Effect of lumping on controllability and observability

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1 Introduction

Modeling of real phenomena involving a large number of variables generally poses a hard problem. There are a lot of solutions to circumvent this problem. The technique we address here is linear lumping [5]: the reduction of the number of variables by grouping them via a linear function and so the lumped variables are certain linear combinations of the original ones. By controllability of a system we mean that it can be brought from any state to any other state in a finite time. Furthermore, observability of a system means that we can determine the initial state of the system from the knowledge of an input-output pair over a certain period of time. In this paper we study the effect of lumping on such properties of the system as controllability and observability.

2 Basic notions of lumping and control theory

In this section we collect the basis of the mathematical theory of controllability [1, page 16], observability [1, page 26] and lumping [5, page 1534].

Let $n, r, p \in \mathbb{N}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}, C \in \mathbb{R}^{p\times n}$ and let us investigate:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

with u denoting the bounded and measurable control function and y the observation function.

Definition 1 *A linear system with a state-space description given by (1) is said to be **completely controllable** if, starting from any position x_0 in $\mathbb{R}^n$, the state vector x at any initial time t_0 can be brought to any other position x_1 in $\mathbb{R}^n$ in a finite amount of time by some control function u.*

Theorem 1 *A linear system described by (1) is completely controllable if and only if the $n \times rn$ matrix $W_{AB} := [B \mid AB \mid \ldots \mid A^{n-1}B]$ has rank n.*

Let $t_0, t_1 \in \mathbb{R}; t_0 < t_1$.

Definition 2 *A linear system with the state-space description (1)-(2) has the observability property on an interval* (t_0, t_1)*, if any input-output pair* $(u(t), y(t))$*,* $t_0 \leq t \leq t_1$*, uniquely determines an initial state* $x(t_0)$*. Furthermore (1), (2) is said to be observable at an initial time* t_0*, if it has the observability property on some interval* (t_0, t_1) *where* $t_1 > t_0$*. It is said to be* ***completely observable****, if it is observable at every initial time* t_0*.*

Theorem 2 *A linear system described by (1)-(2) is completely observable if and only if the the* $n \times pn$ *matrix* $V_{CA} := [C^\top \mid A^\top C^\top \mid \ldots \mid (A^\top)^{n-1} C^\top]$ *has rank* n*.*

Let us consider

$$\begin{aligned} \dot{x}(t) &= A^\top x(t) + C^\top v(t) && (3) \\ z(t) &= B^\top x(t) && (4) \end{aligned}$$

with v denoting the bounded and measurable control function and z the observation function.

Theorem 3 *The linear systems (1)-(2) and (3)-(4), are* ***dual to each other*** *in the sense that (1)-(2) is completely observable if and only if (3) is completely controllable and (1) is completely controllable if and only if (3)-(4) is completely observable* [1, page 32].

Let us introduce a slightly modified notion of lumpability [5, page 1534].

Definition 3 *Suppose* $k \in \mathbb{N}$*,* $k \leq n$*,* $M \in \mathbb{R}^{k \times n}$*,* $\operatorname{rank}(M) = k$*. If for all solutions* x *to (1), the function*

$$\hat{x}(t) := Mx(t) \qquad (5)$$

obeys a differential equation

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + MBu(t) \qquad (6)$$

with $\hat{A} \in \mathbb{R}^{k \times k}$*, then (1) is said to be* ***exactly lumpable*** *to (6) by* M*.*

Remark 1 (1) is exactly lumpable to (6) by M if and only if there exists an $\hat{A} \in \mathbb{R}^{k \times k}$, such that $\hat{A}M = MA$.

Remark 2 (1) is exactly lumpable to (6) by M if and only if the subspace spanned by the row vectors of M is $A^\top$-invariant [4, page 1415]. (We note that the row vectors of M are linearly independent, as $\operatorname{rank}(M) = k$.)

Definition 4 *Let $S \in \mathbb{R}^{n\times n}$ be a real nonsingular matrix such that $A = SJS^{-1}$ with $J \in \mathbb{R}^{n\times n}$ being the Jordan form of A. Furthermore, let*

$$s_{11}, \ldots, s_{1h_1}; s_{21}, \ldots, s_{2h_2}; \ldots; s_{q1}, \ldots, s_{qh_q}$$

be the columns of S where h_i is the size of the i^{th} Jordan block. It can be seen that any subspace of the form

$$\mathrm{Span}\{s_{11}, \ldots, s_{1m_1}, s_{21}, \ldots, s_{2m_2}, \ldots, s_{q1}, \ldots, s_{qm_q}\} \tag{7}$$

*for some choice of integers m_i, $0 \le m_i \le h_i$, $0 \le i \le q$ is A-invariant. (Here $m_i = 0$ is interpreted in the sense that the vectors $s_{i1}, \ldots, s_{i,h_i}$ do not appear in (7) at all. We note that if h_i is the size of a Jordan block corresponding to a pair of conjugate nonreal eigenvalues then it is even and so is m_i.) Such A-invariant subspaces are called **marked** (with respect to the given basis s_{ij})* [3, page 83].

