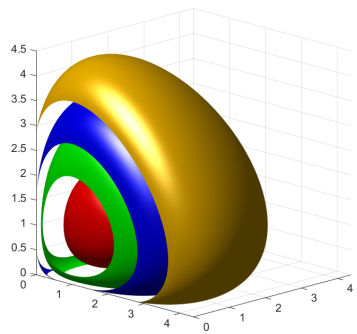
A FLUX-BASED APPROACH FOR ANALYZING THE DISGUISED TORIC LOCUS OF REACTION NETWORKS

Balázs Boros

UNIVERSITY OF SZEGED

Formal Reaction Kinetics and Related Questions
March 3, 2026



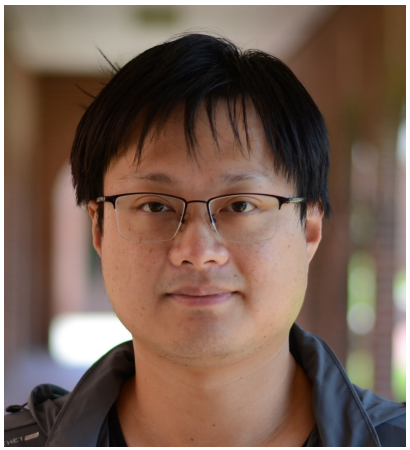
Joint work with

Gheorghe Craciun

Madison, Wisconsin

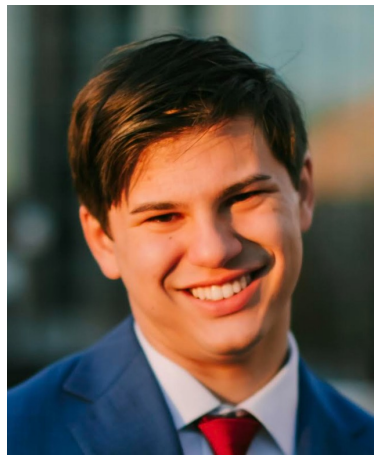
Oskar Henriksson

Dresden, Germany



Jiaxin Jin

Lafayette, Louisiana



Diego Rojas La Luz

Madison, Wisconsin

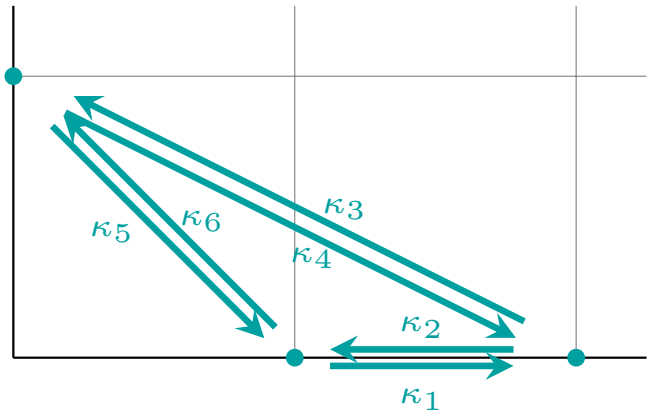
References

- BB–Craciun–Henriksson–Jin–Rojas
  A flux-based approach for analyzing the disguised toric locus of reaction networks
  https://arxiv.org/pdf/2510.03621

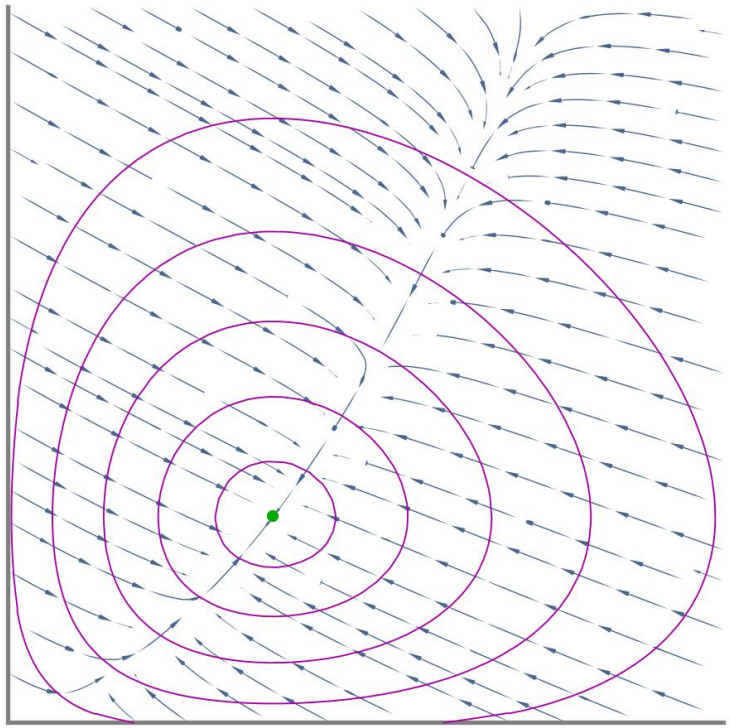
- https://github.com/balazsboros/reaction_networks

- https://sites.google.com/view/balazsboros/talks

TRIANGLE



$$\dot{x} = \kappa_1 x - \kappa_2 x^2 - 2\kappa_3 x^2 + 2\kappa_4 y + \kappa_5 y - \kappa_6 x^2$$
$$\dot{y} = \kappa_3 x^2 - \kappa_4 y - \kappa_5 y + \kappa_6 x^2$$



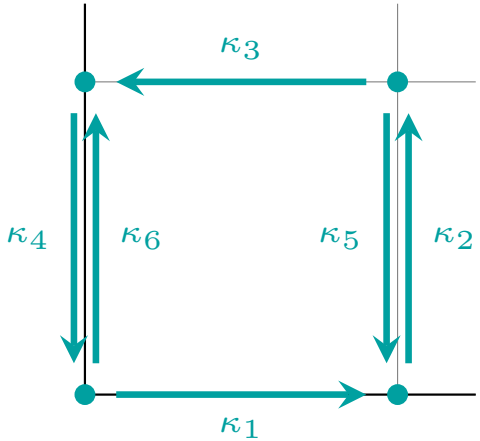
- Horn–Jackson function:

$$V(x,y) = x(\log \tfrac{x}{x^*} - 1) + y(\log \tfrac{y}{y^*} - 1)$$

- Deficiency-Zero Theorem:

$$\delta = 0 \Longrightarrow V \text{ is a global Lyapunov function for all } \boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$$

PARTLY REVERSIBLE SQUARE



$$\dot{x} = \kappa_1 - \kappa_3 xy$$
$$\dot{y} = \kappa_2 x - \kappa_5 xy - \kappa_4 y + \kappa_6$$

- Deficiency-One Theorem:
  $\delta = 1 \Longrightarrow$ there exists a unique positive equilibrium for all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$

- The positive equilibrium (x^*, y^*) is vertex-balanced (VB) if
  $$\kappa_2 x^* = \kappa_1 + \kappa_5 x^* y^*, \qquad \kappa_4 y^* = \kappa_3 x^* y^* + \kappa_6,$$
  $$\kappa_2 x^* = \kappa_3 x^* y^* + \kappa_5 x^* y^*, \qquad \kappa_4 y^* = \kappa_1 + \kappa_6.$$

- Matrix-Tree Theorem:
  VB if and only if $\frac{\kappa_2 \kappa_4}{\kappa_1 \kappa_3} = (1 + \frac{\kappa_5}{\kappa_3})(1 + \frac{\kappa_6}{\kappa_1})$

- Other thms: the pos. equil. is globally stable for all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$

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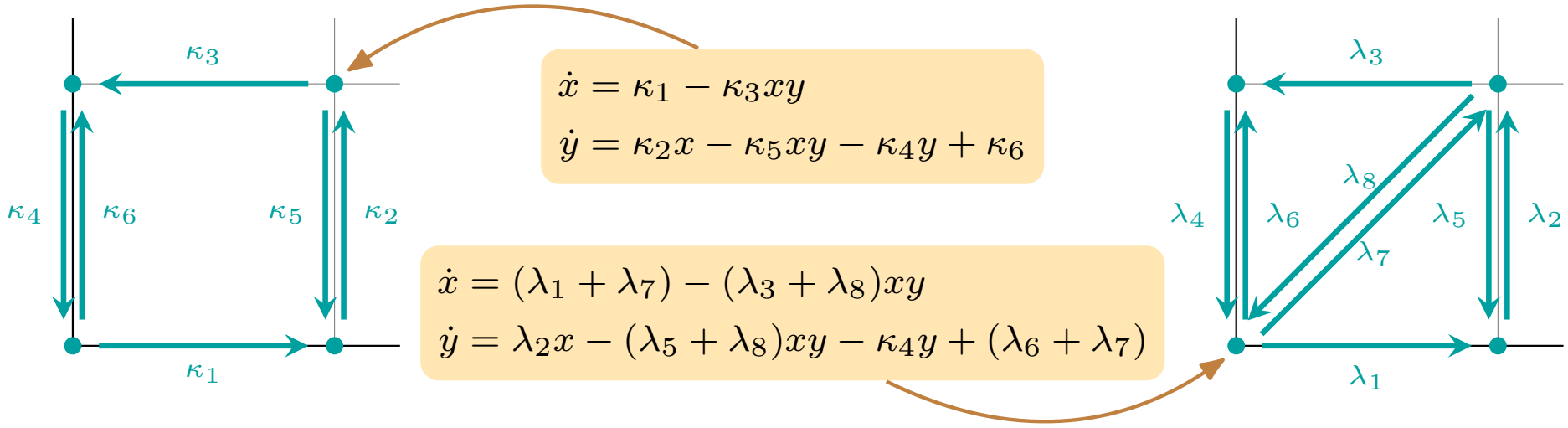

Find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which there exist $\boldsymbol{\lambda} \in \mathbb{R}^8_{\geq 0}$ and $(x, y) \in \mathbb{R}^2_{>0}$ s.t.

$$\begin{aligned} \kappa_1 &= \lambda_1 + \lambda_7, & \kappa_2 &= \lambda_2, & \kappa_3 &= \lambda_3 + \lambda_8, \\ \kappa_6 &= \lambda_6 + \lambda_7, & \kappa_4 &= \lambda_4, & \kappa_5 &= \lambda_5 + \lambda_8; \end{aligned} \tag{DE}$$

$$\begin{aligned} \lambda_4 y + \lambda_8 xy &= \lambda_1 + \lambda_6 + \lambda_7, & \lambda_1 + \lambda_5 xy &= \lambda_2 x, \\ \lambda_2 x + \lambda_7 &= (\lambda_3 + \lambda_5 + \lambda_8)xy, & \lambda_3 xy + \lambda_6 &= \lambda_4 y. \end{aligned} \tag{VB}$$

Solution: $(1 - \frac{\kappa_5}{\kappa_3})(1 - \frac{\kappa_6}{\kappa_1}) \leq \frac{\kappa_2 \kappa_4}{\kappa_1 \kappa_3} \leq (1 + \frac{\kappa_5}{\kappa_3})(1 + \frac{\kappa_6}{\kappa_1})$

Uniform distr. on $\{\boldsymbol{\kappa} \in \mathbb{R}^6_{>0} \mid \sum_{i=1}^6 \kappa_i = 1\} \longrightarrow$ 58% (disguised VB)

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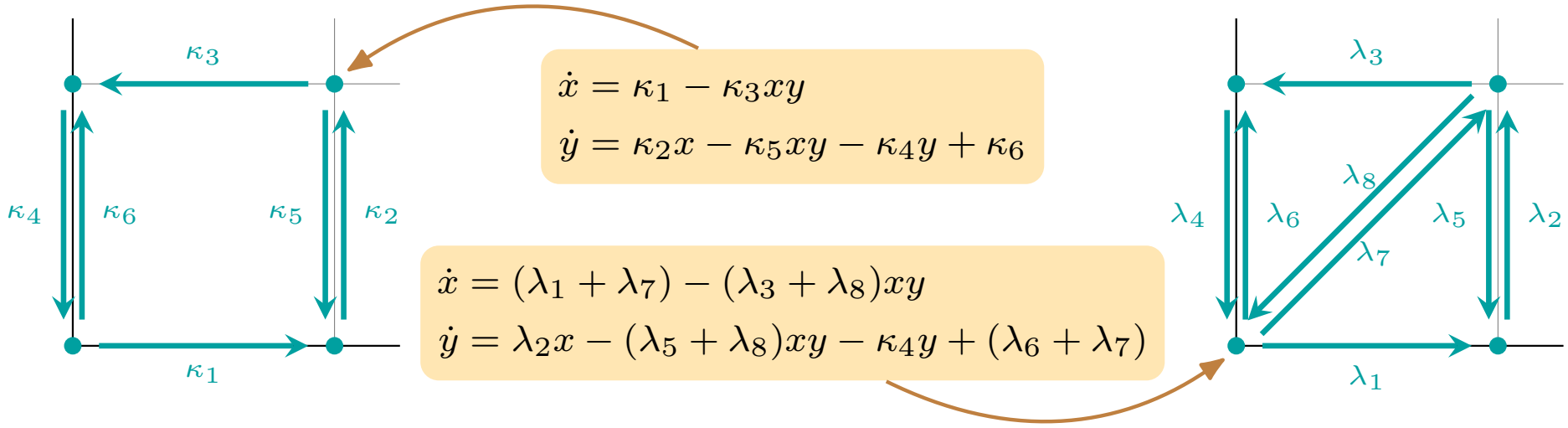

