Effect of linear lumping on controllability and observability

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Lumping, controllability and observability

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Abstract

The effect of linear lumping, linear transformation to reduce the number of state variables on controllability and observability of linear differential equations has been studied. Controllability of the original system implies the controllability of the lumped system. Examples taken from reaction kinetics illustrate our results.

KEY WORDS

completely controllable, lumping matrix, compartmental system

MSC

80A30, 93Bxx, 34A30

1 Introduction

Dealing with modeling of real questions the large number of variables is generally a problem. To have a model which can easily be treated, one possible way is to reduce the number of variables by a method called lumping [7]. By controllability of a system we mean that it can be brought from any position to any other position in a finite amount of time. Furthermore, observability of a system means that we can determine the initial state of the system from the knowledge of an input-output pair over a certain period of time. In this paper we study the effect of linear lumping on such properties of the system as controllability and observability and apply the results with special reference to compartmental systems.

2 Basic notions of lumping and control theory

In this section we collect the basis of the mathematical theory of controllability [2, page 16], observability [2, page 26] and lumping [7, page 1534]. Before turning to the formal definitions we mention that controllability means that any prescribed concentration can be attained using an appropriate control input, observability has the meaning that one can reconstruct the history of the process when knowing the present concentration composition.

Let $n, r, p \in \mathbb{N}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}, C \in \mathbb{R}^{p\times n}$ and let us investigate:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

with $x(t)$ in R^n as the dependent variable of the linear differential equations at time t in R and $u(t)$ in R^r denoting the bounded and measurable control function and $y(t)$ in R^p the observation function.

Definition 1 *A linear system with a state-space description given by (1) is said to be* ***completely controllable*** *if, starting from any position* x_0 *in* R^n*, the state vector* x *at any initial time* t_0 *can be brought to any other position* x_1 *in* $\mathbb{R}^n$ *in a finite amount of time by some control function* u*.*

Theorem 1 *A linear system described by (1) is completely controllable if and only if the* $n \times rn$ *matrix* $W_{AB} := [B \mid AB \mid \dots \mid A^{n-1}B]$ *has rank* n*.*

Let $t_0, t_1 \in \mathbb{R}; t_0 < t_1$.

Definition 2 *A linear system with the state-space description (1) has the observability property on an interval* (t_0, t_1)*, if any input-output pair* $(u(t), y(t))$*,* $t_0 \leq t \leq t_1$*, uniquely determines an initial state* $x(t_0)$*. Furthermore (1) is said to be observable at an initial time* t_0 *if it has the observability property on some interval* (t_0, t_1) *where* $t_1 > t_0$*. It is said to be* ***completely observable*** *if it is observable at every initial time* t_0*.*

Theorem 2 *A linear system described by (1) is completely observable if and only if the the* $n \times pn$ *matrix* $V_{CA} := [C^\top \mid A^\top C^\top \mid \dots \mid (A^\top)^{n-1}C^\top]$ *has rank* n*.*

Let us consider

$$\dot{x}(t) = A^\top x(t) + C^\top v(t) \qquad (3)$$

$$z(t) = B^\top x(t) \qquad (4)$$

with $v(t)$ in R^p denoting the bounded and measurable control function and $z(t)$ in $R^r$the observation function.

Theorem 3 *The linear systems (1) and (3) respectively described above are* ***dual to each other*** *in the sense that (1) is completely observable if and only if (3) is completely controllable and (1) is completely controllable if and only if (3) is completely observable* [2, page 32].

Let us introduce a slightly modified notion of lumpability [6, page 1262].

Definition 3 *Suppose* $l \in \mathbb{N}$*,* $l \leq n$*, and* $M \in \mathbb{R}^{l \times n}$*,* $\mathrm{rank}(M) = l$*. If for all solutions* $x(t)$ *to (1)*

$$\hat{x}(t) := Mx(t) \qquad (5)$$

obeys a differential equation

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{u}(t) \qquad (6)$$

with $\hat{A} \in \mathbb{R}^{l \times l}$*, and* $\hat{u}(t) := MBu(t)$ *then (1) is said to be* ***exactly lumpable*** *to (6) by* M*. The pair of matrices* M *and* $\hat{A}$ *is sometimes referred to as a lumping scheme.*

