Using Symbolic Computation to Analyze Zero-Hopf Bifurcations of Polynomial Differential Systems

Bo Huang

LMIB – School of Mathematical Sciences

Beihang University

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Outline

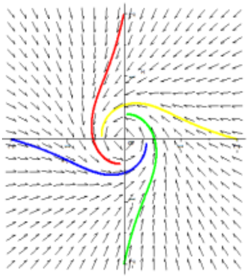
Introduction

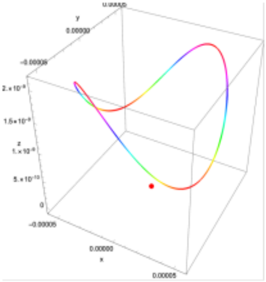
We deal with polynomial differential systems in $\mathbb{R}^n$ of the form

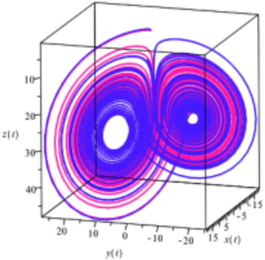
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}), \quad \mathbf{f} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n, \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ are variables, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ are real parameters, and $\mathbf{f} = (f_1, \ldots, f_n)$ are polynomials in $\mathbb{R}[\mathbf{x}]$.

Many applications: physics, biology, chemistry and engineering.







– Zero-Hopf bifurcation of limit cycles

Hopf bifurcation

The Hopf Bifurcation Theorem [Guckenheimer and Holmes, 1983]

Let

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}), \quad \mathbf{x} \in \mathbb{R}^n, \quad \boldsymbol{\mu} \in \mathbb{R}^d \tag{2}$$

be a smooth n-dimensional vector field depending upon d parameters with the following properties:

1. The system has a smooth curve of equilibria: $\mathbf{f}(\underline{\mathbf{x}}, \boldsymbol{\mu}) = 0$.
2. The characteristic equation $|\lambda I - D\mathbf{f}(\underline{\mathbf{x}}, \underline{\boldsymbol{\mu}})| = 0$, where $D\mathbf{f}(\underline{\mathbf{x}}, \underline{\boldsymbol{\mu}})$ is the Jacobian matrix of the system at $(\underline{\mathbf{x}}, \underline{\boldsymbol{\mu}})$, has a pair of imaginary roots $(\lambda(\underline{\mathbf{x}}, \underline{\boldsymbol{\mu}}), \bar{\lambda}(\underline{\mathbf{x}}, \underline{\boldsymbol{\mu}}))$, and no other roots with zero real parts.
   $\dashrightarrow$ $(\underline{\mathbf{x}}, \underline{\boldsymbol{\mu}})$ Hopf equilibria
3. The real part $\mathrm{Re}(\lambda(\underline{\mathbf{x}}, \boldsymbol{\mu}))$ of $\lambda(\underline{\mathbf{x}}, \boldsymbol{\mu})$ satisfies
$$\left.\frac{\partial \mathrm{Re}(\lambda(\underline{\mathbf{x}}, \boldsymbol{\mu}))}{\partial \mu_i}\right|_{\mu_i = \underline{\boldsymbol{\mu}}} \neq 0. \tag{3}$$

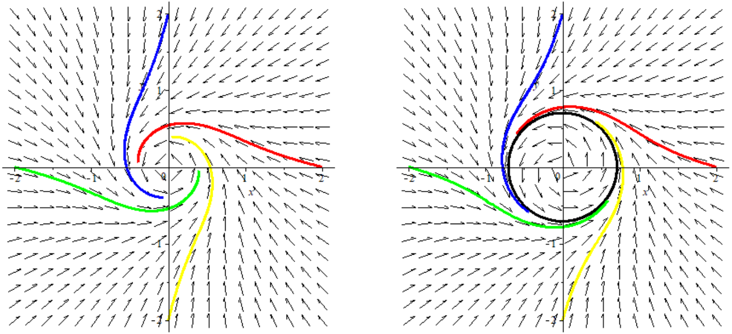
Then there is a smooth submanifold P of dimension $d+1$ containing $(\underline{\mathbf{x}}, \underline{\boldsymbol{\mu}})$ that is a union of periodic orbits and equilibrium points of $\mathbf{f}$.

Hopf bifurcation

A planar differential system with one parameter:

$$\dot{x} = \mu x - y - x(x^2 + y^2), \quad \dot{y} = x + \mu y - y(x^2 + y^2). \tag{4}$$

1) :$O = (0, 0)$, **2)** :$\mu = 0 \Longrightarrow \lambda_{1,2} = \pm i$, **3)** :$\partial \mathrm{Re}(\lambda(O, \mu))/\partial\mu\big|_{\mu=0} = 1 \neq 0.$



(a) Stable focus ($\mu = 0$) (b) Limit cycle ($\mu > 0$)

An isolated periodic solution of system (4) is called a limit cycle.