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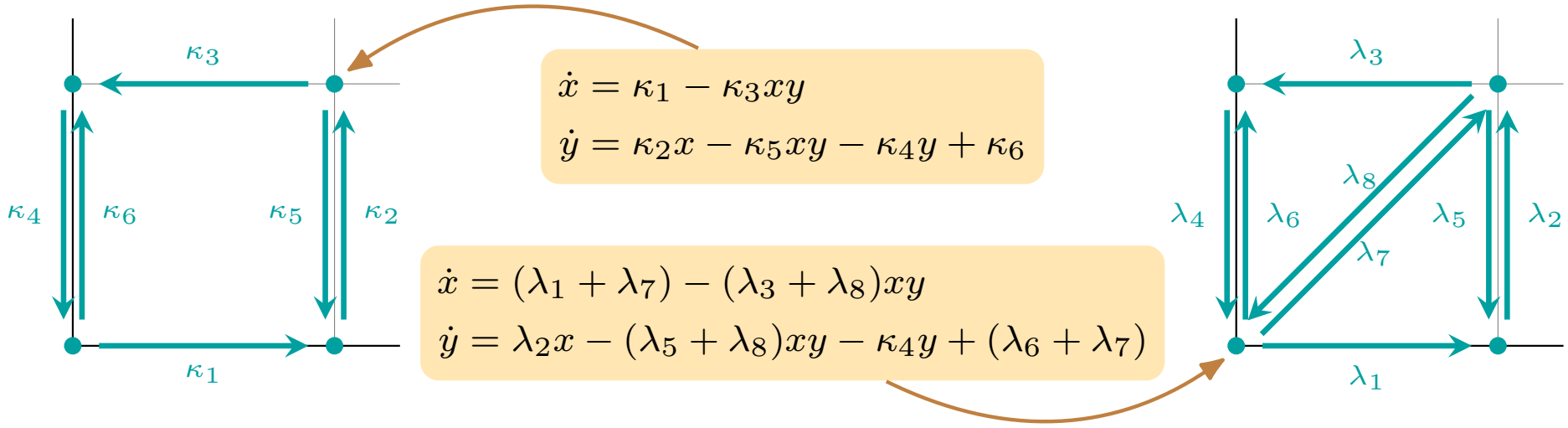

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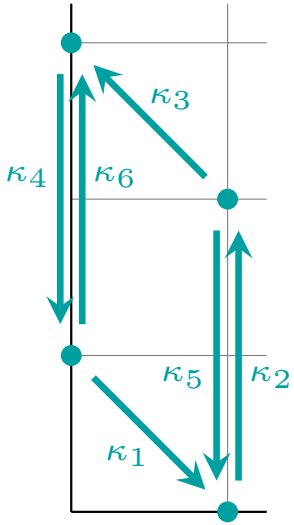
PARTLY REVERSIBLE PARALLELOGRAM



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- Matrix-Tree Theorem:

  VB if and only if $\frac{\kappa_2 \kappa_4}{\kappa_1 \kappa_3} = (1 + \frac{\kappa_5}{\kappa_3})(1 + \frac{\kappa_6}{\kappa_1})$

- stable pos. equil. and stable limit cycle coexist for some $\boldsymbol{\kappa}$

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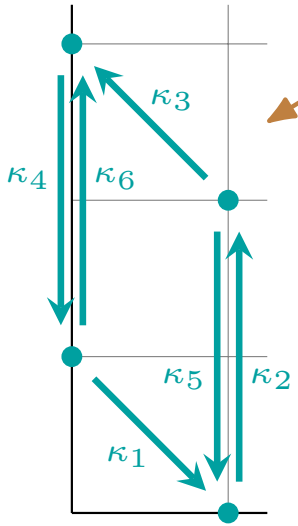

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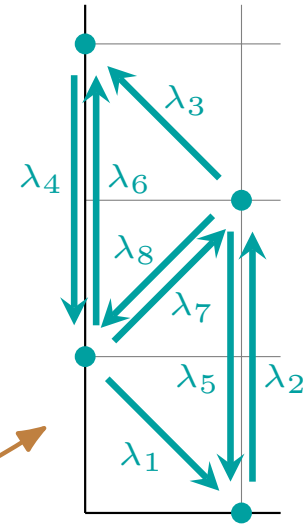

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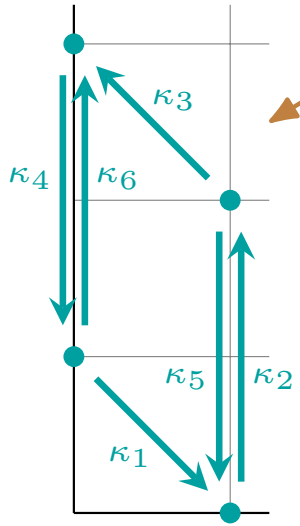

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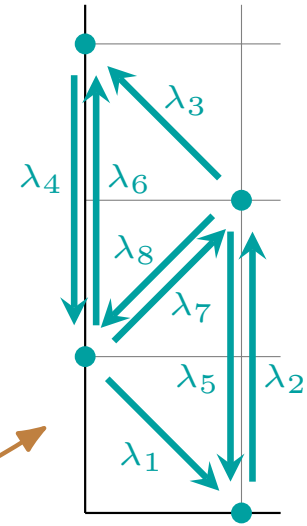

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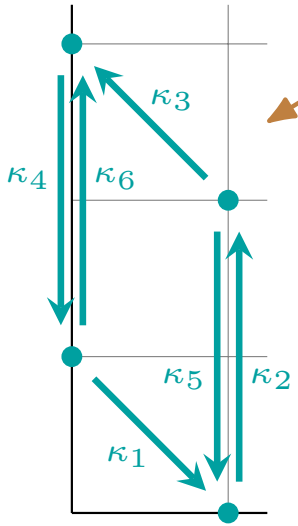

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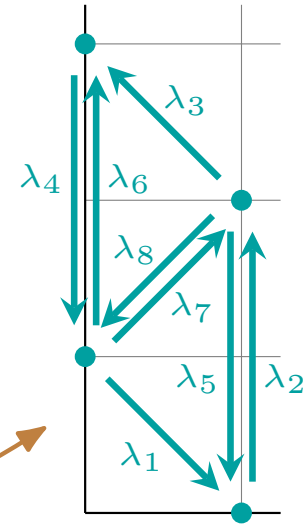

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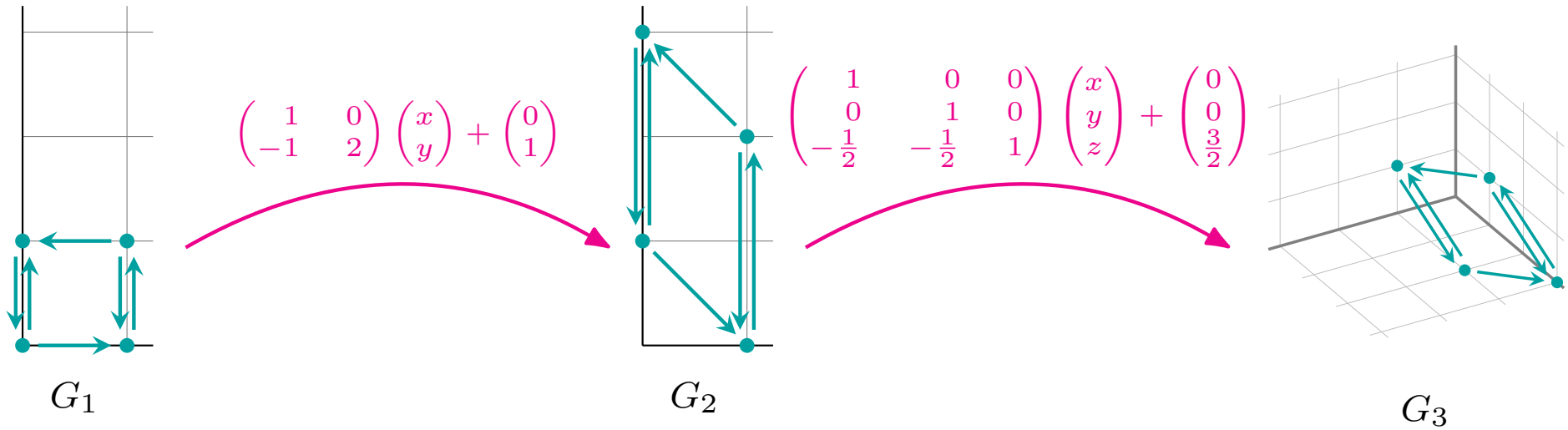
AFFINE TRANSFORMATIONS



THEOREM (HAQUE–SATRIANO–SOREA–YU 2023)

$\boldsymbol{A}\colon \mathbb{R}^n \to \mathbb{R}^n$ invertible affine transformation $\Longrightarrow \mathcal{K}^{\mathrm{dt}}(\boldsymbol{A}(G)) = \mathcal{K}^{\mathrm{dt}}(G)$

disguised toric loci:

$$\mathcal{K}^{\mathrm{dt}}(G_1) = \mathcal{K}^{\mathrm{dt}}(G_2) = \mathcal{K}^{\mathrm{dt}}(G_3) =$$
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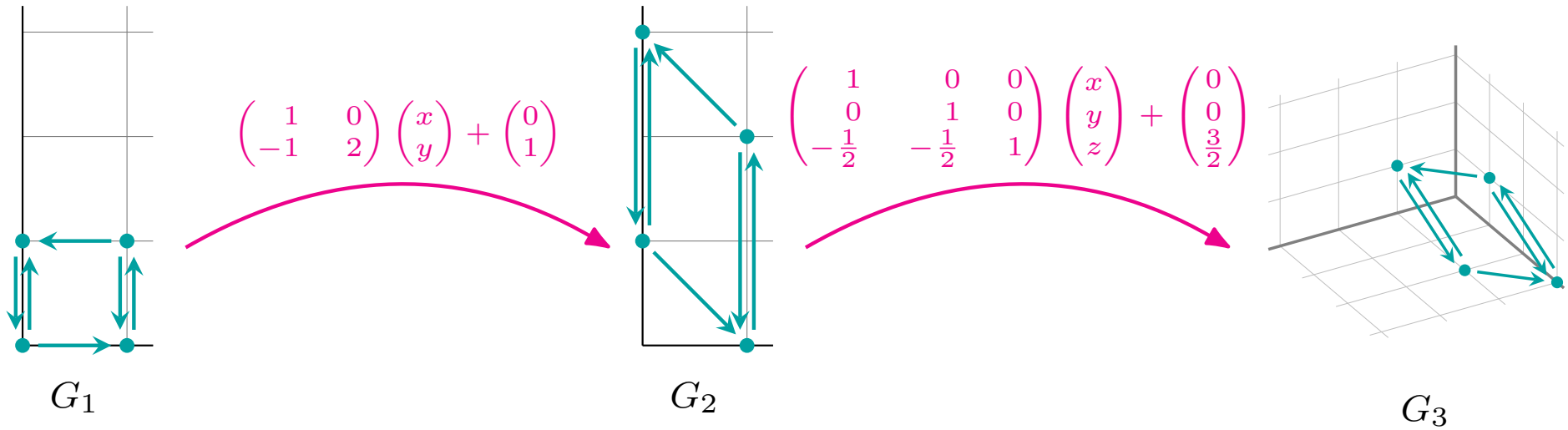

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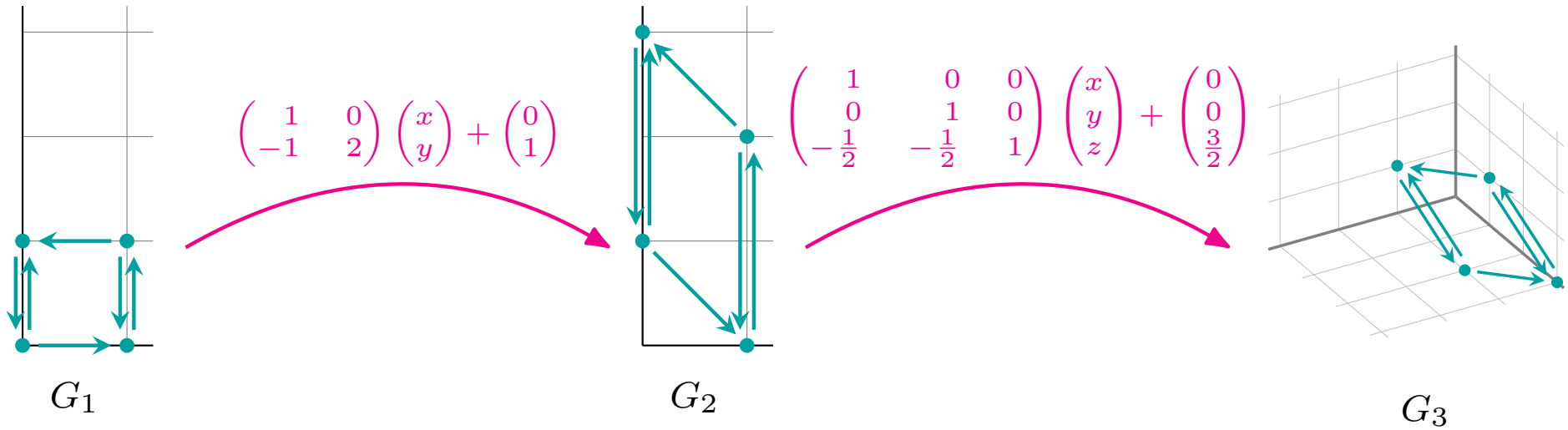

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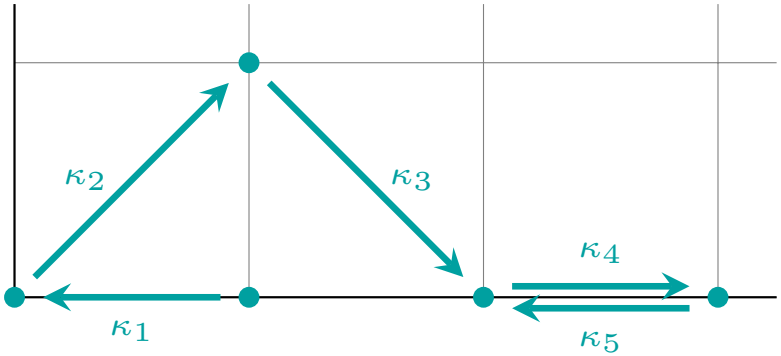
A BISTABLE NETWORK