Remark 3 The following example shows that, in general, not every A-invariant subspace is marked. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We shall verify that the A-invariant subspace $\mathcal{M} = \mathrm{Span}\{e_1, e_2\}$ is not marked in any Jordan basis of A. Indeed, the Jordan form of A is

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so any Jordan basis of A is of the form f_1, f_2, f_3, g, where $Af_1 = Ag = 0$, $Af_2 = f_1$, $Af_3 = f_2$. If $\mathcal{M}$ were marked with respect to this basis, we would have either $\mathcal{M} = \mathrm{Span}\{f_1, g\}$ or $\mathcal{M} = \mathrm{Span}\{f_1, f_2\}$. The former is impossible because $A|_{\mathcal{M}} \neq 0$, and the latter is also impossible because it would imply $\mathcal{M} \subset \mathrm{Im}(A)$, which is not true since $e_2 \notin \mathrm{Im}(A)$ [3, page 83].

Theorem 4 *If the geometric multiplicity of every eigenvalue of A equals either 1 or the corresponding algebraic multiplicity, then every A-invariant subspace is marked* [3, page 84].

Remark 4 Since J is also the Jordan form of $A^\top$ [3, page 54], we conclude that if the hypotheses of Theorem 4 hold, then every $A^\top$-invariant subspace is also marked with respect to a basis in which $A^\top$ is in Jordan form.

3 Effect of lumping on controllability

3.1 The case of $k < n$

Theorem 5 *Let us assume that (1) is exactly lumpable to (6) by M and the linear system (1) is completely controllable. Then the linear system (6) is also completely controllable.*

Proof. Let (1) be completely controllable. Then according to Theorem 1 we have that $\operatorname{rank}(W_{AB}) = n$. Furthermore, using the fact that

$$\hat{A}M = MA, \ldots, \hat{A}^{k-1}M = MA^{k-1},$$

we get that

$$\hat{W}_{AB} := [MB \mid \hat{A}MB \mid \ldots \mid (\hat{A})^{k-1}MB] = M[B \mid AB \mid \ldots \mid A^{k-1}B].$$

This implies that (6) is completely controllable if and only if the $k \times rk$ matrix

$$M[B \mid AB \mid \ldots \mid A^{k-1}B]$$

has rank k by Theorem 1. Let us assume that the rank of $\hat{W}_{AB}$ is less than k, then there is a nonzero vector $b \in \mathbb{R}^k$ with $b^\top \hat{W}_{AB} = 0 \in \mathbb{R}^{rk}$, or equivalently $b^\top MB = b^\top \hat{A}MB = \ldots = b^\top (\hat{A})^{k-1}MB = 0 \in \mathbb{R}^r$. Application of the Cayley–Hamilton Theorem now gives

$$(\hat{A})^k = \gamma_1(\hat{A})^{k-1} + \gamma_2(\hat{A})^{k-2} + \ldots + \gamma_k E$$

where E represents the $k \times k$ unit matrix and $\gamma_1, \ldots, \gamma_k$ are suitable constants, so $b^\top(\hat{A})^k MB = 0 \in \mathbb{R}^r$. By induction on j we can derive that

$$b^\top(\hat{A})^{k+j}MB = 0 \in \mathbb{R}^r$$

for $j = 1, 2, \ldots, n-1-k$ also, thus

$$b^\top[MB \mid \hat{A}MB \mid \ldots \mid (\hat{A})^{n-1}MB] = b^\top M[B \mid AB \mid \ldots \mid A^{n-1}B] = 0 \in \mathbb{R}^{rn}.$$

Since $\operatorname{rank}(M) = k$, therefore $b^\top M \neq 0 \in \mathbb{R}^n$, thus we get that the rows of the matrix $W_{AB} = [B \mid AB \mid \ldots \mid A^{n-1}B]$ are linearly dependent, which is a contradiction. Consequently, the matrix $\hat{W}_{AB}$ has to have full rank, hence (6) is completely controllable.

Remark 5 The complete controllability of system (6) does not imply the complete controllability of system (1). Since the intersection of the subspace of the solutions to the equation $b^\top W_{AB} = 0 \in \mathbb{R}^{rn}$ with the subspace generated by the rows of M may consist of the zero vector only, so despite the fact that the rank of $\hat{W}_{AB}$ is k, the rank of W_{AB} can be less than n.