Hopf bifurcation

Hopf bifurcation $\hookrightarrow$ $[\pm ib\ (b \neq 0), \quad \mathrm{Re}(\lambda_j) \neq 0]$

J. Guckenheimer, M. Myers, B. Sturmfels. Computing Hopf Bifurcation I. *SIAM Journal of Numerical Analysis*, 1997

J. Guckenheimer, M. Myers. Computing Hopf Bifurcation II: Three Examples from Neurophysiology. *SIAM Journal on Scientific Computing*, 1996

W. Govaerts, J. Guckenheimer, A. Khibnik. Defining Functions for Multiple Hopf Bifurcations. *SIAM Journal of Numerical Analysis*, 1997

M. El Kahoui, A. Weber. Deciding Hopf Bifurcations by Quantifier Elimination in a Software-component Architecture. *Journal of Symbolic Computation*, 2000

$$\Downarrow$$

Zero-Hopf bifurcation: $\hookrightarrow$ $[\pm ib\ (b \neq 0), \quad \lambda_j = 0, \quad j \in \{3, \ldots, n\}]$

Complete zero-Hopf bifurcation: $\hookrightarrow$ $[\pm ib\ (b \neq 0), \quad \lambda_3 = \cdots \lambda_n = 0]$

Zero-Hopf bifurcation

We assume that system (1) has a singularity at the origin. In this case, system (1) of degree at most N can be written in the form

$$
\begin{aligned}
\dot{x}_1 &= f_1 = -bx_2 + \sum_{m\geq 2}^{N} \sum_{i_1+\cdots+i_n=m} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}, \\
\dot{x}_2 &= f_2 = bx_1 + \sum_{m\geq 2}^{N} \sum_{i_1+\cdots+i_n=m} b_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}, \\
\dot{x}_s &= f_s = \sum_{m\geq 2}^{N} \sum_{i_1+\cdots+i_n=m} c_{i_1,\ldots,i_n,s} x_1^{i_1} \cdots x_n^{i_n},
\end{aligned} \tag{5}
$$

where $s = 3, \ldots, n$, $b \neq 0$, $a_{i_1,i_2,\ldots,i_n}$, $b_{i_1,i_2,\ldots,i_n}$ and $c_{i_1,i_2,\ldots,i_n,s}$ are real parameters.

Goal: *How many limit cycles can bifurcate from the origin, as a zero-Hopf equilibrium of system* (5), *when the system is perturbed inside the class of differential systems of the same form?*

The main technique: The averaging method $\hookrightarrow$ needs a small parameter ε

– Other methods: Center manifold theory & Normal form theory

[Guckenheimer & Holmes 1993], [Kuznetsov 2004], [Han & Yu 2012]

Our problem

We consider the following perturbations of system (5)

$$\begin{aligned}\dot{x}_1 &= f_1 + p_1(x_1, \ldots, x_n, \varepsilon),\\ \dot{x}_2 &= f_2 + p_2(x_1, \ldots, x_n, \varepsilon),\\ \dot{x}_s &= f_s + p_s(x_1, \ldots, x_n, \varepsilon), \quad s = 3, \ldots, n,\end{aligned} \tag{6}$$

where

$$p_s = \sum_{j=1}^{k} \varepsilon^j \sum_{m^* \geq 1}^{N} \sum_{i_1 + \cdots + i_n = m^*} p_{s,j,i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

the constants $p_{s,j,i_1,\ldots,i_n}$ are real, and ε is a small parameter.

Problem 1. We are interested in the maximum number of limit cycles of system (6) for $|\varepsilon|$ sufficiently small, which bifurcate from the origin in a zero-Hopf bifurcation.

Techniques. The method of averaging & Symbolic computation

The method of averaging

History of averaging

- [Fatou 1928], [Bogoliubov & Krylov 1930's], [Bogoliubov 1945], [Hale 1963,1980], ...
- A survey: [Sanders & Verhulst 1985], [Sanders, Verhulst & Murdock 2007], [Llibre, Moeckel & Simó 2015]

J. Llibre, D. Novaes, M. Teixeira. *Higher order averaging theory for finding periodic solutions via Brouwer degree*, Nonlinearity, 2014

Usually, the averaging method deals with differential systems in the following standard form

$$\frac{d\mathbf{x}}{dt} = \sum_{i=0}^{k} \varepsilon^i \mathbf{F}_i(t, \mathbf{x}) + \varepsilon^{k+1} \mathbf{R}(t, \mathbf{x}, \varepsilon), \tag{7}$$

where $\mathbf{F}_i : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 0, 1, \ldots, k$, and $\mathbf{R} : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, and T-periodic in the variable t, D being an open subset of $\mathbb{R}^n$.

The method of averaging

The process of using the averaging method [Huang & Yap, J. Symbolic Comput., 2021]

STEP 1. Write the perturbed system (6) in the standard form of averaging (7) up to k-th order in ε.

Ideas:

$$(x_1, x_2, \ldots, x_n) = (\varepsilon X_1, \varepsilon X_2, \ldots, \varepsilon X_n) \quad \Longleftarrow \quad \text{rescale the variables} \quad \Longrightarrow \quad dX_i/dt$$

$$X_1 = R\cos\theta,\ X_2 = R\sin\theta,\ X_s = X_s\ (s = 3, \ldots, n) \quad \Longleftarrow \quad \text{change of variables}$$

$$\frac{dR}{d\theta} = \frac{dR/dt}{d\theta/dt} = \varepsilon F_{1,1}(\theta, \boldsymbol{\eta}) + \cdots + \varepsilon^k F_{k,1}(\theta, \boldsymbol{\eta}) + \mathcal{O}(\varepsilon^{k+1}) \quad \Longleftarrow \quad \text{Taylor expansion}$$

$$\frac{dX_s}{d\theta} = \frac{dX_s/dt}{d\theta/dt} = \varepsilon F_{1,s}(\theta, \boldsymbol{\eta}) + \cdots + \varepsilon^k F_{k,s}(\theta, \boldsymbol{\eta}) + \mathcal{O}(\varepsilon^{k+1}) \quad \Longleftarrow \quad \text{Taylor expansion}$$

where $\boldsymbol{\eta} = (R, X_3, \ldots, X_n)$.