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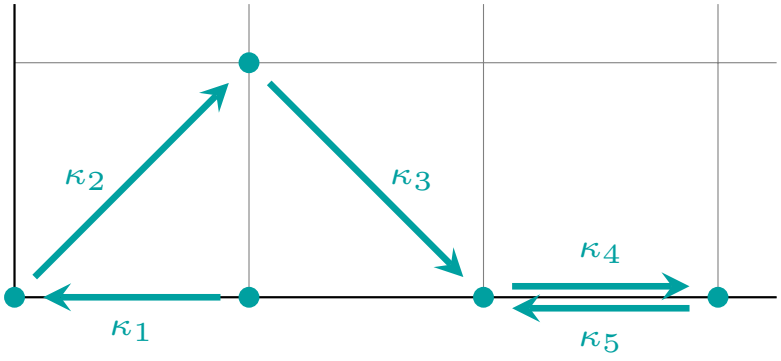
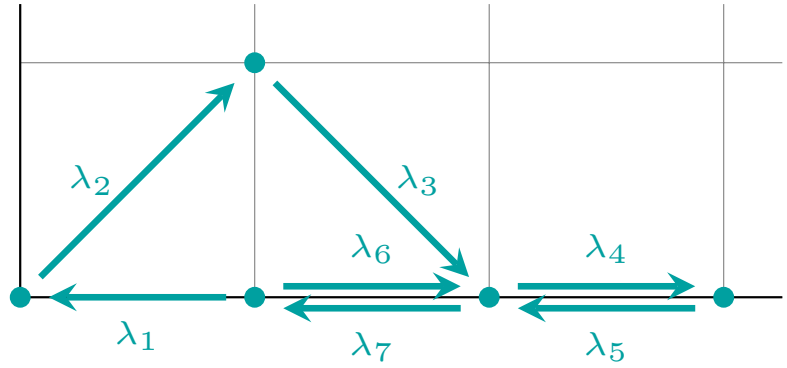
3 pos. equil. (2 sinks, 1 saddle)
homoclinic orbit
limit cycle

A BISTABLE NETWORK





$$\dot{x} = -\kappa_1 x + \kappa_2 + \kappa_3 xy + \kappa_4 x^2 - \kappa_5 x^3$$
$$\dot{y} = \kappa_2 - \kappa_3 xy$$

$$\dot{x} = (\lambda_6 - \lambda_1)x + \lambda_2 + \lambda_3 xy + (\lambda_4 - \lambda_7)x^2 - \lambda_5 x^3$$
$$\dot{y} = \lambda_2 - \lambda_3 xy$$

3 pos. equil. (2 sinks, 1 saddle)
homoclinic orbit
limit cycle

$$\lambda_1 = \frac{\kappa_2}{x^*} \quad \lambda_2 = \kappa_2 \quad \lambda_3 = \kappa_3 \quad \lambda_4 = \kappa_5 x^*$$
$$\lambda_5 = \kappa_5 \quad \lambda_6 = \frac{\kappa_2}{x^*} - \kappa_1 \quad \lambda_7 = \kappa_5 x^* - \kappa_4$$

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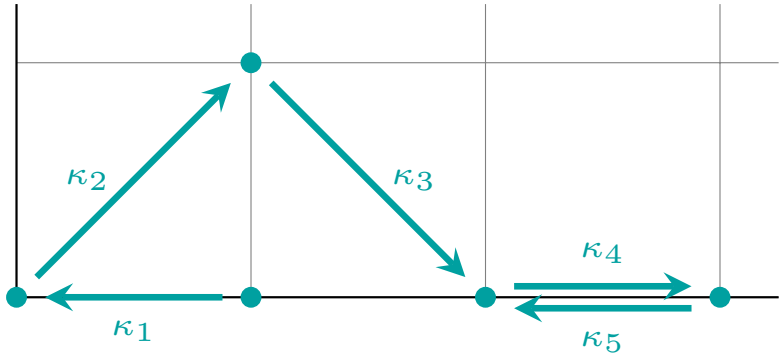

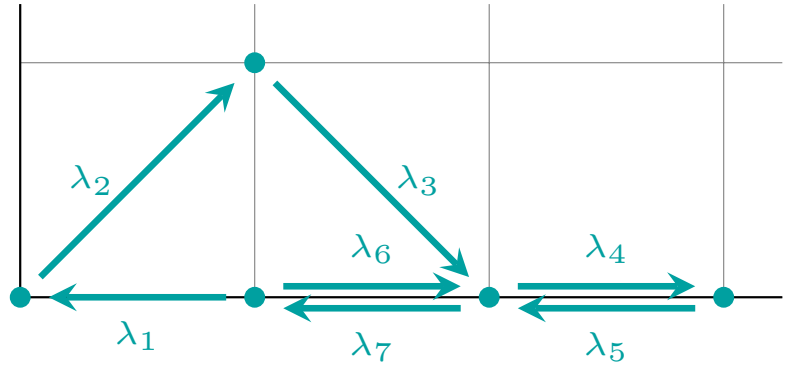

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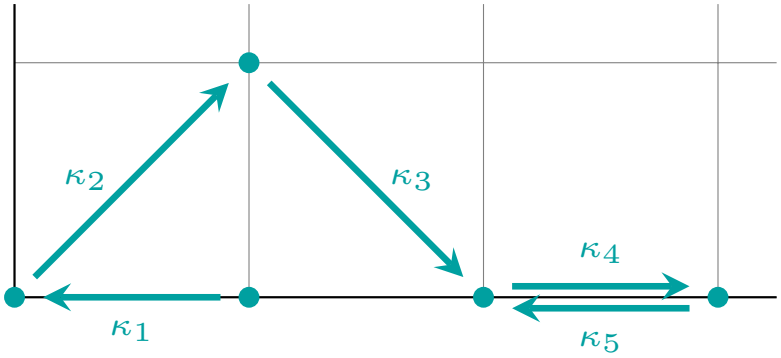

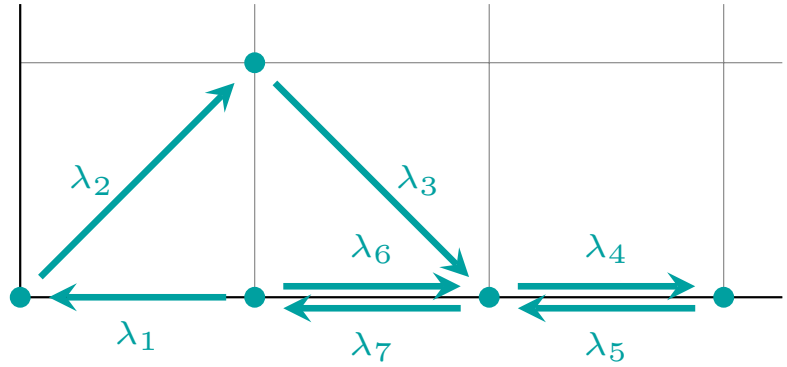

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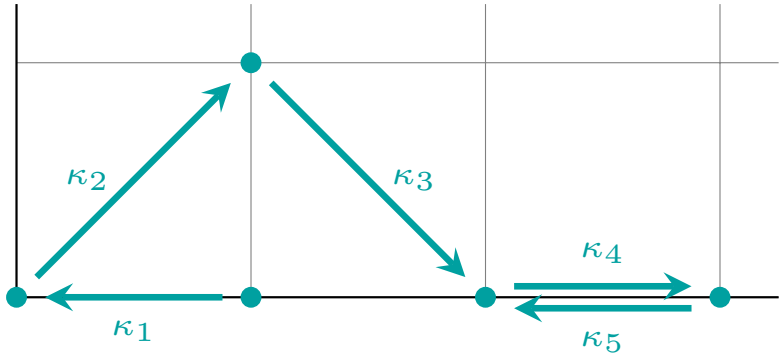

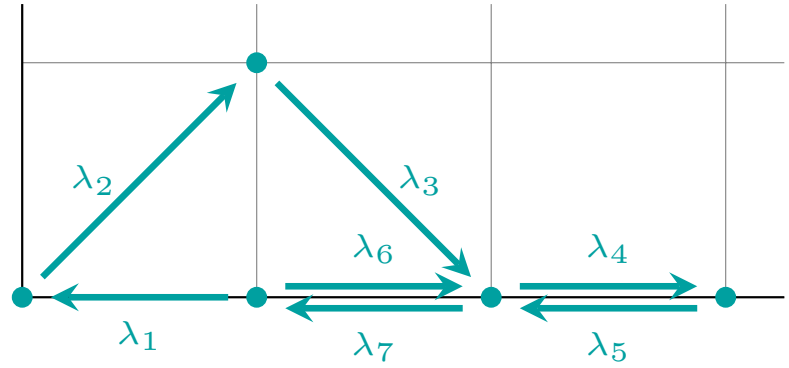

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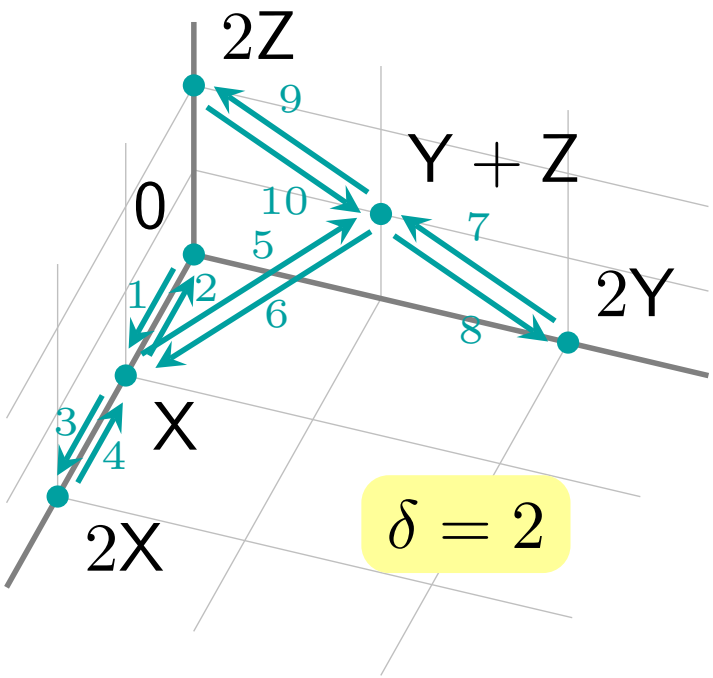
- disguised VB if and only if $\frac{\kappa_4}{\kappa_5} + \frac{\kappa_1^2}{\kappa_2 \kappa_5} \leq \frac{\kappa_2}{\kappa_1}$
- uniform distr. on $\{\boldsymbol{\kappa} \in \mathbb{R}^5_{>0} \mid \sum_{i=1}^5 \kappa_i = 1\} \rightarrow$ 35% (disguised VB)

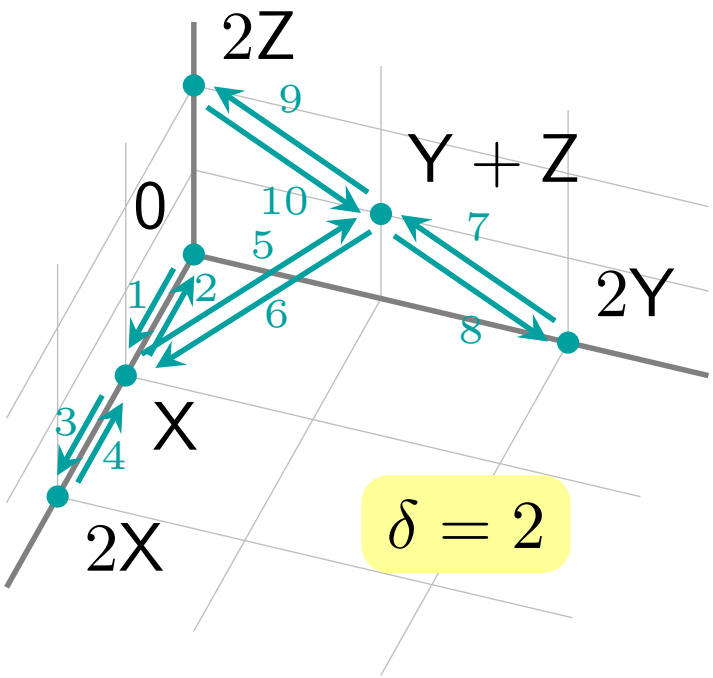
JOHNSTON–SIEGEL–SZEDERKÉNYI TETRAHEDRON



$$\dot{x} = \kappa_1 + (\kappa_3 - \kappa_2 - \kappa_5)x - \kappa_4 x^2 + \kappa_6 yz$$
$$\dot{y} = \kappa_5 x - \kappa_7 y^2 + (\kappa_8 - \kappa_9 - \kappa_6)yz + \kappa_{10} z^2$$
$$\dot{z} = \kappa_5 x + \kappa_7 y^2 - (\kappa_8 - \kappa_9 + \kappa_6)yz - \kappa_{10} z^2$$

vertex-balanced $\iff \frac{\kappa_1}{\kappa_2} = \frac{\kappa_3}{\kappa_4}$ & $\frac{\kappa_7}{\kappa_8} = \frac{\kappa_9}{\kappa_{10}}$

JOHNSTON–SIEGEL–SZEDERKÉNYI TETRAHEDRON



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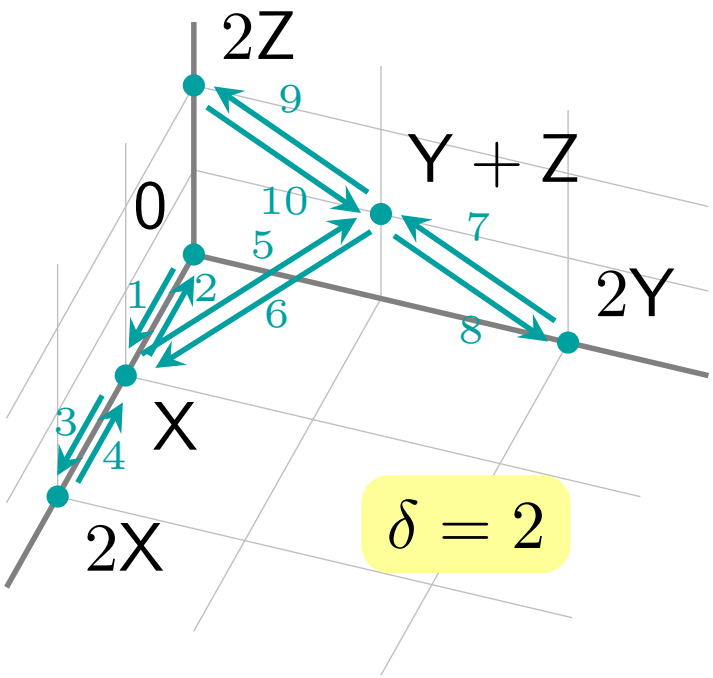
$\lambda_1 = \kappa_1$ $\quad \lambda_3 = \kappa_4 x^*$ $\quad \lambda_5 = \kappa_5$ $\quad \lambda_7 = \kappa_7$ $\quad \lambda_9 = \kappa_{10}\frac{z^*}{y^*}$