Remark 1 It can be shown that (1) is exactly lumpable to (6) by M if and only if there exists $\hat{A} \in \mathbb{R}^{l\times l}$, such that $\hat{A}M = MA$, so we get the lumping scheme consisting of M and $\hat{A} = MAM^\top(MM^\top)^{-1}$. We note that there exists matrix $(MM^\top)^{-1} \in \mathbb{R}^{l\times l}$ since $\text{rank}(M) = l$ and $M^\top(MM^\top)^{-1}$ is a generalized inverse of M satisfying $MM^\top(MM^\top)^{-1} = I_{l\times l}$ with $I_{l\times l}$ being the l-identity matrix. It can be proved that there exists a matrix $\hat{A}$ such that $\hat{A}M = MA$ if

$$M = N \begin{bmatrix} f_1^\top \\ \vdots \\ f_l^\top \end{bmatrix}$$

where f_i $(i = 1, \ldots, l)$ are any independent, real eigenvectors of the matrix $A^\top$ and the matrix $N \in \mathbb{R}^{l\times l}$ is nonsingular [1, page 117].

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Theorem 4 *Let us assume that (1) is exactly lumpable to (6) by M and the linear system (1) is completely controllable then the linear system (6) is also completely controllable.*

Proof. Let (1) be completely controllable then according to Theorem 1 we know that $\text{rank}(W_{AB}) = n$. Furthermore, using the fact that

$$\hat{A}M = MA, \ldots, \hat{A}^{l-1}M = MA^{l-1}$$

we get

$$\hat{W}_{AB} := [MB \mid \hat{A}MB \mid \ldots \mid (\hat{A})^{l-1}MB] = M[B \mid AB \mid \ldots \mid A^{l-1}B].$$

This implies that (6) is completely controllable if and only if the $l\times rl$ matrix

$$M[B \mid AB \mid \ldots \mid A^{l-1}B]$$

has rank l by Theorem 1. Let us assume that the rank of $\hat{W}_{AB}$ is less than l, then there is a nonzero vector $b \in \mathbb{R}^l$ with $b^\top\hat{W}_{AB} = 0 \in \mathbb{R}^{rl}$ or equivalently $b^\top MB = b^\top\hat{A}MB = \ldots = b^\top(\hat{A})^{l-1}MB = 0 \in \mathbb{R}^r$. Application of the Cayley–Hamilton Theorem now gives

$$(\hat{A})^l = \gamma_1(\hat{A})^{l-1} + \gamma_2(\hat{A})^{l-2} + \ldots + \gamma_l I_{l\times l}$$

where $I_{l\times l}$ represents the $l\times l$ unit matrix and $\gamma_1, \ldots, \gamma_l$ are suitable constants, so $b^\top(\hat{A})^l MB = 0 \in \mathbb{R}^r$. With induction we can derive that $b^\top(\hat{A})^{l+j}MB = 0 \in \mathbb{R}^r$ for $(j = 1, 2, \ldots, n-1-l)$ also, thus

$$b^\top[MB \mid \hat{A}MB \mid \ldots \mid (\hat{A})^{n-1}MB] = b^\top M[B \mid AB \mid \ldots \mid A^{n-1}B] = 0 \in \mathbb{R}^{rn}.$$

Since $\text{rank}(M) = l$, therefore $b^\top M \neq 0 \in \mathbb{R}^n$ thus we get that the rows of matrix $W_{AB} = [B \mid AB \mid \ldots \mid A^{n-1}B]$ are linearly dependent, which is a contradiction. Consequently matrix $\hat{W}_{AB}$ has to have full rank and hence (6) is completely controllable.

Remark 2 The complete controllability of system (6) does not imply the complete controllability of system (1). Since even if the rank of $\hat{W}_{AB}$ is l, the rank of W_{AB} can be less then n. A concrete example will also be given below on page 7.

Remark 3 In the case of $l = n$ we get that M is a non-singular $n \times n$ matrix, thus

$$\begin{aligned}\text{rank}(\hat{W}_{AB}) = & \ \text{rank}(M[B \mid AB \mid \ldots \mid A^{l-1}B]) \ = \\ = & \ \ \text{rank}([B \mid AB \mid \ldots \mid A^{n-1}B]) \ = \text{rank}(W_{AB}).\end{aligned}$$

Therefore if (1) is completely controllable then (6) is also completely controllable and vice versa.