The method of averaging

Let L be a positive integer, let $\mathbf{x} = (x_1, \ldots, x_n) \in D$, $t \in \mathbb{R}$ and $\mathbf{y}_j = (y_{j1}, \ldots, y_{jn}) \in \mathbb{R}^n$ for $j = 1, \ldots, L$. The definition of the L-multilinear map is

$$\partial^L \mathbf{F}(t, \mathbf{x}) \bigodot_{j=1}^{L} \mathbf{y}_j = \sum_{i_1, \ldots, i_L = 1}^{n} \frac{\partial^L \mathbf{F}(t, \mathbf{x})}{\partial x_{i_1} \cdots \partial x_{i_L}} y_{1 i_1} \cdots y_{L i_L}. \tag{8}$$

The simple zeros of the ith $(i = 1, 2, \ldots, k)$ order averaged function

$$\mathbf{f}_i(\mathbf{z}) = \frac{\mathbf{y}_i(T, \mathbf{z})}{i!}, \tag{9}$$

controls the limit cycles of the differential system (7).

The method of averaging

Here $\mathbf{y}_i : \mathbb{R} \times D \to \mathbb{R}^n$, for $i = 1, 2, \ldots, k$, are defined recurrently by the following integral equations

$$\mathbf{y}_1(t, \mathbf{z}) = \int_0^t \mathbf{F}_1(\theta, \mathbf{z}) d\theta,$$

$$\mathbf{y}_i(t, \mathbf{z}) = i! \int_0^t \Big(\mathbf{F}_i(\theta, \mathbf{z}) + \sum_{\ell=1}^{i-1} \sum_{S_\ell} \frac{1}{b_1! \, b_2! 2!^{b_2} \cdots b_\ell! \ell!^{b_\ell}} \times \partial^L \mathbf{F}_{i-\ell}(\theta, \mathbf{z}) \bigodot_{j=1}^{\ell} \mathbf{y}_j(\theta, \mathbf{z})^{b_j} \Big) d\theta, \tag{10}$$

where S_ℓ is the set of all ℓ-tuples of nonnegative integers $[b_1, b_2, \ldots, b_\ell]$ satisfying $b_1 + 2b_2 + \cdots + \ell b_\ell = \ell$ and $L = b_1 + b_2 + \cdots + b_\ell$.

Nonlinearity **27** (2014) 563 J Llibre *et al*

$$y_4(t, z) = \int_0^t \Big(24F_4(s, z) + 24\partial F_3(s, z) y_1(s, z) + 12\partial^2 F_2(s, z) y_1(s, z)^2 + 12\partial F_2(s, z) y_2(s, z) + 12\partial^2 F_1(s, z) y_1(s, z) \odot y_2(s, z) + 4\partial^3 F_1(s, z) y_1(s, z)^3 + 4\partial F_1(s, z) y_3(s, z) \Big) \mathrm{d}s,$$

The method of averaging

STEP 2. Derive the symbolic expression of $\mathbf{f}_i(\mathbf{z})$ by (9).

Key substep: Compute the exact formula for $\mathbf{y}_i(t, \mathbf{z})$. Symbolic program

Idea: Using the following Bell polynomial [Novaes 2017]

$$B_{p,q}(x_1, \ldots, x_{p-q+1}) = \sum_{\widetilde{S}_{p,q}} \frac{p!}{b_1! b_2! \cdots b_{p-q+1}!} \prod_{j=1}^{p-q+1} \left(\frac{x_j}{j!}\right)^{b_j}, \tag{11}$$

where $\widetilde{S}_{\ell,m}$ is the set of all $(p-q+1)$-tuples of nonnegative integers $[b_1, b_2, \ldots, b_{p-q+1}]$ satisfying $b_1 + 2b_2 + \cdots + (p-q+1)b_{p-q+1} = p$, and $b_1 + b_2 + \cdots + b_{p-q+1} = q$.

The method of averaging

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Then equation (10) becomes

$$\mathbf{y}_1(t, \mathbf{z}) = \int_0^t \mathbf{F}_1(\theta, \mathbf{z}) d\theta,$$

$$\mathbf{y}_i(t, \mathbf{z}) = i! \int_0^t \left(\mathbf{F}_i(\theta, \mathbf{z}) + \sum_{\ell=1}^{i-1} \sum_{m=1}^{\ell} \frac{1}{\ell!} \partial^m \mathbf{F}_{i-\ell}(\theta, \mathbf{z}) B_{\ell,m}\left(\mathbf{y}_1(\theta, \mathbf{z}), \ldots, \mathbf{y}_{\ell-m+1}(\theta, \mathbf{z})\right)\right)$$

(12)

Averaging theorem

Theorem 1 (Llibre–Novaes–Teixeira 2014)

