless (Markov) property of the exponential distribution:** Prove that if X follows an exponential distribution then $\mathbb{P}(X > t + s \,|\, X > s) = \mathbb{P}(X > t)$ for any $t, s > 0$.

6. Let $X \sim \mathcal{E}xp(\lambda)$ be a memoryless lifetime and let Y be the discretized version of it: $Y := k$, if the decay happens in the $[k-1, k)$ interval ($k = 1, 2, \dots$). Prove that $Y \sim \mathcal{G}(1 - e^{-\lambda})$.

7. In Chernobil (1986) a radioactive decay of an isotope started. Give the proportion of this isotope that is still present in 2000, if $\lambda = 1/140(years)$. Give the proportion of the isotopes present in 2000, that decayed until 2008. Give the halving time of this isotope.

8. According to the data of the past 20 years, there were 40 months without accident on a certain section of highway M1. What is the probability that at least 3 accidents will happen in the next month?

9. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ be Gaussian random variable. Calculate $\mathbb{P}(\mu - \sigma < X < \mu + \sigma)$ and $\mathbb{P}(\mu - 2\sigma < X < \mu + 2\sigma)$.

10. The body height of university students in Hungary is approximately $X \sim \mathcal{N}(170, 25)$ in centimeters. Calculate $\mathbb{P}(165 < X < 173)$.

11. The lifetime of an transistor is exponential with expected lifetime 4/3 year. Two such transistors are operated in parallel (independently). Give the probability that both go wrong in one and a half year. If one of them is working after a year, give the probability that it will still work after another half year?