Remark 4 Let us assume that (1) is exactly lumpable to (6) by M then (3) is exactly lumpable to

$$\dot{\hat{x}}(t) = \tilde{A}\hat{x}(t) + \ MC^\top v(t) \tag{7}$$

by M, where $\tilde{A} \in \mathbb{R}^{l \times l}$ such that $MA^\top = \tilde{A}M$, so $\tilde{A} = MA^\top M^\top (MM^\top)^{-1}$. If we assume that (1) is completely observable then (3) is completely controllable by Theorem 3, so (7) is also completely controllable by the application of Theorem 4.

Remark 5 The complete controllability of system (7) does not imply the complete observability of system (1). Since if the controllability matrix of system (7) is $[MC^\top \mid \tilde{A}MC^\top \mid \ldots \mid (\tilde{A})^{l-1}MC^\top] = M[C^\top \mid A^\top C^\top \mid \ldots \mid (A^\top)^{l-1}C^\top]$ where we can use $\tilde{A}M = MA^\top, \ldots, \tilde{A}^{l-1}M = M(A^\top)^{l-1}$ and has rank l, the rank of V_{CA} can be less then n. A concrete example will also be given below on page 8.

4 Examples

In this section we will give examples to illustrate our results.

Let us consider the following chemical reaction, a special case of a compartmental system [3, page 3]:

$$\mathcal{X}_1 \xrightarrow{k} \mathcal{X}_2 \quad \mathcal{X}_2 \xrightarrow{k} \mathcal{X}_1 \quad \mathcal{X}_2 \xrightarrow{k} \mathcal{X}_3 \quad \mathcal{X}_3 \xrightarrow{k} \mathcal{X}_2 \tag{8}$$

where $\mathcal{X}_i \quad (i = 1, 2, 3)$ is the ith chemical component or species and the positive real number k is the uniform reaction rate constant. The example may be degenerate from the point of view of kinetics, however it may be considered as a three stage approximation of diffusion in a tube. We are interested in the time evolution of the quantities of chemical components. If we assume that the physical circumstances are ideal, i.e. in the given reaction the temperature, the pressure, and the volume of the vessel are constant we can build up the mass action-type model of the reaction (8):

$$\begin{aligned}\dot{x}_1(t) &= -kx_1(t) + kx_2(t)\\ \dot{x}_2(t) &= kx_1(t) - 2kx_2(t) + kx_3(t)\\ \dot{x}_3(t) &= kx_2(t) - kx_3(t)\end{aligned} \tag{9}$$

where $x_i(t) \quad (i = 1, 2, 3)$ is to be interpreted as the concentration of the species $\mathcal{X}_i$ at time t. Equation (9) is said to be the induced kinetic differential equation of (8).

To construct the lumping matrix M we use the following facts: if every element of the matrix $M = N \begin{bmatrix} f_1^\top \\ \vdots \\ f_l^\top \end{bmatrix}$ is nonnegative, moreover for every row of M there exists an element from that row which is the only nonzero element of its column, then the lumped system of the induced kinetic differential equation of a reaction is also a induced kinetic differential equation of a reaction [4].

According to these and using the fact that eigenvectors of

$$A^\top = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}$$

are

$$f_1^\top = [1, 1, 1], \quad f_2^\top = [1, 0, -1], \quad f_3^\top = [-1, 2, -1]$$

set

$$M := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

in all the examples of this section. In this case the new variables are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_2 + 2x_3 \end{bmatrix}$$

and the lumped system is

$$\begin{aligned}\dot{\hat{x}}_1 &= -\frac{k}{2}\hat{x}_1 + \frac{k}{2}\hat{x}_2\\ \dot{\hat{x}}_2 &= \frac{k}{2}\hat{x}_1 - \frac{k}{2}\hat{x}_2\end{aligned}$$

which is the induced kinetic differential equation of the reaction

$$\hat{\mathcal{X}}_1 \overset{k/2}{\longrightarrow} \hat{\mathcal{X}}_2 \quad \hat{\mathcal{X}}_2 \overset{k/2}{\longrightarrow} \hat{\mathcal{X}}_1.$$

Let us remark that the new variables can be considered as groups of the old ones measured together, since they are nonnegative linear combinations of the old ones.