3.2 The case of $k = n$

Since in this case M is a non-singular $n \times n$ matrix, thus

$$\begin{aligned}\operatorname{rank}(\hat{W}_{AB}) = & \operatorname{rank}(M[B \mid AB \mid \ldots \mid A^{k-1}B]) & = \\ = & \operatorname{rank}([B \mid AB \mid \ldots \mid A^{n-1}B]) & = \operatorname{rank}(W_{AB}).\end{aligned}$$

Therefore if (1) is completely controllable, then (6) is also completely controllable and vice versa.

4 Effect of lumping on observability

Theorem 6 *Let us assume that (1) is exactly lumpable to (6) by M and that the geometric multiplicity of every eigenvalue of A equals the corresponding algebraic multiplicity. Then there exist $\tilde{M} \in \mathbb{R}^{k\times n}$ and $\tilde{A} \in \mathbb{R}^{k\times k}$ such that $\tilde{M}A^\top = \tilde{A}\tilde{M}$.*

Proof. Since we assume that (1) is exactly lumpable to (6) by M so the subspace $\mathcal{M}$ spanned by the row vectors of M is $A^\top$-invariant by Remark 2. Furthermore since for A the hypotheses of Theorem 4 hold, application of Remark 4 now gives that $\mathcal{M}$ is marked with respect to a basis in which $A^\top$ is in Jordan form. Let us denote these basis vectors by

$$f_{11}, \ldots, f_{1h_1}; f_{21}, \ldots, f_{2h_2}; \ldots; f_{q1}, \ldots, f_{qh_q} \tag{8}$$

where h_i is the size of the i^{th} Jordan block and then let

$$\mathcal{M} = \operatorname{Span}\{f_{11}, \ldots, f_{1l_1}, f_{21}, \ldots, f_{2l_2}, \ldots, f_{q1}, \ldots, f_{ql_q}\} \tag{9}$$

where $0 \le l_i \le h_i$, $1 \le i \le q$, $\sum_{i=1}^{q} l_i = k$ as $\dim(\mathcal{M}) = k$. (We note that since now for every eigenvalue λ of A holds that all partial multiplicities of λ are equal to unity thus $h_i \in \{1, 2\}$. $h_i = 2$ if and only if it is the size of a Jordan block corresponding to a pair of conjugate nonreal eigenvalues and in this case $l_i \in \{0, 2\}$.) Moreover let $F \in \mathbb{R}^{n\times n}$ such that its columns consist of the vectors (8) (i.e. $A^\top = FJF^{-1}$) and let us denote by e_i the unit vector of $\mathbb{R}^n$ with 1 in the ith place. Then the basis (9) of $\mathcal{M}$ can be given by FI where $I \in \mathbb{R}^{n\times k}$ and its columns are the vectors

$$e_1, \ldots, e_{l_1}, e_{h_1+1}, \ldots, e_{h_1+l_2}, \ldots, e_{\sum_{i=1}^{q-1} h_i+1}, \ldots, e_{\sum_{i=1}^{q-1} h_i+l_q}.$$

Hence we can find a nonsingular $N \in \mathbb{R}^{k\times k}$ such that

$$M = N(FI)^\top = NI^\top F^\top.$$

Using the fact that $A^\top = FJF^{-1}$, we get $F^\top A = J^\top F^\top$ and so

$$MA = NI^\top F^\top A = NI^\top J^\top F^\top.$$

From the definition of the matrix I, it can be seen that

$$I^\top J^\top I I^\top F^\top = I^\top J^\top F^\top$$

and

$$I^\top J I I^\top F^{-1} = I^\top J F^{-1} \tag{10}$$

similarity. Thus we get that

$$\begin{aligned} MA = NI^\top J^\top F^\top = NI^\top J^\top I I^\top F^\top = & \quad NI^\top J^\top I N^{-1} N I^\top F^\top \\ = & \quad NI^\top J^\top I N^{-1} M. \end{aligned}$$

This implies that $\hat{A} = NI^\top J^\top I N^{-1}$ as $MA = \hat{A}M$ by Remark 1.

Now let us consider the matrix

$$\tilde{M} = (N^{-1})^\top I^\top F^{-1} \quad \text{and} \quad \tilde{A} = \hat{A}^\top = (N^{-1})^\top I^\top J I N^\top.$$

We shall verify that $\tilde{M}A^\top = \tilde{A}\tilde{M}$. Indeed, using the fact that $A = (A^\top)^\top = (F^{-1})^\top J^\top F^\top$, we get $F^{-1}A^\top = JF^{-1}$ and so

$$\tilde{M}A^\top = (N^{-1})^\top I^\top F^{-1} A^\top = (N^{-1})^\top I^\top J F^{-1}.$$

Application of (10) gives

$$\begin{aligned} \tilde{M}A^\top = & \quad (N^{-1})^\top I^\top J F^{-1} = (N^{-1})^\top I^\top J I I^\top F^{-1} = \\ = & \quad (N^{-1})^\top I^\top J I N^\top (N^{-1})^\top I^\top F^{-1} = (N^{-1})^\top I^\top J I N^\top \tilde{M} = \tilde{A}\tilde{M}. \end{aligned}$$

Remark 6 It can be seen that if the the hypotheses of Theorem 6 hold, then there exists $\tilde{M} \in \mathbb{R}^{k\times n}$ such that $\tilde{M}A^\top = \hat{A}^\top \tilde{M}$.