Assume that the following conditions hold:

1. *for each* $i = 1, 2, \ldots, k$ *and* $t \in \mathbb{R}$, *the function* $\mathbf{F}_i(t, \mathbf{x})$ *is of class* $\mathcal{C}^{k-i}$, $\partial^{k-i}\mathbf{F}_i$ *is locally Lipschitz in* $\mathbf{x}$, *and* $\mathbf{R}$ *is a continuous function locally Lipschitz in* $\mathbf{x}$*;*
2. *for some* $j \in \{1, 2, \ldots, k\}$, $\mathbf{f}_i = 0$ *for* $i = 1, 2, \ldots, j-1$ *and* $\mathbf{f}_j \neq 0$*;*
3. *for some* $\mathbf{z}^* \in D$ *with* $\mathbf{f}_j(\mathbf{z}^*) = 0$ *we have* $\det(J_{\mathbf{f}_j}(\mathbf{z}^*)) \neq 0$.

Then, for any $|\varepsilon| > 0$ *sufficiently small, there exists a* T*-periodic solution* $\mathbf{x}(t, \varepsilon)$ *of* (7) *such that* $\mathbf{x}(0, \varepsilon) \to \mathbf{z}^*$ *when* $\varepsilon \to 0$.

STEP 3. Determine the exact upper bound for the number $H_k(n, N)$ of real isolated solutions of $\mathbf{f}_k(\mathbf{z})$.

Some remarks

\# of limit cycles $\dashrightarrow$ # of real isolated solutions of $\mathbf{f}_k(\mathbf{z})$

Remark 1

Some advanced techniques from symbolic computation, such as **Gröbner basis** *[Buchberger 1985]*, **triangular decomposition** *[Wu 2000], [Wang 2001]*, **quantifier elimination** *[Collins 1975], [Collins & Hong 1991]*, *and* **real solution classification** *[Yang & Xia 2005] may be used to perform the task.*

Problem 2: Given a perturbed differential system (6), how can we determine the symbolic expression of $\mathbf{f}_i$ for $i = 1, \ldots, k$? We provide an algorithmic approach to the solution (based on algorithms for **STEPS 1,2**).

Maple code: `https://github.com/Bo-Math/zero-Hopf`

Note: Updating the standard form of averaging by using

$f_1 \equiv f_2 \equiv \cdots \equiv f_{k-1} \equiv 0 \implies$ improve efficiency of the algorithm.

Main results

Theorem 2 ($H_1(n,N) = 2^{n-3}$)

For $k=1$ and $|\varepsilon|>0$ sufficiently small, there are systems of the form (6) having exactly $\ell \in \{0,1,\ldots,2^{n-3}\}$ limit cycles bifurcating from the origin at $\varepsilon = 0$.

$\Longrightarrow$ polynomial equations

$$a_{1,1,0,\mathbf{0}_{n-2}} + b_{1,0,1,\mathbf{0}_{n-2}} + \sum_{j=3}^{n}(a_{1,0,\mathbf{e}_j} + b_{0,1,\mathbf{e}_j})X_j = 0, \quad \text{Bézout bound} = 2^{n-3}$$

$$(c_{2,0,\mathbf{0}_{n-2},s} + c_{0,2,\mathbf{0}_{n-2},s})R^2 + 2\sum_{3\le j_1\le j_2\le n} c_{0,0,\mathbf{e}_{j_1 j_2},s}X_{j_1}X_{j_2} + 2\sum_{j=3}^{n} c_{1,0,0,\mathbf{e}_j,s}X_j = 0, \quad s = 3,\ldots,n.$$

Main results

Theorem 2 ($H_1(n, N) = 2^{n-3}$)

For $k = 1$ and $|\varepsilon| > 0$ sufficiently small, there are systems of the form (6) having exactly $\ell \in \{0, 1, \ldots, 2^{n-3}\}$ limit cycles bifurcating from the origin at $\varepsilon = 0$.

$\Longrightarrow$ polynomial equations

$$a_{1,1,0,\mathbf{0}_{n-2}} + b_{1,0,1,\mathbf{0}_{n-2}} + \sum_{j=3}^{n}(a_{1,0,\mathbf{e}_j} + b_{0,1,\mathbf{e}_j})X_j = 0, \quad \text{Bézout bound} = 2^{n-3}$$

$$(c_{2,0,\mathbf{0}_{n-2},s} + c_{0,2,\mathbf{0}_{n-2},s})R^2 + 2\sum_{3 \le j_1 \le j_2 \le n} c_{0,0,\mathbf{e}_{j_1 j_2},s}X_{j_1}X_{j_2} + 2\sum_{j=3}^{n} c_{1,0,0,\mathbf{e}_j,s}X_j = 0, \quad s = 3, \ldots, n.$$

Denote the Jacobian of the function $\mathbf{f}_k(\boldsymbol{\eta})$ by $J_{\mathbf{f}_k}(\boldsymbol{\eta})$. That is,

$$J_{\mathbf{f}_k}(\boldsymbol{\eta}) = \begin{bmatrix} \frac{\partial f_{k,1}}{\partial R} & \frac{\partial f_{k,1}}{\partial X_3} & \cdots & \frac{\partial f_{k,1}}{\partial X_n} \\ \frac{\partial f_{k,3}}{\partial R} & \frac{\partial f_{k,3}}{\partial X_3} & \cdots & \frac{\partial f_{k,3}}{\partial X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{k,n}}{\partial R} & \frac{\partial f_{k,n}}{\partial X_3} & \cdots & \frac{\partial f_{k,n}}{\partial X_n} \end{bmatrix}.$$

Main results

Let $D_k(\boldsymbol{\eta})$ be the determinate of the Jacobian $J_{\mathbf{f}_k}(\boldsymbol{\eta})$.