4.1 Examples of the effect of lumping on controllability

First, let us consider the differential equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

then

$$W_{AB} = [B \mid AB \mid A^2B] = \begin{bmatrix} 1 & 0 & 0 & -k & k & 0 & 2k^2 & -3k^2 & k^2 \\ 0 & 1 & 0 & k & -2k & k & -3k^2 & 6k^2 & -3k^2 \\ 0 & 0 & 1 & 0 & k & -k & k^2 & -3k^2 & 2k^2 \end{bmatrix}.$$

This matrix has rank 3, therefore this system is completely controllable. The lumped system is

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & -\frac{k}{2} \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

So $\hat{W}_{AB} = [MB \mid \hat{A}MB] = \begin{bmatrix} 2 & 1 & 0 & -k & 0 & k \\ 0 & 1 & 2 & k & 0 & -k \end{bmatrix}$ which has rank 2, therefore in this case the lumped system is also completely controllable in accordance with Theorem 4.

Second, let us consider the control differential equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Since in this case the matrix $W_{AB} = \begin{bmatrix} 1 & 1 & 0 & 0 & -k & 0 & 0 & k^2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & k & 0 & 0 & -k^2 & 0 \end{bmatrix}$ has only rank 2, thus this system is not completely controllable. At the same time the lumped system is

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & -\frac{k}{2} \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 0 \\ 3 & -2 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Therefore $\hat{W}_{AB} = \begin{bmatrix} 3 & 2 & 0 & 0 & -2k & 0 \\ 3 & -2 & 0 & 0 & 2k & 0 \end{bmatrix}$ and this matrix has rank 2, so the lumped system is still completely controllable in this case.

4.2 Examples for the effect of lumping on observability

First, let us consider the observation system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e. the observation matrix $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. So $V_{CA} = [C^\top \mid A^\top C^\top \mid (A^\top)^2 C^\top] = \begin{bmatrix} 1 & 0 & 0 & k & -k^2 & -2k^2 \\ 1 & 1 & -k & -k & 3k^2 & 3k^2 \\ 0 & 1 & k & 0 & -2k^2 & -k^2 \end{bmatrix}$ which has rank 3, hence this system is completely observable by Theorem 2. The lumped system is characterized by $\tilde{A} = \begin{bmatrix} -\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & -\frac{k}{2} \end{bmatrix}$ and $MC^\top = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ by Remark 4. So $[MC^\top \mid \tilde{A}MC^\top] = \begin{bmatrix} 3 & 1 & -k & k \\ 1 & 3 & k & -k \end{bmatrix}$ which has rank 2, therefore in this case the lumped system is completely controllable

Second, let us consider the observation system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e. now $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. Since in this case the matrix

$$V_{CA} = \begin{bmatrix} 2 & 0 & -k & k & k^2 & -k^2 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & k & -k & -k^2 & k^2 \end{bmatrix}$$

which has rank only 2, thus this system is not completely observable. At the same time the lumped system is $\tilde{A} = \begin{bmatrix} -\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & -\frac{k}{2} \end{bmatrix}$ and $MC^\top = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$. Therefore $[MC^\top \mid \tilde{A}MC^\top] = \begin{bmatrix} 5 & 1 & -2k & 2k \\ 1 & 5 & 2k & -2k \end{bmatrix}$ and this matrix has rank 2, so the lumped system is completely controllable in this case.

5 Appendices

5.1 Appendix1

Very often problems in the biological, physical, and social sciences can reduced to problems involving matrices which, due to certain constraints, have some special structure. One of the most common situations is where the matrix A_1 in question has nonpositive off-diagonal and nonnegative diagonal entries, that is, $A_1 \in \mathbb{R}^{n\times n}$ is a finite matrix of type

$$A_1 = \begin{bmatrix} a_{11} & -a_{12} & -a_{13} & \dots & -a_{1n} \\ -a_{21} & a_{22} & -a_{23} & \dots & -a_{2n} \\ -a_{31} & -a_{32} & a_{33} & \dots & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \dots & a_{nn} \end{bmatrix}$$