$\lambda_2 = \kappa_1 \frac{1}{x^*}$ $\quad \lambda_4 = \kappa_4$ $\quad \lambda_6 = \kappa_5 \frac{x^*}{y^* z^*}$ $\quad \lambda_8 = \kappa_7 \frac{y^*}{z^*}$ $\quad \lambda_{10} = \kappa_{10}$

JOHNSTON–SIEGEL–SZEDERKÉNYI TETRAHEDRON



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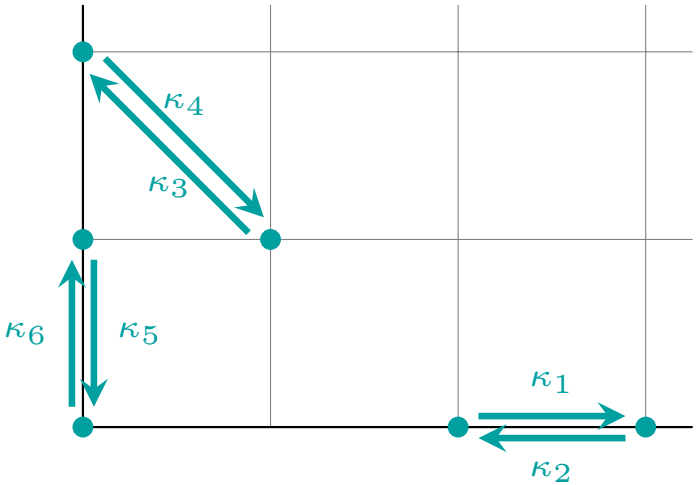
$\lambda_2 = \kappa_1 \frac{1}{x^*}$ $\quad \lambda_4 = \kappa_4$ $\quad \lambda_6 = \kappa_5 \frac{x^*}{y^* z^*}$ $\quad \lambda_8 = \kappa_7 \frac{y^*}{z^*}$ $\quad \lambda_{10} = \kappa_{10}$

- the two ODEs are identical for all $\boldsymbol{\kappa} \in \mathbb{R}^{10}_{>0}$
- 2nd ODE is vertex-balanced
- $\implies$ the Horn–Jackson function is a global Lyapunov function
- $\implies$ global asymptotic stability for all $\boldsymbol{\kappa} \in \mathbb{R}^{10}_{>0}$

Reversible Lotka–Volterra autocatalator (LVA)



$$\dot{x} = \kappa_1 x^2 - \kappa_2 x^3 - \kappa_3 xy + \kappa_4 y^2$$
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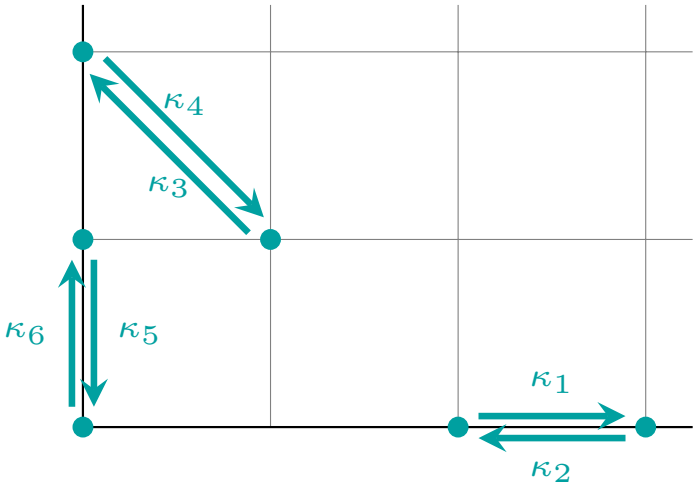
5 positive equilibria (3 sinks, 2 saddles)
homoclinic orbit
limit cycle

- Simon 1992: unique globally stable pos. equil. if $\frac{\kappa_2\kappa_4\kappa_6}{\kappa_1\kappa_3\kappa_5} > 1$
- vertex-balanced $\iff \frac{\kappa_2\kappa_4\kappa_6}{\kappa_1\kappa_3\kappa_5} = 1$
- disguised vertex-balanced $\iff \frac{\kappa_2\kappa_4\kappa_6}{\kappa_1\kappa_3\kappa_5} \geq 1$

Reversible Lotka–Volterra autocatalator (LVA)



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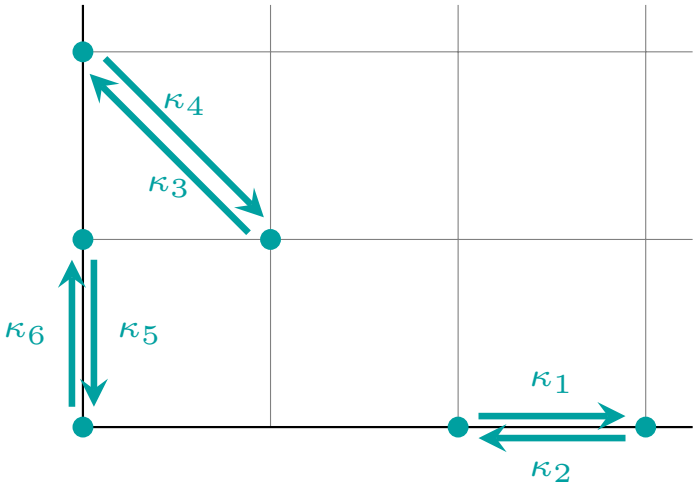

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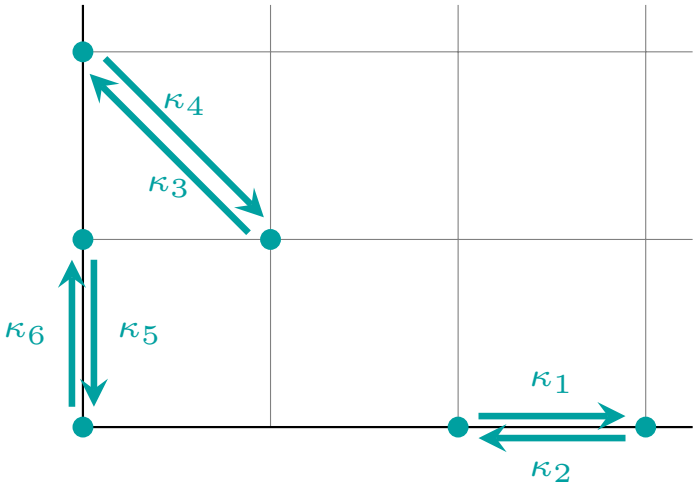

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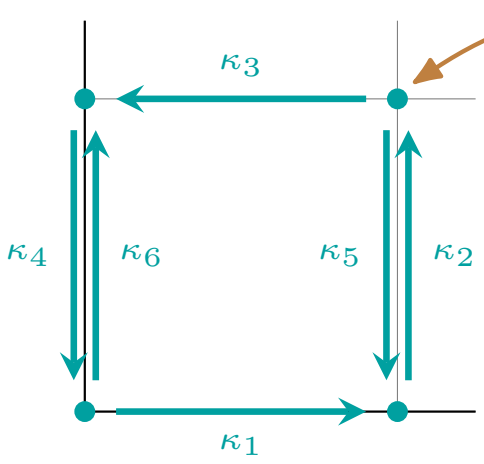
Computational aspects (partly reversible square)



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$$\dot{x} = (\lambda_1 + \lambda_7) - (\lambda_3 + \lambda_8)xy$$
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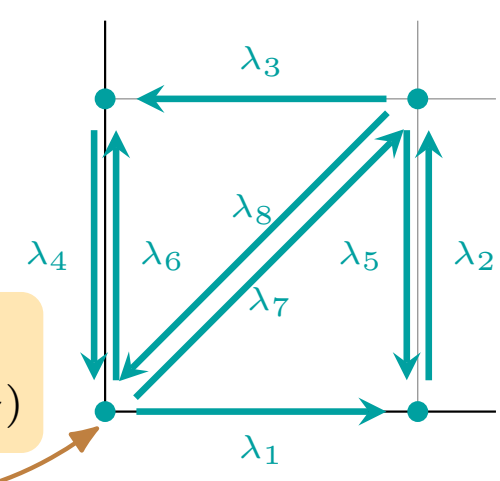

Find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which there exist

$\boldsymbol{\lambda} \in \mathbb{R}^8_{\geq 0}$ and $(x, y) \in \mathbb{R}^2_{>0}$ such that

$$\kappa_1 = \lambda_1 + \lambda_7, \quad \kappa_2 = \lambda_2, \quad \kappa_3 = \lambda_3 + \lambda_8,$$
$$\kappa_6 = \lambda_6 + \lambda_7, \quad \kappa_4 = \lambda_4, \quad \kappa_5 = \lambda_5 + \lambda_8;$$

$$\lambda_4 y + \lambda_8 xy = \lambda_1 + \lambda_6 + \lambda_7, \qquad \lambda_1 + \lambda_5 xy = \lambda_2 x,$$
$$\lambda_2 x + \lambda_7 = (\lambda_3 + \lambda_5 + \lambda_8)xy, \quad \lambda_3 xy + \lambda_6 = \lambda_4 y.$$

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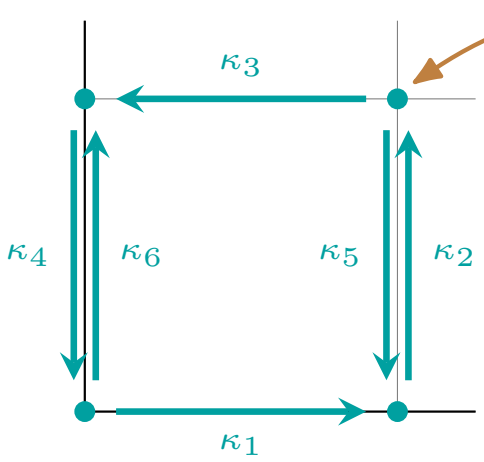

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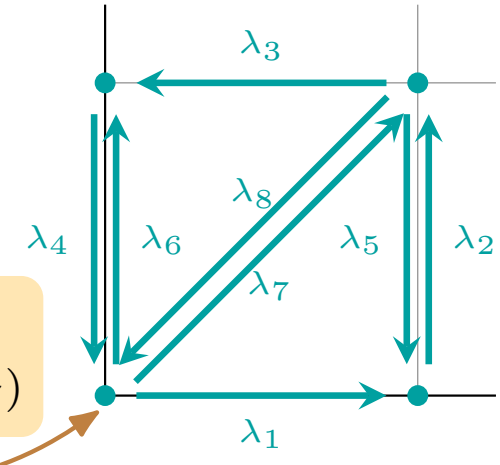

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$$\kappa_6 = \lambda_6 + \lambda_7, \quad \kappa_4 = \lambda_4, \quad \kappa_5 = \lambda_5 + \lambda_8;$$

$$\lambda_4 y + \lambda_8 xy = \lambda_1 + \lambda_6 + \lambda_7, \qquad \lambda_1 + \lambda_5 xy = \lambda_2 x,$$
$$\lambda_2 x + \lambda_7 = (\lambda_3 + \lambda_5 + \lambda_8)xy, \quad \lambda_3 xy + \lambda_6 = \lambda_4 y.$$

Find all $\boldsymbol{\beta} \in \mathbb{R}^6_{>0}$ for which there exists a $\boldsymbol{\gamma} \in \mathbb{R}^8_{\geq 0}$ such that

$$\beta_1 = \gamma_1 + \gamma_7, \quad \beta_2 = \gamma_2, \quad \beta_3 = \gamma_3 + \gamma_8,$$
$$\beta_6 = \gamma_6 + \gamma_7, \quad \beta_4 = \gamma_4, \quad \beta_5 = \gamma_5 + \gamma_8;$$

$$\gamma_4 + \gamma_8 = \gamma_1 + \gamma_6 + \gamma_7, \quad \gamma_1 + \gamma_5 = \gamma_2,$$
$$\gamma_2 + \gamma_7 = \gamma_3 + \gamma_5 + \gamma_8, \quad \gamma_3 + \gamma_6 = \gamma_4.$$

$$\beta_1 = \kappa_1 \quad \beta_2 = \kappa_2 x \quad \beta_3 = \kappa_3 xy \quad \beta_4 = \kappa_4 y \quad \beta_5 = \kappa_5 xy \quad \beta_6 = \kappa_6$$
$$\gamma_1 = \lambda_1 \quad \gamma_2 = \lambda_2 x \quad \gamma_3 = \lambda_3 xy \quad \gamma_4 = \lambda_4 y \quad \gamma_5 = \lambda_5 xy \quad \gamma_6 = \lambda_6 \quad \gamma_7 = \lambda_7 \quad \gamma_8 = \lambda_8 xy$$

COMPUTATIONAL ASPECTS (PARTLY REVERSIBLE SQUARE)