Theorem 3

For $|\varepsilon| > 0$ sufficiently small, system (6) up to the kth-order averaging has exactly ℓ limit cycles bifurcating from the origin if the following semi-algebraic system

$$\begin{cases} \bar{f}_{k,1}(\boldsymbol{\eta},\boldsymbol{\mu}) = \bar{f}_{k,3}(\boldsymbol{\eta},\boldsymbol{\mu}) = \cdots = \bar{f}_{k,n}(\boldsymbol{\eta},\boldsymbol{\mu}) = 0, \\ R > 0, \quad \bar{D}_k(\boldsymbol{\eta},\boldsymbol{\mu}) \neq 0, \quad b \neq 0 \end{cases} \tag{13}$$

has exactly ℓ distinct real solutions with respective to the variables $\boldsymbol{\eta}$. Here $\bar{f}_{k,j}(\boldsymbol{\eta},\boldsymbol{\mu})$ ($j = 1, 3, \ldots, n$), and $\bar{D}_k(\boldsymbol{\eta},\boldsymbol{\mu})$ are respectively the numerator of the functions $f_{k,j}(\boldsymbol{\eta})$ and $D_k(\boldsymbol{\eta})$, with $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)$ are parameters appearing in the averaged functions.

Software packages: QEPCAD + DV + DISCOVERER + RegularChains

⇠ CAD [Collins & Hong 1991] discriminant varieties [Lazard & Rouillier 2007]

real solution classification [Yang & Xia 2005]

STEP 3 ⇠ BKK Bound + Theorem 3

Main results

Mixed volumes and Bernstein's theorem

David A. Cox, John B. Little, Don O'Shea. Using Algebraic Geometry, Second Edition, Graduate Texts in Mathematics, Volume 185 Springer, 2005

Chap. 7 Polytopes, Resultants, and Equations

Table: Some values of the BKK bounds of $H_2(n, N)$.

		n					
		3	4	5	6	7	$\cdots$
N	2	3	9	27	81	243	
	3	3	9	27	81	243	
	4	3	9	27	81	243	
	5	3	9	27	81	243	
	6	3	9	27	81	243	
	7	3	9	27	81	243	
	$\vdots$						$\ddots$

Emiris and Canny's algorithm: https://github.com/iemiris/MixedVolume-SparseResultants

Conjecture: BKK bound $= H_2(n, N) = 3^{n-2}$

Experiments

Table: Computational times (in seconds) of the function $\textsc{OrderKFormula}(k, n)$.

		k				
		1	2	3	4	5
n	2	0.	0.024	0.034	0.055	0.107
	3	0.	0.027	0.052	0.137	0.685
	4	0.	0.031	0.076	0.399	4.071
	5	0.	0.033	0.125	1.033	21.699
	6	0.	0.039	0.203	2.563	103.486
	7	0.002	0.046	0.322	5.527	440.564

```
> OrderKFormula(3, 2);
```

$$\Bigg[\int_0^t \Bigg(6\, F_{3,\,1}(\theta, z_1, z_2) + 6\left(\frac{\partial}{\partial z_1} F_{2,\,1}(\theta, z_1, z_2)\right) y_{1,\,1}(\theta, z_1, z_2) + 6\left(\frac{\partial}{\partial z_2} F_{2,\,1}(\theta, z_1, z_2)\right) y_{1,\,2}(\theta, z_1, z_2) + 3\left(\frac{\partial}{\partial z_1} F_{1,\,1}(\theta, z_1, z_2)\right) y_{2,\,1}(\theta, z_1, z_2) + 3\left(\frac{\partial}{\partial z_2} F_{1,\,1}(\theta, z_1, z_2)\right) y_{2,\,2}(\theta, z_1, z_2) + 3\left(\frac{\partial^2}{\partial z_1^2} F_{1,\,1}(\theta, z_1, z_2)\right) y_{1,\,1}(\theta, z_1, z_2)^2 + 6\left(\frac{\partial^2}{\partial z_2\, \partial z_1} F_{1,\,1}(\theta, z_1, z_2)\right) y_{1,\,1}(\theta, z_1, z_2)\, y_{1,\,2}(\theta, z_1, z_2) + 3\left(\frac{\partial^2}{\partial z_2^2} F_{1,\,1}(\theta, z_1, z_2)\right) y_{1,\,2}(\theta, z_1, z_2)^2\Bigg) \mathrm{d}\theta, \int_0^t \Bigg(6\, F_{3,\,2}(\theta, z_1, z_2) + 6\left(\frac{\partial}{\partial z_1} F_{2,\,2}(\theta, z_1, z_2)\right) y_{1,\,1}(\theta, z_1, z_2) + 6\left(\frac{\partial}{\partial z_2} F_{2,\,2}(\theta, z_1, z_2)\right) y_{1,\,2}(\theta, z_1, z_2) + 3\left(\frac{\partial}{\partial z_1} F_{1,\,2}(\theta, z_1, z_2)\right) y_{2,\,1}(\theta, z_1, z_2) + 3\left(\frac{\partial}{\partial z_2} F_{1,\,2}(\theta, z_1, z_2)\right) y_{2,\,2}(\theta, z_1, z_2) + 3\left(\frac{\partial^2}{\partial z_1^2} F_{1,\,2}(\theta, z_1, z_2)\right) y_{1,\,1}(\theta, z_1, z_2)^2 + 6\left(\frac{\partial^2}{\partial z_2\, \partial z_1} F_{1,\,2}(\theta, z_1, z_2)\right) y_{1,\,1}(\theta, z_1, z_2)\, y_{1,\,2}(\theta, z_1, z_2) + 3\left(\frac{\partial^2}{\partial z_2^2} F_{1,\,2}(\theta, z_1, z_2)\right) y_{1,\,2}(\theta, z_1, z_2)^2\Bigg) \mathrm{d}\theta\Bigg]$$