where the $a_{ij} \in \mathbb{R}$ are nonnegative. Since A_1 can then be expressed in the form

$$A_1 = sI_{n\times n} - D, \quad s > 0, \quad D \geq 0. \tag{10}$$

$s \in \mathbb{R}, \quad A_1 \in \mathbb{R}^{n\times n}, \quad D \in \mathbb{R}^{n\times n}, \quad \mathrm{I}_{n\times n}$ the $n \times n$ unit matrix.
Definition 4 *Any matrix A_1 of the form (10) for which $s \geq \varrho(D)$, the spectral radius of D, is called an $M-matrix$. Nonsingular $M-matrices$, that is, those of the form (10) for which $s > \varrho(D)$. We note that $\varrho(D) = max_{h=1,2,\dots n} \mid \lambda_h \mid$, λ_h there is eigenvalue of D.*

Theorem 5 *The nonnegative matrix $T \in \mathbb{R}^{n\times n}$ is convergent, $lim_{h\to+\infty}T^h$ exists and is the zero matrix, that is, $\varrho(T) < 1$, if $(I_{n\times n} - T)^{-1}$ exists and*

$$(I_{n\times n} - T)^{-1} = \sum_{h=1}^{+\infty} T^h > 0.$$

Proof. If T is convergent then we get this for $(I_{n\times n} - T)^{-1}$ for the identity

$$(I_{n\times n} - T)(I_{n\times n} + T + \ldots + T^h) = (I_{n\times n} - T^{h+1}), \quad h \geq 0,$$

by letting h approach infinity.

Remark 6 *Suppose that A_1 is a nonsingular M-matrix. Letting $T = D/s$, it follows that $\varrho(T) < 1$, since $\varrho(T) = \frac{1}{s}\varrho(D)$, so by Theorem 5*

$$A_1^{-1} = (I_{n\times n} - T)^{-1}/s > 0.$$

Thus A_1 is inverse-positive: that is A_1^{-1} exists and $A_1^{-1} > 0$.

Remark 7 *Singular M-matrices, that is, matrices A_1 of the form $A_1 = \varrho(D)I_{n\times n} - D, \quad D \geq 0$. Let A_1 be an M-matrix (singular or nonsingular), then for any $\varepsilon > 0$*

$$A_1 + \varepsilon I_{n\times n} = sI_{n\times n} - D + \varepsilon I_{n\times n} = (s+\varepsilon)I_{n\times n} - D = s_\varepsilon I_{n\times n} - D,$$

where $s_\varepsilon = s + \varepsilon > \varrho(D)$ since $s \geq \varrho(D)$. Thus $A_1 + \varepsilon I_{n\times n}$ is nonsingular M-matrix.

5.2 Appendix2

First, let us consider the matrix

$$A_1 = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 2k \end{bmatrix} - \begin{bmatrix} k & k & 0 \\ k & 0 & k \\ 0 & k & k \end{bmatrix} = 2kI_{3\times 3} - D, \quad k > 0$$

so $s = 2k > 0$ and since $\varrho(D) = max[\mid k \mid, \mid 2k \mid, \mid -k \mid] = 2k$, thus A_1 is singular M-matrix. Second, let us consider the matrix

$$A_1 = \begin{bmatrix} \frac{k}{2} & -\frac{k}{2} \\ -\frac{k}{2} & \frac{k}{2} \end{bmatrix} = \begin{bmatrix} \frac{k}{2} & 0 \\ 0 & \frac{k}{2} \end{bmatrix} - \begin{bmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{bmatrix} = \frac{k}{2}I_{2\times 2} - D, \quad k > 0$$

so $s = \frac{k}{2} > 0$ and since $\varrho(D) = max[\mid \frac{k}{2} \mid, \mid -\frac{k}{2} \mid] = \frac{k}{2}$, thus in this case A_1 is also singular M-matrix.

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Discussion and outlook

In the present paper we investigated the effect of nonnegative linear lumping or controllability and observability of linear differential equations, with special reference to compartmental systems. Previously, [5] investigated a similar problem: local observability and local controllability of reactions. Possible further topics are: investigation of the effect of exact nonlinear lumping on compartmental systems, or the effect of nonlinear lumping on local controllability and observability.

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