Find all $\boldsymbol{\beta} \in \mathbb{R}^6_{>0}$ for which there exists a $\boldsymbol{\gamma} \in \mathbb{R}^8_{\geq 0}$ such that

$$\beta_1 = \gamma_1 + \gamma_7, \quad \beta_2 = \gamma_2, \quad \beta_3 = \gamma_3 + \gamma_8, \qquad \gamma_4 + \gamma_8 = \gamma_1 + \gamma_6 + \gamma_7, \quad \gamma_1 + \gamma_5 = \gamma_2,$$
$$\beta_6 = \gamma_6 + \gamma_7, \quad \beta_4 = \gamma_4, \quad \beta_5 = \gamma_5 + \gamma_8; \qquad \gamma_2 + \gamma_7 = \gamma_3 + \gamma_5 + \gamma_8, \quad \gamma_3 + \gamma_6 = \gamma_4.$$

- This is the positive part of a *polyhedral cone* in $\mathbb{R}^6_{\geq 0}$!
- Call it the *disguised toric flux cone*:

$$\{\boldsymbol{\beta} \in \mathbb{R}^6_{>0} \mid \beta_1 = \beta_3,\ \beta_2 - \beta_5 = \beta_4 - \beta_6,\ |\beta_4 - \beta_6| \leq \beta_1 \leq \beta_2 + \beta_6\}$$

- Find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which there exists an $(x, y) \in \mathbb{R}^2_{>0}$ such that

$$\kappa_1 = \kappa_3 xy, \quad \kappa_2 x - \kappa_5 xy = \kappa_4 y - \kappa_6, \quad |\kappa_4 y - \kappa_6| \leq \kappa_1 \leq \kappa_2 x + \kappa_6.$$

- Equivalently, find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which a pos. equil. (x, y) exists w/

$$x \geq \tfrac{\kappa_1 - \kappa_6}{\kappa_2} \quad \text{and} \quad \tfrac{\kappa_6 - \kappa_1}{\kappa_4} \leq y \leq \tfrac{\kappa_1 + \kappa_6}{\kappa_4}.$$

- Disguised vertex-balanced if and only if

$$(1 - \tfrac{\kappa_5}{\kappa_3})(1 - \tfrac{\kappa_6}{\kappa_1}) \leq \tfrac{\kappa_2 \kappa_4}{\kappa_1 \kappa_3} \leq (1 + \tfrac{\kappa_5}{\kappa_3})(1 + \tfrac{\kappa_6}{\kappa_1})$$

COMPUTATIONAL ASPECTS (PARTLY REVERSIBLE SQUARE)

Find all $\boldsymbol{\beta} \in \mathbb{R}^6_{>0}$ for which there exists a $\boldsymbol{\gamma} \in \mathbb{R}^8_{\geq 0}$ such that

$$\beta_1 = \gamma_1 + \gamma_7, \quad \beta_2 = \gamma_2, \quad \beta_3 = \gamma_3 + \gamma_8, \qquad \gamma_4 + \gamma_8 = \gamma_1 + \gamma_6 + \gamma_7, \quad \gamma_1 + \gamma_5 = \gamma_2,$$
$$\beta_6 = \gamma_6 + \gamma_7, \quad \beta_4 = \gamma_4, \quad \beta_5 = \gamma_5 + \gamma_8; \qquad \gamma_2 + \gamma_7 = \gamma_3 + \gamma_5 + \gamma_8, \quad \gamma_3 + \gamma_6 = \gamma_4.$$

- This is the positive part of a *polyhedral cone* in $\mathbb{R}^6_{\geq 0}$!
- Call it the *disguised toric flux cone*:

$$\{\boldsymbol{\beta} \in \mathbb{R}^6_{>0} \mid \beta_1 = \beta_3,\ \beta_2 - \beta_5 = \beta_4 - \beta_6,\ |\beta_4 - \beta_6| \leq \beta_1 \leq \beta_2 + \beta_6\}$$

- Find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which there exists an $(x, y) \in \mathbb{R}^2_{>0}$ such that

$$\kappa_1 = \kappa_3 xy, \quad \kappa_2 x - \kappa_5 xy = \kappa_4 y - \kappa_6, \quad |\kappa_4 y - \kappa_6| \leq \kappa_1 \leq \kappa_2 x + \kappa_6.$$

- Equivalently, find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which a pos. equil. (x, y) exists w/

$$x \geq \tfrac{\kappa_1 - \kappa_6}{\kappa_2} \quad \text{and} \quad \tfrac{\kappa_6 - \kappa_1}{\kappa_4} \leq y \leq \tfrac{\kappa_1 + \kappa_6}{\kappa_4}.$$

- Disguised vertex-balanced if and only if

$$(1 - \tfrac{\kappa_5}{\kappa_3})(1 - \tfrac{\kappa_6}{\kappa_1}) \leq \tfrac{\kappa_2 \kappa_4}{\kappa_1 \kappa_3} \leq (1 + \tfrac{\kappa_5}{\kappa_3})(1 + \tfrac{\kappa_6}{\kappa_1})$$

COMPUTATIONAL ASPECTS (PARTLY REVERSIBLE SQUARE)

Find all $\boldsymbol{\beta} \in \mathbb{R}^6_{>0}$ for which there exists a $\boldsymbol{\gamma} \in \mathbb{R}^8_{\geq 0}$ such that

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$$\beta_6 = \gamma_6 + \gamma_7, \quad \beta_4 = \gamma_4, \quad \beta_5 = \gamma_5 + \gamma_8; \qquad \gamma_2 + \gamma_7 = \gamma_3 + \gamma_5 + \gamma_8, \quad \gamma_3 + \gamma_6 = \gamma_4.$$

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- Find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which there exists an $(x, y) \in \mathbb{R}^2_{>0}$ such that

$$\kappa_1 = \kappa_3 xy, \quad \kappa_2 x - \kappa_5 xy = \kappa_4 y - \kappa_6, \quad |\kappa_4 y - \kappa_6| \leq \kappa_1 \leq \kappa_2 x + \kappa_6.$$

- Equivalently, find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which a pos. equil. (x, y) exists w/

$$x \geq \tfrac{\kappa_1 - \kappa_6}{\kappa_2} \quad \text{and} \quad \tfrac{\kappa_6 - \kappa_1}{\kappa_4} \leq y \leq \tfrac{\kappa_1 + \kappa_6}{\kappa_4}.$$

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$$(1 - \tfrac{\kappa_5}{\kappa_3})(1 - \tfrac{\kappa_6}{\kappa_1}) \leq \tfrac{\kappa_2 \kappa_4}{\kappa_1 \kappa_3} \leq (1 + \tfrac{\kappa_5}{\kappa_3})(1 + \tfrac{\kappa_6}{\kappa_1})$$

COMPUTATIONAL ASPECTS (PARTLY REVERSIBLE SQUARE)

Find all $\boldsymbol{\beta} \in \mathbb{R}^6_{>0}$ for which there exists a $\boldsymbol{\gamma} \in \mathbb{R}^8_{\geq 0}$ such that

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$$\beta_6 = \gamma_6 + \gamma_7, \quad \beta_4 = \gamma_4, \quad \beta_5 = \gamma_5 + \gamma_8; \qquad \gamma_2 + \gamma_7 = \gamma_3 + \gamma_5 + \gamma_8, \quad \gamma_3 + \gamma_6 = \gamma_4.$$

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$$\{\boldsymbol{\beta} \in \mathbb{R}^6_{>0} \mid \beta_1 = \beta_3,\ \beta_2 - \beta_5 = \beta_4 - \beta_6,\ |\beta_4 - \beta_6| \leq \beta_1 \leq \beta_2 + \beta_6\}$$

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$$\kappa_1 = \kappa_3 xy, \quad \kappa_2 x - \kappa_5 xy = \kappa_4 y - \kappa_6, \quad |\kappa_4 y - \kappa_6| \leq \kappa_1 \leq \kappa_2 x + \kappa_6.$$

- Equivalently, find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which a pos. equil. (x, y) exists w/

$$x \geq \tfrac{\kappa_1 - \kappa_6}{\kappa_2} \quad \text{and} \quad \tfrac{\kappa_6 - \kappa_1}{\kappa_4} \leq y \leq \tfrac{\kappa_1 + \kappa_6}{\kappa_4}.$$

- Disguised vertex-balanced if and only if

$$(1 - \tfrac{\kappa_5}{\kappa_3})(1 - \tfrac{\kappa_6}{\kappa_1}) \leq \tfrac{\kappa_2 \kappa_4}{\kappa_1 \kappa_3} \leq (1 + \tfrac{\kappa_5}{\kappa_3})(1 + \tfrac{\kappa_6}{\kappa_1})$$

Computational aspects (partly reversible square)

Find all $\boldsymbol{\beta} \in \mathbb{R}^6_{>0}$ for which there exists a $\boldsymbol{\gamma} \in \mathbb{R}^8_{\geq 0}$ such that

$$\beta_1 = \gamma_1 + \gamma_7, \quad \beta_2 = \gamma_2, \quad \beta_3 = \gamma_3 + \gamma_8, \qquad \gamma_4 + \gamma_8 = \gamma_1 + \gamma_6 + \gamma_7, \quad \gamma_1 + \gamma_5 = \gamma_2,$$
$$\beta_6 = \gamma_6 + \gamma_7, \quad \beta_4 = \gamma_4, \quad \beta_5 = \gamma_5 + \gamma_8; \qquad \gamma_2 + \gamma_7 = \gamma_3 + \gamma_5 + \gamma_8, \quad \gamma_3 + \gamma_6 = \gamma_4.$$

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- Find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which there exists an $(x, y) \in \mathbb{R}^2_{>0}$ such that

$$\kappa_1 = \kappa_3 xy, \quad \kappa_2 x - \kappa_5 xy = \kappa_4 y - \kappa_6, \quad |\kappa_4 y - \kappa_6| \leq \kappa_1 \leq \kappa_2 x + \kappa_6.$$

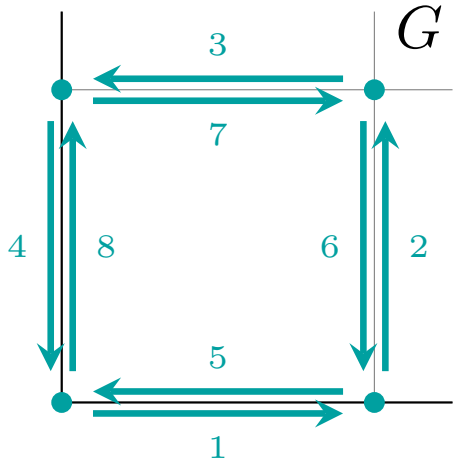
- Equivalently, find all $\boldsymbol{\kappa} \in \mathbb{R}^6_{>0}$ for which a pos. equil. (x, y) exists w/

$$x \geq \tfrac{\kappa_1 - \kappa_6}{\kappa_2} \quad \text{and} \quad \tfrac{\kappa_6 - \kappa_1}{\kappa_4} \leq y \leq \tfrac{\kappa_1 + \kappa_6}{\kappa_4}.$$

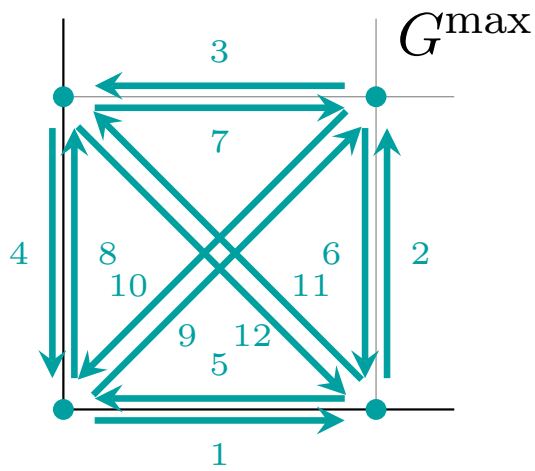
- Disguised vertex-balanced if and only if

$$(1 - \tfrac{\kappa_5}{\kappa_3})(1 - \tfrac{\kappa_6}{\kappa_1}) \leq \tfrac{\kappa_2 \kappa_4}{\kappa_1 \kappa_3} \leq (1 + \tfrac{\kappa_5}{\kappa_3})(1 + \tfrac{\kappa_6}{\kappa_1})$$

REVERSIBLE SQUARE



$$\dot{x} = \kappa_1 - \kappa_5 x + \kappa_7 y - \kappa_3 xy$$
$$\dot{y} = \kappa_2 x - \kappa_6 xy + \kappa_8 - \kappa_4 y$$

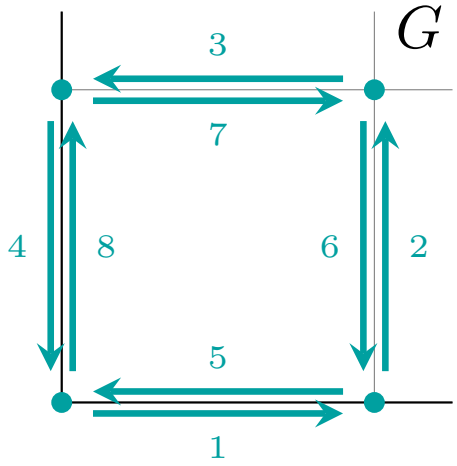


$$\mathcal{F}^{\mathrm{eq}}(G) = \{\boldsymbol{\beta} \in \mathbb{R}^8_{>0} \mid \beta_1 - \beta_5 = \beta_3 - \beta_7 \ \ \& \ \ \beta_2 - \beta_6 = \beta_4 - \beta_8\}$$