Experiments

3D jerk system

$$\begin{aligned}
&\dot{x} = y, \quad \dot{y} = z, \\
&\dot{z} = -az - bx + cy + xy^2 - x^3, \quad a, b, c \in \mathbb{R}.
\end{aligned} \tag{14}$$

The origin is a zero-Hopf equilibrium when $a = b = 0$ and $c < 0$.
We consider the vector (a, b, c) given by the second order averaging

$$\begin{aligned}
&a = \varepsilon a_1 + \varepsilon^2 a_2, \quad b = \varepsilon b_1 + \varepsilon^2 b_2, \\
&c = -\beta^2 + \varepsilon c_1 + \varepsilon^2 c_2, \quad \beta \neq 0,
\end{aligned}$$

where the constants a_i, b_i and c_i are all real parameters. Then the jerk system becomes

$$\begin{aligned}
&\dot{x} = y, \quad \dot{y} = z, \\
&\dot{z} = -(\varepsilon a_1 + \varepsilon^2 a_2)z - (\varepsilon b_1 + \varepsilon^2 b_2)x \\
&\qquad + (-\beta^2 + \varepsilon c_1 + \varepsilon^2 c_2)y + xy^2 - x^3.
\end{aligned} \tag{15}$$

Experiments

Theorem 4

The following statements hold for $|\varepsilon| > 0$ sufficiently small.

(i) *The first-order averaging does not provide any information about limit cycles that bifurcate from the origin.*

(ii) *System* (15) *has, up to the second-order averaging, at most 3 limit cycles bifurcating from the origin, and this number can be reached if one of the following 2 conditions holds:*

$$\begin{aligned} \mathcal{C}_0 &= [R_1 < 0,\ R_2 < 0,\ 0 < R_3,\ 0 < R_4] \wedge \bar{\mathcal{C}}, \\ \mathcal{C}_1 &= [0 < R_1,\ 0 < R_2,\ 0 < R_3,\ R_4 < 0] \wedge \bar{\mathcal{C}}, \end{aligned} \tag{16}$$

where

$$R_1 = \beta^2 - 3, \quad R_2 = \beta^2 a_2 + 2b_2, \quad R_3 = 2\beta^2 a_2 - b_2,$$
$$R_4 = \beta^2 a_2 - b_2, \quad \bar{\mathcal{C}} = [\beta \neq 0, R_1 \neq 0, R_2 \neq 0, R_3 \neq 0, R_4 \neq 0].$$

Note: the second-order averaged functions cannot be identically zero!

Experiments

A class of generalized Lorenz systems

Consider the following integrable deformation of Lorenz system:

$$\begin{aligned} \dot{x} &= a(y-x) + dy(z-c), \\ \dot{y} &= cx - xz - y, \\ \dot{z} &= -bz + xy + sx, \end{aligned} \tag{17}$$

where a, b, c, d, s are real parameters.

The origin is a zero-Hopf equilibrium when $a=-1$, $b=0$ and $c^2d+c-1>0$.
Now consider the vector (a,b,c,d,s) given by the third order averaging

$$\begin{aligned} & a = -1 + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 a_3, \quad b = \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 b_3, \\ & c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 c_3, \quad s = s_0 + \varepsilon s_1 + \varepsilon^2 s_2 + \varepsilon^3 s_3, \\ & d = \frac{\beta^2 - c_0 + 1}{c_0^2} + \varepsilon d_1 + \varepsilon^2 d_2 + \varepsilon^3 d_3, \quad \beta > 0, \end{aligned}$$

where the constants a_i, b_i, c_i, d_i and s_i are all real parameters with $c_0 \neq 0$.

Experiments

Theorem 5

The following statements hold for $|\varepsilon| > 0$ sufficiently small.

