1.

$y' + 2y = -x + 4,$ $\quad y' - 2y = 0,$ $\quad y_h = ce^{2x},$

$c(x) = \int \frac{-x+4}{e^{2x}} dx,$ $\quad c(x) = \frac{2x-7}{4} e^{-2x},$ $\quad y_p = \frac{2x-7}{4},$

$y = y_p + y_h = \frac{2x-7}{4} + ce^{2x}.$

2.

$\underbrace{e^y}_{M(x,y)} dx + \underbrace{(xe^y + 2y)}_{N(x,y)} dy = 0,$ $\quad \frac{\partial M(x,y)}{\partial y} = e^y = \frac{\partial N(x,y)}{\partial x}$

$\frac{\partial F(x,y)}{\partial x} = M(x,y),$ $\quad F(x,y) = \int M(x,y)\ dx$

$F(x,y) = xe^y + h(y)$ $\quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$

$xe^y + \frac{\partial h(y)}{\partial y} = xe^y + 2y$ $\quad h(y) = y^2$

$F(x,y) = xe^y + y^2 \equiv C.$

3.

$y'' + y = 0,$ $\quad y_h = c_1 \cos x + c_2 \sin x,$

$y_p = e^x(A \cos x + B \sin x)$ $\quad y_p'' = e^x(2B \cos x - 2A \sin x)$

$A + 2B = 0,\ -2A + B = 1$ $\quad A = -2/5,\ B = 1/5$

$y_p = e^x(-\frac{2}{5} \cos x + \frac{1}{5} \sin x)$ $\quad y = y_h + y_p = (c_1 - \frac{2}{5}e^x) \cos x + (c_2 + \frac{1}{5}e^x) \sin x.$

4. Each permutation of A, B, C occurs with equal probability therefore the desired probability is $1/3! = 1/6$.

5. Let event A : there is a tail among the 3 flips, B : there is a head among the 3 flips. $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{1-\frac{2}{2^3}}{1-\frac{1}{2^3}} = 6/7.$

6. Denote X the number of times a certain firm is investigated during a 5 year period. Then $X \sim \mathrm{Bin}(5, 0.05)$.

   a) $1 - P(X = 0) = 1 - 0.95^5 = 0.226$.
   b) $1 - P(X = 0) - P(X = 1) = 1 - 0.95^5 - \binom{5}{1} 0.05^1 \cdot 0.95^4 = 0.023$.

   Remark. It is not reasonable to use Poisson distribution here because the number of experiments (5) is very small.

7. Let X denote the nicotine level of a random smoker. Then $X \sim \mathrm{N}(315, 131^2)$.

   (a) $P(150 < X < 400) = P\left(\frac{150-315}{131} < \frac{X-315}{131} < \frac{400-315}{131}\right) = \Phi(\frac{85}{131}) - \Phi(-\frac{165}{131}) = 07422 - (1 - 0.8962) = 0.6384$.
   (b) $P(X > a) = 0.05$, that is, $P(X < a) = 0.95$, whence $P\left(\frac{X-315}{131} < \frac{a-315}{131}\right) = 0.95$, from where $\Phi\left(\frac{a-315}{131}\right) = 0.95$, therefore $a = 530.5$.

8. Rewriting the system we obtain

$$\begin{array}{rcccl} (D+1)x & + & y & = & 0 \\ 2Dx & + & (D+1)y & = & 0 \end{array}$$

from where

$$\Delta = \begin{vmatrix} D+1 & 1 \\ 2D & D+1 \end{vmatrix} = D^2 + 1,$$

whence

$$x'' + x = 0, \ y'' + y = 0.$$

These are second order homogeneous linear differential equations with constant coefficients, hence

$$x(t) = c_1 \cos t + c_2 \sin t, \ y(t) = c_3 \cos t + c_4 \sin t.$$

Plugging these into, say, the first equation of the original system we obtain $c_3 = -c_1 - c_2, c_4 = c_1 - c_2$. Now the general solution of the system is

$$x(t) = c_1 \cos t + c_2 \sin t, \ y(t) = (-c_1 - c_2) \cos t + (c_1 - c_2) \sin t.$$

9. If X denotes the number of defected items then $X \sim \mathrm{Bin}(500, 0.02)$. Applying the central limit theorem we have $P(X \geq 15) \approx 1 - \Phi\left(\frac{14.5 - 0.02 \cdot 500}{\sqrt{500 \cdot 0.02 \cdot 0.98}}\right) = 1 - \Phi(1.44) = 0.0749$. (the exact probability is $\sum_{i=15}^{500} \binom{500}{i} 0.02^i \cdot 0.98^{500-i} = 1 - \sum_{i=0}^{14} \binom{500}{i} 0.02^i \cdot 0.98^{500-i} = 0.081$)