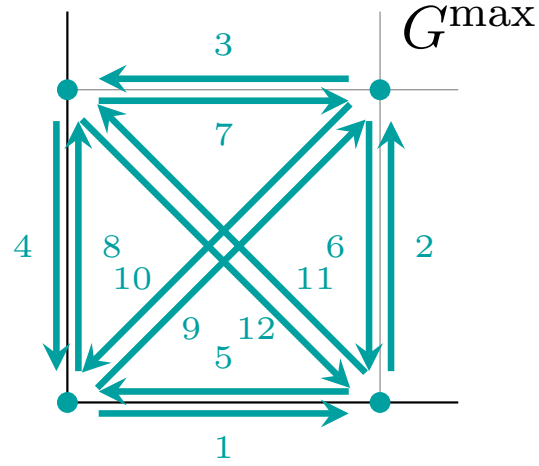
$$\mathcal{F}^{\mathrm{dt}}(G) = \left\{ \boldsymbol{\beta} \in \mathcal{F}^{\mathrm{eq}}(G) \;\middle|\; \begin{array}{l} \max(\beta_1, \beta_8) - \beta_4 - \beta_5 \le \min(\beta_3, \beta_6) \\ \max(\beta_2, \beta_5) - \beta_1 - \beta_6 \le \min(\beta_4, \beta_7) \end{array} \right\}$$

$$\mathcal{K}^{\mathrm{dt}}(G) = \left\{ \boldsymbol{\kappa} \in \mathbb{R}^8_{>0} \;\middle|\; \begin{array}{l} (\kappa_1 - \kappa_8)(\kappa_3 - \kappa_6) \le (\kappa_2 + \kappa_5)(\kappa_4 + \kappa_7) \\ (\kappa_2 - \kappa_5)(\kappa_4 - \kappa_7) \le (\kappa_1 + \kappa_8)(\kappa_3 + \kappa_6) \end{array} \right\}$$

REVERSIBLE SQUARE



$$\dot{x} = \kappa_1 - \kappa_5 x + \kappa_7 y - \kappa_3 xy$$
$$\dot{y} = \kappa_2 x - \kappa_6 xy + \kappa_8 - \kappa_4 y$$



$$\mathcal{F}^{\mathrm{eq}}(G) = \{\boldsymbol{\beta} \in \mathbb{R}^8_{>0} \mid \beta_1 - \beta_5 = \beta_3 - \beta_7 \ \ \& \ \ \beta_2 - \beta_6 = \beta_4 - \beta_8\}$$

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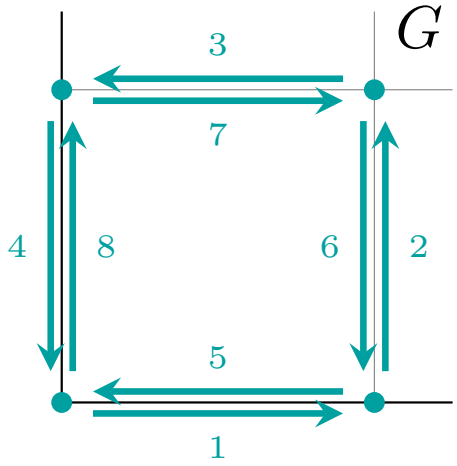
$$\mathcal{K}^{\mathrm{dt}}(G) = \left\{ \boldsymbol{\kappa} \in \mathbb{R}^8_{>0} \;\middle|\; \begin{array}{l} (\kappa_1 - \kappa_8)(\kappa_3 - \kappa_6) \le (\kappa_2 + \kappa_5)(\kappa_4 + \kappa_7) \\ (\kappa_2 - \kappa_5)(\kappa_4 - \kappa_7) \le (\kappa_1 + \kappa_8)(\kappa_3 + \kappa_6) \end{array} \right\}$$

REVERSIBLE SQUARE



$$\dot{x} = \kappa_1 - \kappa_5 x + \kappa_7 y - \kappa_3 xy$$
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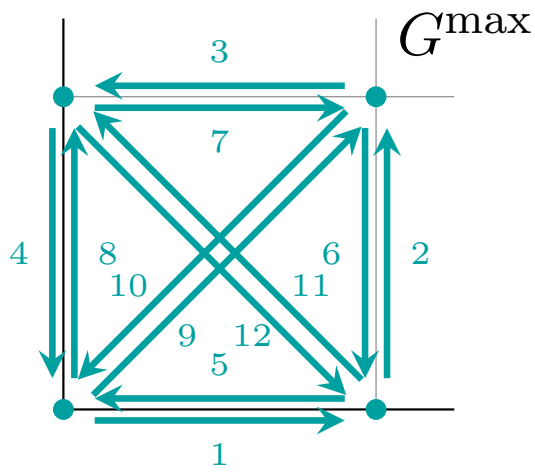

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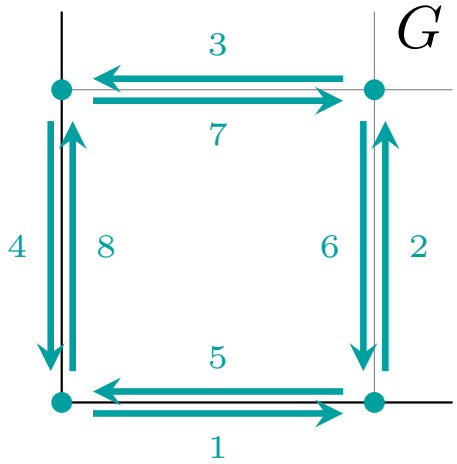
$$\mathcal{K}^{\mathrm{dt}}(G) = \left\{ \boldsymbol{\kappa} \in \mathbb{R}^8_{>0} \;\middle|\; \begin{array}{l} (\kappa_1 - \kappa_8)(\kappa_3 - \kappa_6) \leq (\kappa_2 + \kappa_5)(\kappa_4 + \kappa_7) \\ (\kappa_2 - \kappa_5)(\kappa_4 - \kappa_7) \leq (\kappa_1 + \kappa_8)(\kappa_3 + \kappa_6) \end{array} \right\}$$

REVERSIBLE SQUARE



$$\dot{x} = \kappa_1 - \kappa_5 x + \kappa_7 y - \kappa_3 xy$$
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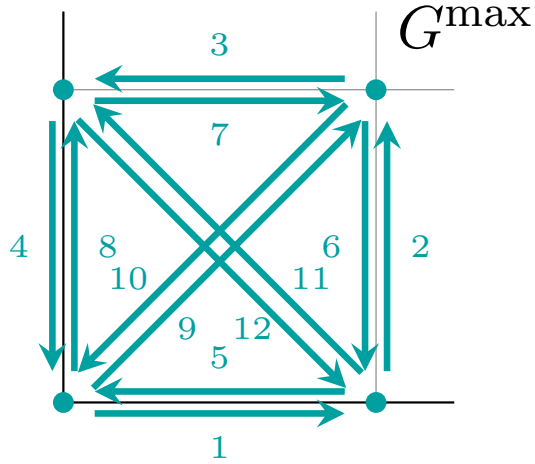
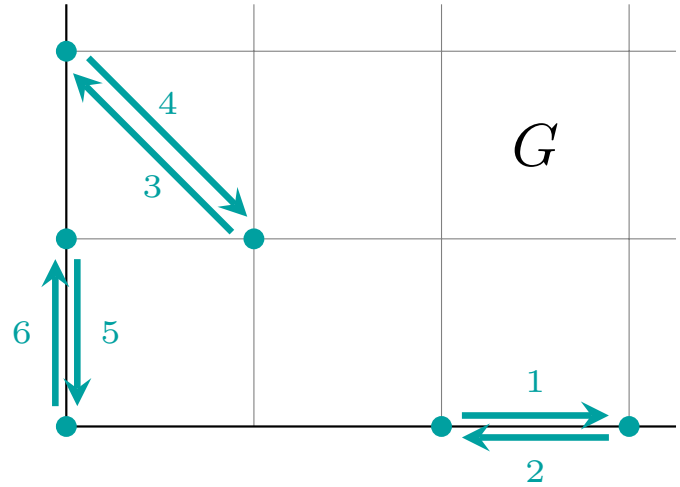


$$\mathcal{F}^{\mathrm{eq}}(G) = \{\boldsymbol{\beta} \in \mathbb{R}^8_{>0} \mid \beta_1 - \beta_5 = \beta_3 - \beta_7 \ \ \& \ \ \beta_2 - \beta_6 = \beta_4 - \beta_8\}$$

$$\mathcal{F}^{\mathrm{dt}}(G) = \left\{ \boldsymbol{\beta} \in \mathcal{F}^{\mathrm{eq}}(G) \;\middle|\; \begin{array}{l} \max(\beta_1, \beta_8) - \beta_4 - \beta_5 \le \min(\beta_3, \beta_6) \\ \max(\beta_2, \beta_5) - \beta_1 - \beta_6 \le \min(\beta_4, \beta_7) \end{array} \right\}$$

$$\mathcal{K}^{\mathrm{dt}}(G) = \left\{ \boldsymbol{\kappa} \in \mathbb{R}^8_{>0} \;\middle|\; \begin{array}{l} (\kappa_1 - \kappa_8)(\kappa_3 - \kappa_6) \le (\kappa_2 + \kappa_5)(\kappa_4 + \kappa_7) \\ (\kappa_2 - \kappa_5)(\kappa_4 - \kappa_7) \le (\kappa_1 + \kappa_8)(\kappa_3 + \kappa_6) \end{array} \right\}$$

Reversible Lotka–Volterra autocatalator (revisited)



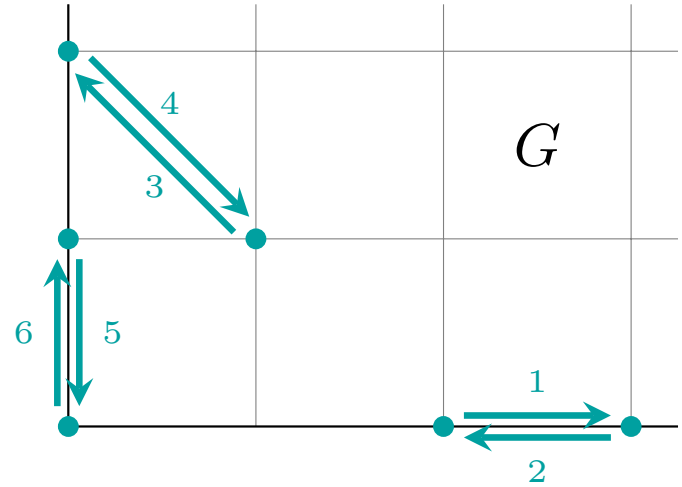
$$\dot{x} = \kappa_1 x^2 - \kappa_2 x^3 - \kappa_3 xy + \kappa_4 y^2$$
$$\dot{y} = \kappa_3 xy - \kappa_4 y^2 - \kappa_5 y + \kappa_6$$

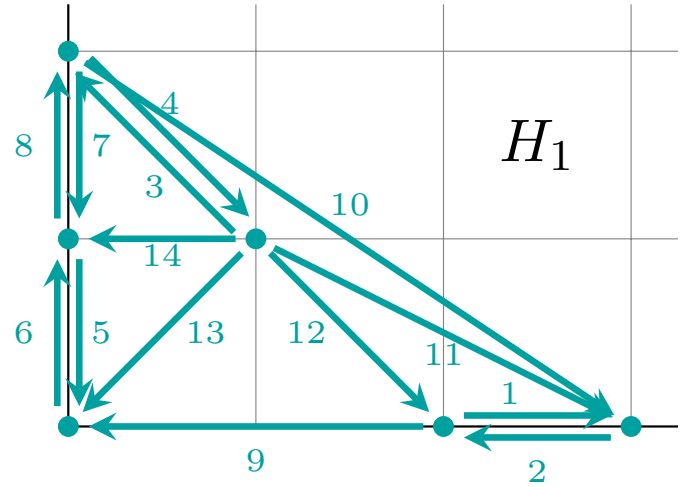
$$\mathcal{F}^{\mathrm{eq}}(G) = \{\boldsymbol{\beta} \in \mathbb{R}^6_{>0} \mid \beta_2 - \beta_1 = \beta_4 - \beta_3 = \beta_6 - \beta_5\}$$

$$\mathcal{F}^{\mathrm{t}}(G) = \{\boldsymbol{\beta} \in \mathcal{F}^{\mathrm{eq}}(G) \mid \beta_2 - \beta_1 = 0\}$$

$$\mathcal{K}^{\mathrm{t}}(G) = \{\boldsymbol{\kappa} \in \mathbb{R}^6_{>0} \mid \tfrac{\kappa_2\kappa_4\kappa_6}{\kappa_1\kappa_3\kappa_5} = 1\}$$

Reversible Lotka–Volterra autocatalator (revisited)





$$\dot{x} = \kappa_1 x^2 - \kappa_2 x^3 - \kappa_3 xy + \kappa_4 y^2$$
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$$\beta_1 = \gamma_1 - 2\gamma_9 \;\&\; \beta_2 = \gamma_2 \text{ (DE)}$$
$$\gamma_2 = \gamma_1 + \gamma_{10} + \gamma_{11} \text{ (VB)}$$

$$\Downarrow$$

$$\beta_2 - \beta_1 = 2\gamma_9 + \gamma_{10} + \gamma_{11} \geq 0$$

$$\mathcal{F}^{\mathrm{eq}}(G) = \{\boldsymbol{\beta} \in \mathbb{R}^6_{>0} \mid \beta_2 - \beta_1 = \beta_4 - \beta_3 = \beta_6 - \beta_5\}$$

$$\mathcal{F}^{\mathrm{t}}(G) = \{\boldsymbol{\beta} \in \mathcal{F}^{\mathrm{eq}}(G) \mid \beta_2 - \beta_1 = 0\}$$