(i) *The first-order averaging does not provide any information about limit cycles that bifurcate from the origin.*

(ii) *System* (17) *has, up to the second-order averaging, at most 1 limit cycle bifurcating from the origin, and this number can be reached if one of the following 2 conditions holds:*

$$\begin{aligned} \mathcal{C}_2 &= [a_2 < 0,\, 2\beta^2 + 2 - c_0 < 0] \wedge [\beta > 0], \\ \mathcal{C}_3 &= [0 < a_2,\, 0 < 2\beta^2 + 2 - c_0] \wedge [\beta > 0, c_0 \neq 0]. \end{aligned} \tag{18}$$

(iii) *System* (17) *has, up to the third-order averaging, at most 3 limit cycles bifurcating from the origin, and this number can be reached if we take the condition $\mathcal{C}^* = [c_1 = d_1 = 1, d_2 = s_2 = 2]$ and the* **sample points** *of (b_3, a_3, δ) listed below, where $\delta = \sqrt{\beta^2 + 1}$.*

The semi-algebraic system (13) contains many parameters $\dashrightarrow$ time consuming

Selected sample points of (b_3, a_3, δ)

60 sample points of (b_3, a_3, δ) with $\delta = \sqrt{\beta^2+1}$		
$b_3 = \frac{3}{512}, a_3 = -\frac{461}{16384}, \delta = \frac{17477}{16384}$	$b_3 = \frac{3}{512}, a_3 = -\frac{117}{8192}, \delta = \frac{34903}{32768}$	$b_3 = \frac{3}{512}, a_3 = -\frac{117}{8192}, \delta = \frac{17465}{16384}$
$b_3 = \frac{3}{512}, a_3 = -\frac{159}{16384}, \delta = \frac{8709}{8192}$	$b_3 = \frac{3}{512}, a_3 = -\frac{159}{16384}, \delta = \frac{8725}{8192}$	$b_3 = \frac{3}{512}, a_3 = -\frac{159}{16384}, \delta = \frac{8733}{8192}$
$b_3 = \frac{3}{512}, a_3 = \frac{127}{32768}, \delta = \frac{8709}{8192}$	$b_3 = \frac{3}{512}, a_3 = \frac{127}{32768}, \delta = \frac{8725}{8192}$	$b_3 = \frac{3}{512}, a_3 = \frac{127}{32768}, \delta = \frac{8733}{8192}$
$b_3 = \frac{3}{512}, a_3 = \frac{69}{8192}, \delta = \frac{34903}{32768}$	$b_3 = \frac{3}{512}, a_3 = \frac{69}{8192}, \delta = \frac{17465}{16384}$	$b_3 = \frac{3}{512}, a_3 = \frac{365}{16384}, \delta = \frac{17477}{16384}$
$b_3 = \frac{1739}{32768}, a_3 = -\frac{795}{4096}, \delta = \frac{34939}{32768}$	$b_3 = \frac{1739}{32768}, a_3 = -\frac{281}{2048}, \delta = \frac{8729}{8192}$	$b_3 = \frac{1739}{32768}, a_3 = -\frac{281}{2048}, \delta = \frac{8733}{8192}$
$b_3 = \frac{1739}{32768}, a_3 = -\frac{345}{4096}, \delta = \frac{1087}{1024}$	$b_3 = \frac{1739}{32768}, a_3 = -\frac{345}{4096}, \delta = \frac{8725}{8192}$	$b_3 = \frac{1739}{32768}, a_3 = -\frac{345}{4096}, \delta = \frac{8733}{8192}$
$b_3 = \frac{1739}{32768}, a_3 = \frac{127}{4096}, \delta = \frac{1087}{1024}$	$b_3 = \frac{1739}{32768}, a_3 = \frac{127}{4096}, \delta = \frac{8725}{8192}$	$b_3 = \frac{1739}{32768}, a_3 = \frac{127}{4096}, \delta = \frac{8733}{8192}$
$b_3 = \frac{1739}{32768}, a_3 = \frac{43}{512}, \delta = \frac{8729}{8192}$	$b_3 = \frac{1739}{32768}, a_3 = \frac{43}{512}, \delta = \frac{8733}{8192}$	$b_3 = \frac{1739}{32768}, a_3 = \frac{9}{64}, \delta = \frac{34939}{32768}$
$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{1429}{4096}, \delta = \frac{279481}{262144}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{275}{1024}, \delta = \frac{279443}{262144}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{275}{1024}, \delta = \frac{34933}{32768}$
$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{313}{2048}, \delta = \frac{2175}{2048}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{313}{2048}, \delta = \frac{17871791}{16777216}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{313}{2048}, \delta = \frac{17885025}{16777216}$
$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{207}{2048}, \delta = \frac{8565}{8192}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{207}{2048}, \delta = \frac{4575160993}{4294967296}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{195}{2048}, \delta = \frac{542193}{524288}$
$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{195}{2048}, \delta = \frac{285947481}{268435456}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = -\frac{195}{2048}, \delta = \frac{4578528035}{4294967296}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{71}{65536}, \delta = \frac{271227}{262144}$
$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{71}{65536}, \delta = \frac{9150319465}{8589934592}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{71}{65536}, \delta = \frac{9157056213}{8589934592}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{115}{16384}, \delta = \frac{1071}{1024}$
$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{115}{16384}, \delta = \frac{4575161051}{4294967296}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{477}{8192}, \delta = \frac{2175}{2048}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{477}{8192}, \delta = \frac{17871791}{16777216}$
$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{477}{8192}, \delta = \frac{17885025}{16777216}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{89}{512}, \delta = \frac{279443}{262144}$	$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{89}{512}, \delta = \frac{558927}{524288}$
$b_3 = \frac{51843751309}{549755813888}, a_3 = \frac{65}{256}, \delta = \frac{279481}{262144}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = -\frac{1429}{4096}, \delta = \frac{279481}{262144}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = -\frac{275}{1024}, \delta = \frac{279443}{262144}$
$b_3 = \frac{25922055879}{274877906944}, a_3 = -\frac{275}{1024}, \delta = \frac{34933}{32768}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = -\frac{5}{32}, \delta = \frac{2175}{2048}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = -\frac{5}{32}, \delta = \frac{4467949}{4194304}$
$b_3 = \frac{25922055879}{274877906944}, a_3 = -\frac{5}{32}, \delta = \frac{8942519}{8388608}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = -\frac{53}{512}, \delta = \frac{2147}{2048}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = -\frac{53}{512}, \delta = \frac{142973799}{134217728}$
$b_3 = \frac{25922055879}{274877906944}, a_3 = \frac{77}{8192}, \delta = \frac{8589}{8192}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = \frac{77}{8192}, \delta = \frac{285947601}{268435456}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = \frac{253}{4096}, \delta = \frac{2175}{2048}$
$b_3 = \frac{25922055879}{274877906944}, a_3 = \frac{253}{4096}, \delta = \frac{17871797}{16777216}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = \frac{253}{4096}, \delta = \frac{17885037}{16777216}$	$b_3 = \frac{25922055879}{274877906944}, a_3 = \frac{89}{512}, \delta = \frac{279443}{262144}$

