A3 exam test solutions, 2020. jan. 28.

1.

$$y' = \frac{y}{x} + \frac{y^2}{x^2}, \qquad u = \frac{y}{x}, \qquad u + u'x = u + u^2,$$

$$u' \cdot \frac{1}{u^2} = \frac{1}{x}, \qquad -\tfrac{1}{u} = \ln|x| + c, \qquad u = -\frac{1}{\ln|x| + c},$$

$$\boxed{y(x) = x \cdot u(x) = -\frac{x}{\ln|x| + c}.}$$

2.

$$y' + \frac{2}{x} \cdot y = x^2 \qquad y_h' + \frac{2}{x} \cdot y_h = 0 \qquad y_h(x) = c \cdot \frac{1}{x^2}$$

$$c(x) = \int \frac{x^2}{1/x^2}\, dx \qquad c(x) = \frac{x^5}{5} \qquad y_p(x) = c(x) \cdot \frac{1}{x^2} = \frac{x^3}{5}$$

$$\boxed{y(x) = y_p(x) + y_h(x) = \frac{x^3}{5} + \frac{c}{x^2}.}$$

3.

$$y' = p(y),\; y'' = p'(y)p(y), \qquad 2y^2p'p = p^3, \qquad p \neq 0$$

$$p' \cdot \frac{1}{p^2} = \frac{1}{2y^2}, \qquad -\frac{1}{p} = -\frac{1}{2y} + c, \quad c = 0,$$

$$y' = 2y, \qquad y = c \cdot e^{2x}, \qquad c = 1$$

$$\boxed{y(x) = e^{2x}.}$$

4. $P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^6 = \boxed{0.665.}$

5. Let A be the event that the first flip is a head and B that exactly two are heads out of the three flips.

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{2} \cdot \binom{2}{1} \frac{1}{2^2}}{\binom{3}{2} \frac{1}{2^3}} = \boxed{\frac{2}{3}.}$$

6. If X is the lifetime of the battery then by the exercise $X \sim \text{Exp}(1/3)$.

   (a) $P(2 < X < 4) = (1 - e^{-4/3}) - (1 - e^{-2/3}) = \boxed{0.250.}$

   (b) By the memoriless property $P(X > 5 \mid X > 3) = P(X > 2) = 1 - (1 - e^{-2/3}) = \boxed{0.513.}$

7. Let X be the rainfall in a year. Then by the exercise $X \sim \mathrm{N}(600{,}75)$.

   (a) $p = P(X > 750) = P\left(\frac{X-600}{75} > \frac{750-600}{75}\right) = 1 - \Phi(2) = \boxed{0.023.}$

   (b) Ha Y év múlva lesz ilyen év, akkor $Y \sim \mathrm{Geo}(p)$, tehát $EY = \frac{1}{0.023} = \boxed{43.9.}$

8. Using the differential operator D we can rewrite the system as:

$$(2D-4)x + (D-1)y = e^t$$

$$(D+3)x + \quad 1 \cdot y \quad = 0$$

from where

$$\Delta = \begin{vmatrix} 2D-4 & D-1 \\ (D+3) & 1 \end{vmatrix} = -D^2 - 1 \neq 0,$$

hence

$$\Delta_x = \begin{vmatrix} e^t & D-1 \\ 0 & 1 \end{vmatrix} = e^t, \quad \Delta_y = \begin{vmatrix} 2D-4 & e^t \\ D+3 & 0 \end{vmatrix} = -4e^t,$$

azaz

$$-x'' - x = e^t, \; -y'' - y = -4e^t.$$

The solutions ofor these differential equations are (for instance by using the method of undetermined coefficients):

$$x(t) = -\frac{1}{2}e^t + c_1 \cos t + c_2 \sin t, \; y(t) = 2e^t + c_3 \cos t + c_4 \sin t.$$

Plugging these back to the second equation of the original system we obtain $c_2 + 3c_1 = -c_3, -c_1 + 3c_2 = -c_4$, hence the general slution for the system is

$$\boxed{x(t) = -\frac{1}{2}e^t + c_1 \cos t + c_2 \sin t, \; y(t) = 2e^t + (-3c_1 - c_2) \cos t + (c_1 - 3c_2) \sin t.}$$

9. If X_i is the number of requests in the ith second then by the exercise $X_i \sim$ $\sim$ Poisson(1.3), $i = 1, \ldots, 60$, therefore $EX = D^2X = 1.3$. If $S = X_1 + \cdots + X_{60}$ then by the CLT

$$P(S \geq 120) =$$
$$= P\left(\frac{S - 60 \cdot 1.3}{\sqrt{60 \cdot 1.3}} \geq \frac{120 - 60 \cdot 1.3}{\sqrt{60 \cdot 1.3}}\right) = 1 - P\left(\frac{S - 60 \cdot 1.3}{\sqrt{60 \cdot 1.3}} \leq \frac{119.5 - 60 \cdot 1.3}{\sqrt{60 \cdot 1.3}}\right) \approx$$
$$\approx 1 - \Phi\left(\frac{119.5 - 60 \cdot 1.3}{\sqrt{60 \cdot 1.3}}\right) = 1 - \Phi(4.699) = \boxed{1.308 \cdot 10^{-6}.}$$

(The exact value is $1 - \sum_{i=0}^{60} \frac{(60 \cdot 1.3)^i}{i!} e^{-60 \cdot 1.3} = 6.163 \cdot 10^{-6}$.)