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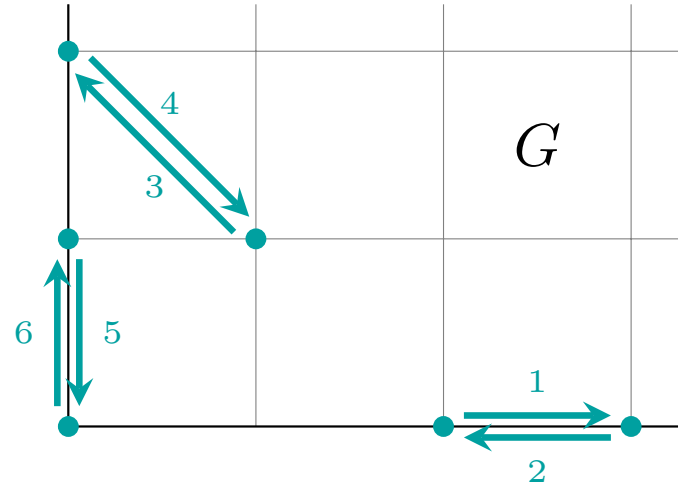
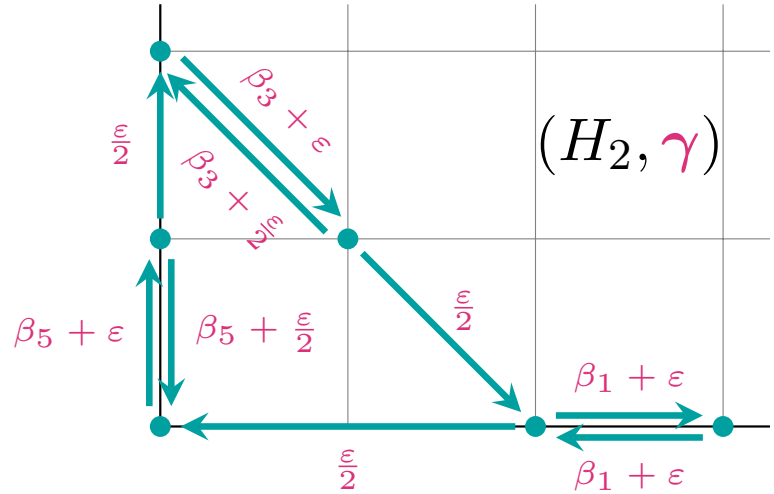
Reversible Lotka–Volterra autocatalator (revisited)





$$\dot{x} = \kappa_1 x^2 - \kappa_2 x^3 - \kappa_3 xy + \kappa_4 y^2$$
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let $\varepsilon = \beta_2 - \beta_1 \geq 0$

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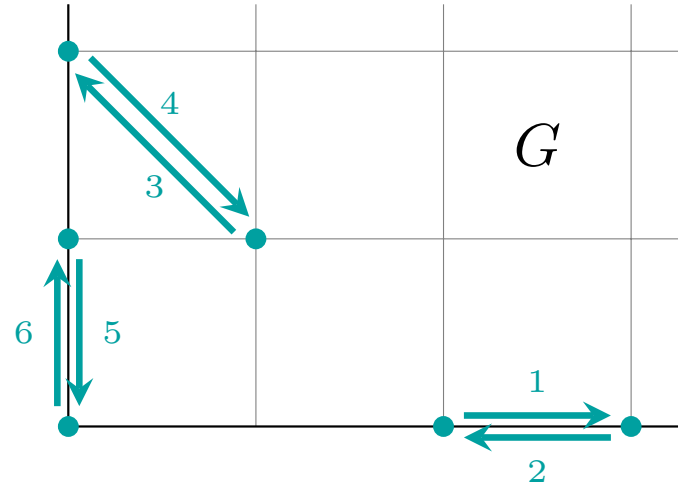
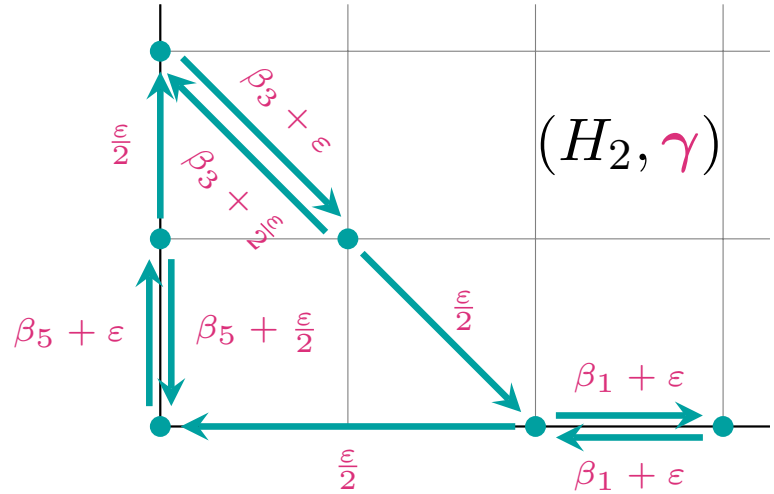
$$\mathcal{K}^{\mathrm{t}}(G) = \{\boldsymbol{\kappa} \in \mathbb{R}^6_{>0} \mid \tfrac{\kappa_2\kappa_4\kappa_6}{\kappa_1\kappa_3\kappa_5} = 1\}$$

Reversible Lotka–Volterra autocatalator (revisited)





$$\dot{x} = \kappa_1 x^2 - \kappa_2 x^3 - \kappa_3 xy + \kappa_4 y^2$$
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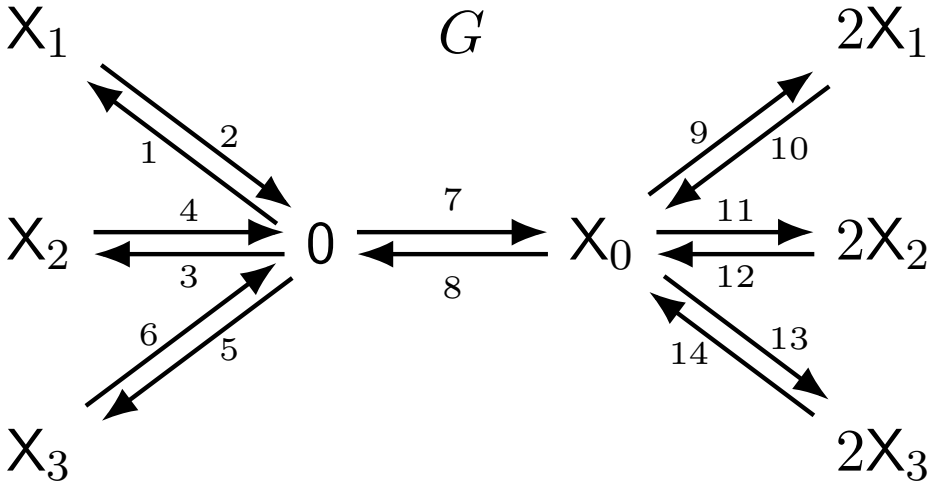
let $\varepsilon = \beta_2 - \beta_1 \geq 0$

$$\mathcal{F}^{\mathrm{eq}}(G) = \{\boldsymbol{\beta} \in \mathbb{R}^6_{>0} \mid \beta_2 - \beta_1 = \beta_4 - \beta_3 = \beta_6 - \beta_5\}$$

$$\mathcal{F}^{\mathrm{t}}(G) = \{\boldsymbol{\beta} \in \mathcal{F}^{\mathrm{eq}}(G) \mid \beta_2 - \beta_1 = 0\} \qquad \mathcal{K}^{\mathrm{t}}(G) = \{\boldsymbol{\kappa} \in \mathbb{R}^6_{>0} \mid \tfrac{\kappa_2\kappa_4\kappa_6}{\kappa_1\kappa_3\kappa_5} = 1\}$$

$$\mathcal{F}^{\mathrm{dt}}(G) = \{\boldsymbol{\beta} \in \mathcal{F}^{\mathrm{eq}}(G) \mid \beta_2 - \beta_1 \geq 0\} \qquad \mathcal{K}^{\mathrm{dt}}(G) = \{\boldsymbol{\kappa} \in \mathbb{R}^6_{>0} \mid \tfrac{\kappa_2\kappa_4\kappa_6}{\kappa_1\kappa_3\kappa_5} \geq 1\}$$

A FOUR-DIMENSIONAL EXAMPLE (CRACIUN–PANTEA 2008)



unique pos. equil. for all $\boldsymbol{\kappa} \in \mathbb{R}^{14}_{>0}$

vertex-balanced (VB):

$$\frac{\kappa_1^2}{\kappa_2^2}\frac{\kappa_{10}}{\kappa_9} = \frac{\kappa_3^2}{\kappa_4^2}\frac{\kappa_{12}}{\kappa_{11}} = \frac{\kappa_5^2}{\kappa_6^2}\frac{\kappa_{14}}{\kappa_{13}} = \frac{\kappa_7}{\kappa_8}$$

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$\dim \mathcal{K}^{\mathrm{dt}}(G) = 14$ vs. $\dim \mathcal{K}^{\mathrm{t}}(G) = 11$

uniform distr. on $\{\boldsymbol{\kappa} \in \mathbb{R}^{14}_{>0} \mid \sum_{i=1}^{14} \kappa_i = 1\} \longrightarrow$ 63% (disguised VB)

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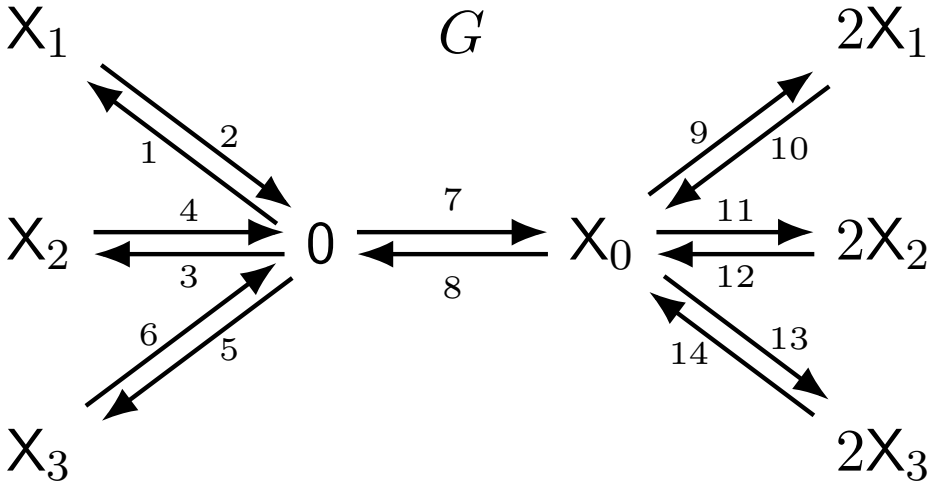

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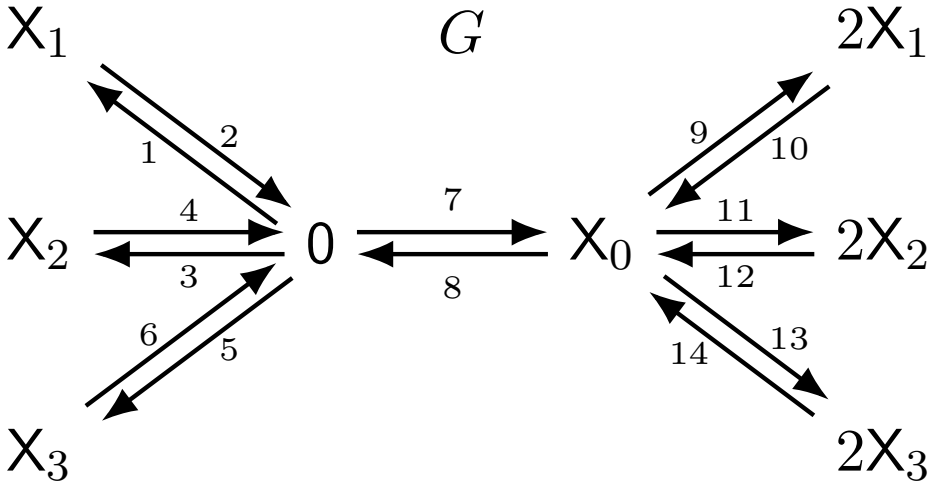

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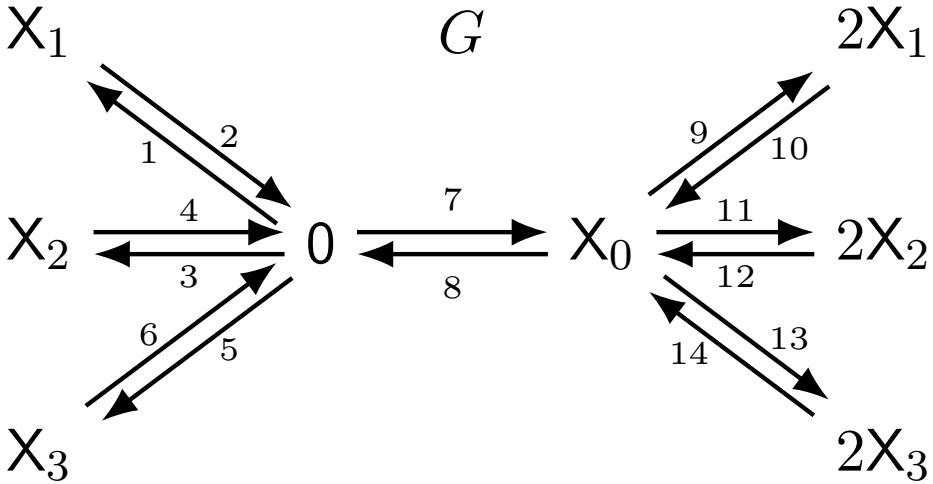

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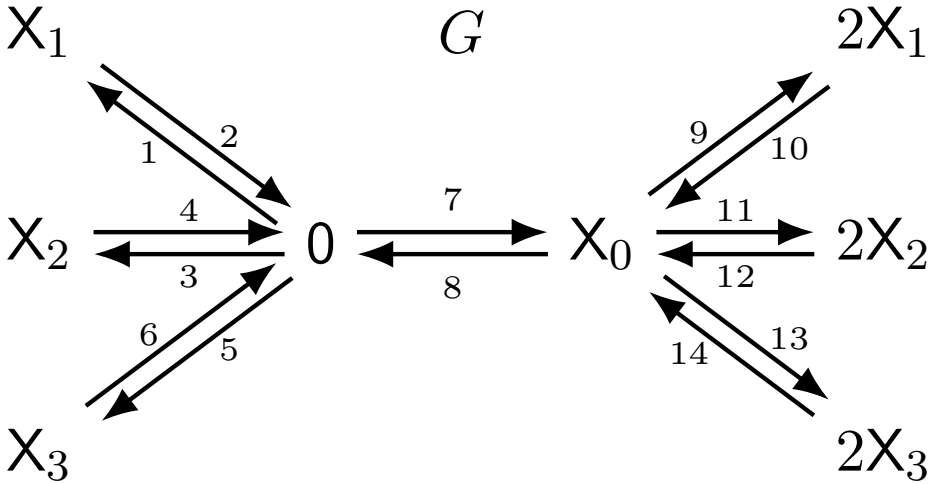