Experiments

Y. Tian, B. Huang. *Local stability and Hopf bifurcations analysis of the Muthuswamy–Chua–Ginoux system*, Nonlinear Dynamics, 109, 1135-1151, 2022

Muthuswamy–Chua–Ginoux (MCG) circuit:

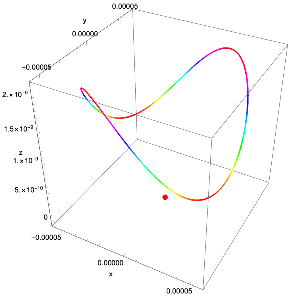
$$\begin{aligned} \dot{x} &= k_1 y, \\ \dot{y} &= k_2 \left(x + f(y) + R(z)\, y \right), \\ \dot{z} &= R(z)\, y^2 - \epsilon z, \end{aligned} \tag{19}$$

where $k_1 = 1/\alpha$, $k_2 = -1/\eta$, $f(y) = ay + by^3$ and $R(z) = cz^2 + dz + s$. We consider the vector $(a, b, c, d, s, k_1, k_2, \epsilon)$ given by

$$a \leftarrow a + \sum_{i=1}^{3} \varepsilon^i a_i, \quad b \leftarrow b + \sum_{i=1}^{3} \varepsilon^i b_i, \quad c \leftarrow c + \sum_{i=1}^{3} \varepsilon^i c_i,$$

$$d \leftarrow d + \sum_{i=1}^{3} \varepsilon^i d_i, \quad s \leftarrow -a, \quad k_1 \leftarrow k_1 + \sum_{i=1}^{3} \varepsilon^i \ell_i,$$

$$k_2 \leftarrow -\frac{\omega^2}{k_1} + \sum_{i=1}^{3} \varepsilon^i m_i, \quad \epsilon \leftarrow \sum_{i=1}^{3} \varepsilon^i \epsilon_i.$$



y
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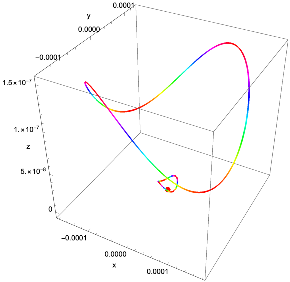
0.00000
−0.00005
z
2. × 10^−9
1.5 × 10^−9
1. × 10^−9
5. × 10^−10
0
x
−0.00005
0.00000
0.00005



y
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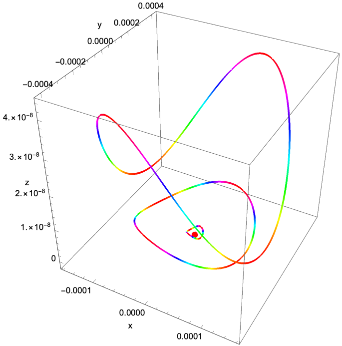
0.0000
−0.0001
z
1.5 × 10^−7
1. × 10^−7
5. × 10^−8
0
x
−0.0001
0.0000
0.0001



y
0.0004
0.0002
0.0000
−0.0002
−0.0004
z
4. × 10^−8
3. × 10^−8
2. × 10^−8
1. × 10^−8
0
x
−0.0001
0.0000
0.0001

Future works

- Generalizations to high-dimensional discontinuous differential systems
- Complexity analysis
- Algorithmic approach to write the linear part of a given system at the origin in its real Jordan normal form or avoid such a step!

  $\Longrightarrow$ Suggestion welcome!
- Provide lower and upper bounds for the number $H_k(n, N)$ $(k \geq 2)$
- Application to reaction networks

  ...

B. Huang. *Using symbolic computation to analyze zero-Hopf bifurcations of polynomial differential systems*, Proceedings of ISSAC 2023, pp. 307–314, 2023

B. Huang, D. Wang. *Zero-Hopf bifurcation of limit cycles in certain differential systems*, arXiv: 2205.14450

Thanks for your attention!