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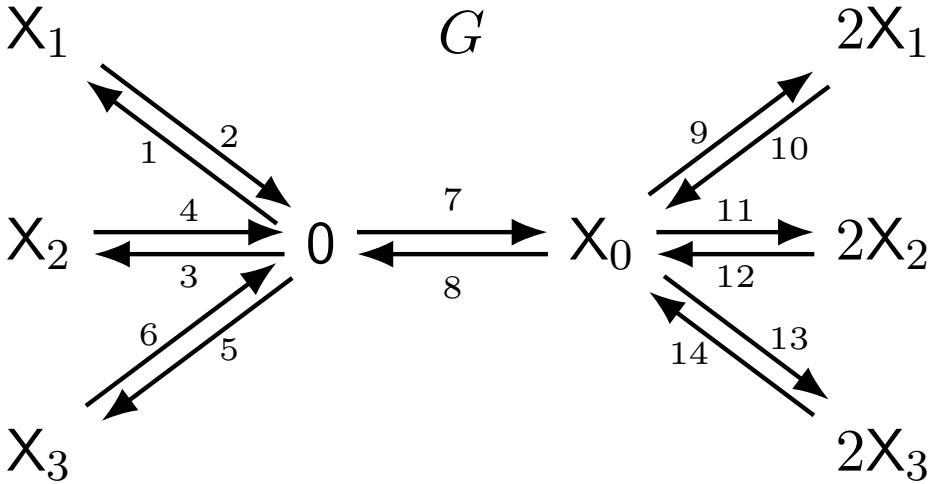

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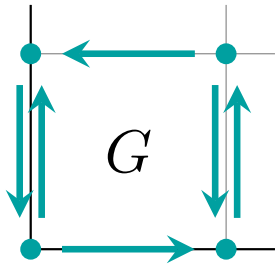

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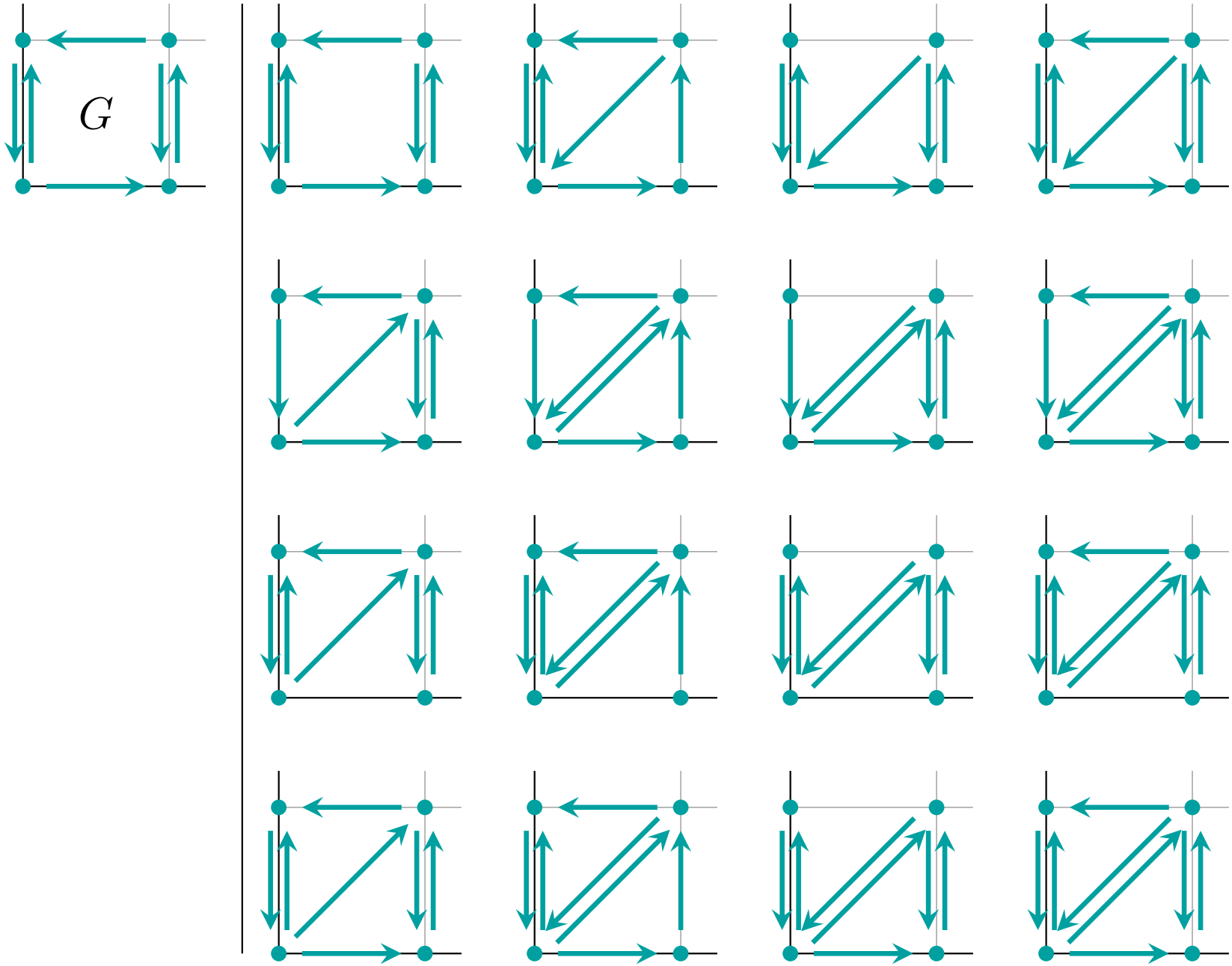

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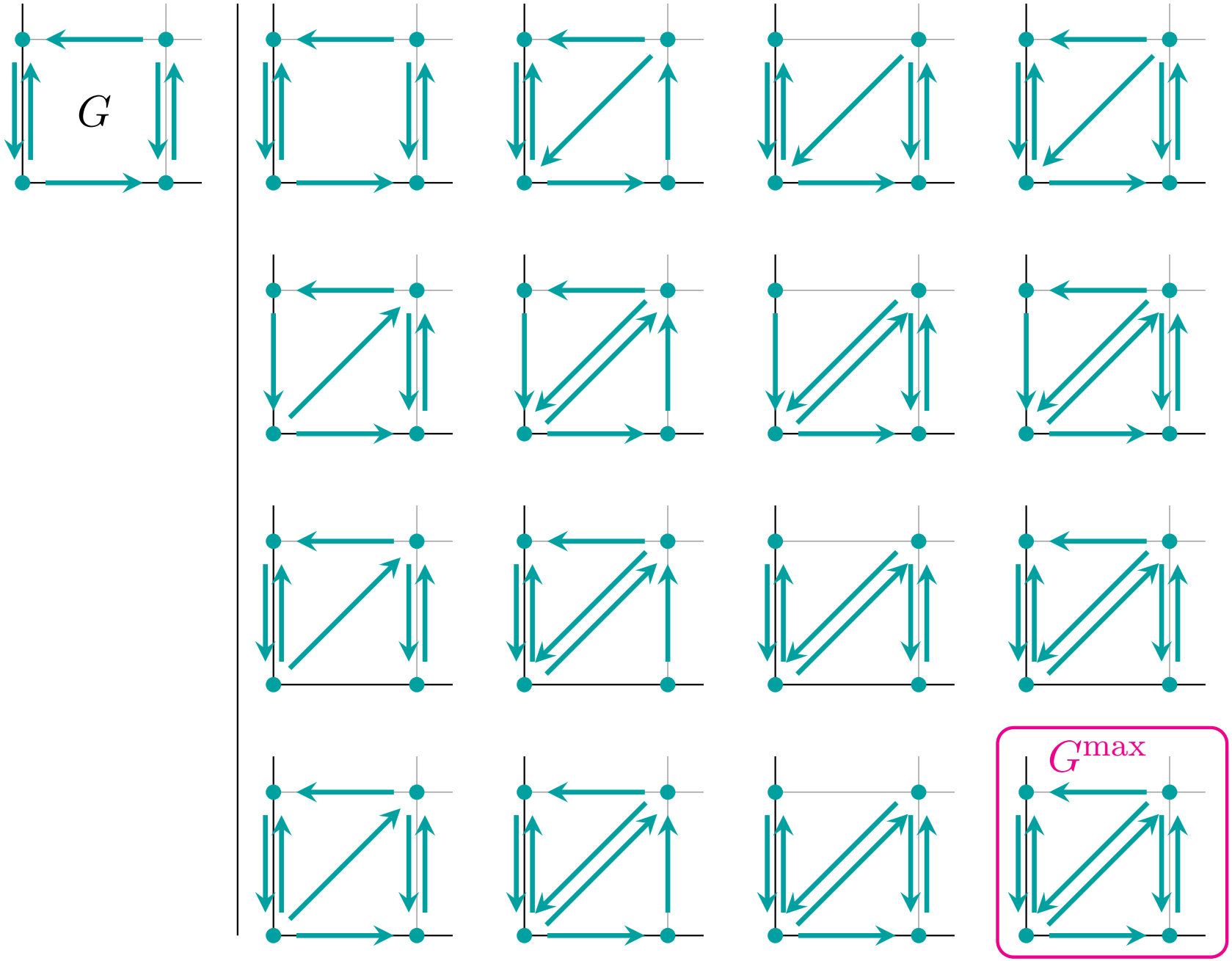

THE DISGUISED TORIC FLUX CONE

$$\mathcal{F}^{\mathrm{dt}}(G) = \{\boldsymbol{\beta} \in \mathbb{R}^{|E|}_{>0} \mid \exists H \text{ and } \exists \boldsymbol{\gamma} \in \mathcal{F}^{\mathrm{t}}(H) \text{ s.t. } (G, \boldsymbol{\beta}) \triangleq (H, \boldsymbol{\gamma})\}$$

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$$\mathcal{F}^{\mathrm{dt}}(G) = \bigcup_{H \subseteq G^{\mathrm{comp}}} \mathcal{F}^{\mathrm{dt}}(G, H) = \bigcup_{H \subseteq G^{\max}} \mathcal{F}^{\mathrm{dt}}(G, H)$$

LEMMA

- $\overline{\mathcal{F}^{\mathrm{dt}}(G)}$ *is a pointed polyhedral cone, and* $\mathcal{F}^{\mathrm{dt}}(G) = (\overline{\mathcal{F}^{\mathrm{dt}}(G)})_{>0}$
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is a homeomorphism

Corollary

Assuming $\mathcal{K}^{\mathrm{dt}}(G) \neq \emptyset$, the following statements hold.

- $\mathcal{K}^{\mathrm{dt}}(G)$ *is a manifold with boundary*
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History

- Identifiability of chemical reaction networks

  Craciun–Pantea 2008

- Computing weakly reversible and complex balanced realizations

  Ács–Csercsik–Hangos–Lipták–Péni–Szederkényi–Tuza–Tuza 2010–2016

  Johnston–Siegel–Szederkényi 2011–2013

- Disguised toric dynamical systems

  Brustenga–Craciun–Deshpande–Jin–Sorea–Yu 2020–2025

  Haque–Satriano–Sorea–Yu 2023

- Disguised toric flux cone

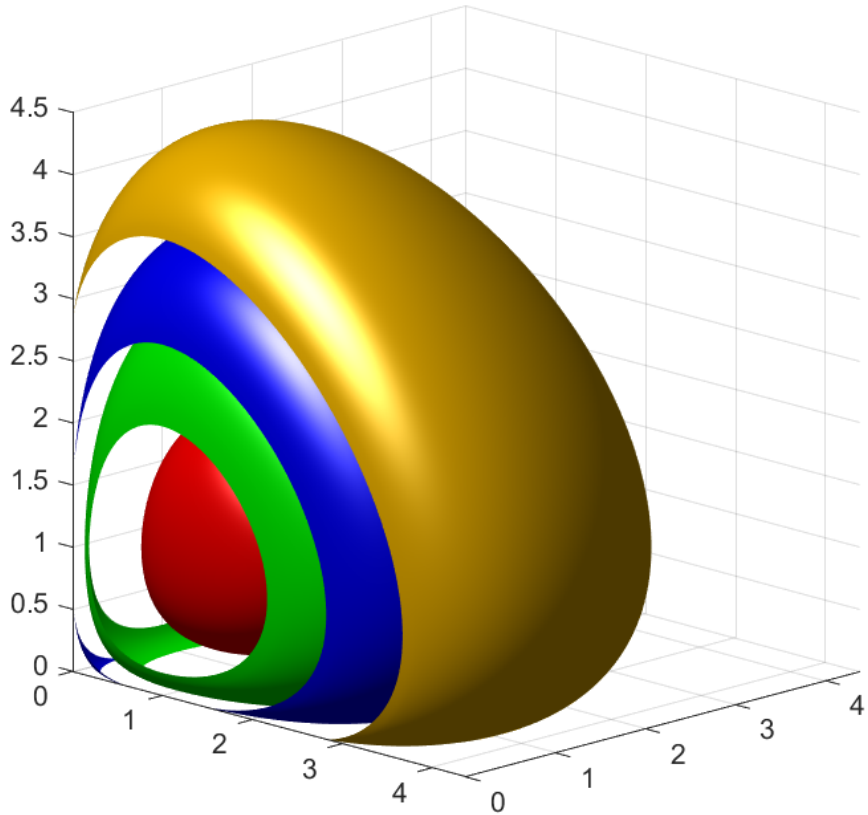
  BB–Craciun–Henriksson–Jin–Rojas 2025

- Multiple follow-up manuscripts in progress

  BB–Craciun–Henriksson–Jin–Rojas 2026+

TAKE-HOME MESSAGE

More often than you'd think, the Horn–Jackson function

$$V(\boldsymbol{x}) = \sum_{i=1}^{n} x_i(\log \tfrac{x_i}{x_i^*} - 1)$$

is a global Lyapunov function.