Theorem 7 *Let us assume that (1) is exactly lumpable to (6) by M and that for every eigenvalue λ of A all partial multiplicities of λ are equal to unity. Furthermore, let us consider system (6) with the observation function*

$$\hat{y}(t) = \hat{C}\hat{x}(t) \tag{11}$$

where $\hat{C} \in \mathbb{R}^{p\times k}$, $\hat{C} := C\tilde{M}^\top$ and $\tilde{M} \in \mathbb{R}^{k\times n}$, $\tilde{M}A^\top = \hat{A}^\top \tilde{M}$.

Then, the complete observability of system (1)-(2) implies the complete observability of the system (6)-(11).

Proof. Let (1)-(2) be completely observable, then (3) is completely controllable by Theorem 3. Furthermore, according to Remark 6 and Remark 1, we know that (3) is exactly lumpable to

$$\dot{\hat{x}}(t) = \hat{A}^\top \hat{x}(t) + \tilde{M}C^\top v(t) \tag{12}$$

by $\tilde{M}$. Since application of Theorem 5 now gives that (12) is also completely controllable, thus (6)-(11) is completely observable by Theorem 3.

Remark 7 Let us assume that (1) is exactly lumpable to (6) by M and that all partial multiplicities of λ are equal to unity for every eigenvalue λ of A.

We know by Theorem 3 that (6)-(11) is completely observable if and only if (12) is completely controllable, and (1)-(2) is completely observable if and only if (3) is completely controllable. But according to Subsection 3.2 and Remark 5, the complete controllability of the system (12) implies the complete controllability of the system (3) if $k = n$, while the implication does not hold if $k < n$. Thus in the case of $k = n$ we can state that if (6)-(11) is completely observable then (1)-(2) is also completely observable, while in the case of $k < n$ the complete observability of system (6)-(11) does not imply the complete observability of the system (1)-(2).

Remark 8 Let us consider the effect of increasing the number of variables, i.e. instead of lumping let us use a transformation $\hat{x}(t) = Mx(t)$ where $M \in \mathbb{R}^{k\times n}$, $k > n$, $\operatorname{rank}(M) = n$. In this case the rows of M become linearly dependent, consequently $\hat{W}_{AB} = M[B \mid AB \mid \ldots \mid A^{k-1}B]$ has a rank less than k, hence (6) is not completely controllable.

Furthermore, let us consider system (6) with the observation function

$$\hat{y}(t) = \bar{C}\hat{x}(t), \tag{13}$$

where $\bar{C} \in \mathbb{R}^{p\times k}$ and $\bar{C}M = C$. Here we have to assume that C has the property that there exists a matrix $\bar{C}$ with $\bar{C}M = C$. (From linear algebra we know that this matrix equation is solvable if and only if every row of C is in the subspace spanned by the rows of M.)

Let us notice that by using this observation function observability could be examined in the case of $k > n$ analogously to the case of $k < n$. Indeed, let us assume that (6)-(13) is completely observable, then

$$\dot{\hat{x}}(t) = \hat{A}^\top\hat{x}(t) + \bar{C}^\top v(t) \tag{14}$$

is completely controllable by Theorem 3. Furthermore since $M^\top\hat{A}^\top = A^\top M^\top$, we know that (14) is exactly lumpable to (3) by $M^\top$. Application of Theorem 5 now gives that (3) is also completely controllable, thus (1)-(2) is completely observable by Theorem 3. Hence we can state that the complete observability of system (6)-(13) implies the complete observability of the system (1)-(2).

Moreover we know that (6)-(13) is completely observable if and only if (14) is completely controllable by Theorem 3, but according to Remark 5 the complete controllability of the system (3) does not imply the complete controllability of the system (14). Hence in the case of $k > n$ the complete observability of system (1)-(2) does not imply the complete observability of system (6)-(13).

5 Examples

In this section we will give examples to illustrate our results.

Let us consider the following chemical reaction:

$$\mathcal{X}_1 \xrightarrow{k} \mathcal{X}_2 \quad \mathcal{X}_2 \xrightarrow{k} \mathcal{X}_1 \quad \mathcal{X}_2 \xrightarrow{k} \mathcal{X}_3 \quad \mathcal{X}_3 \xrightarrow{k} \mathcal{X}_2 \tag{15}$$

where $\mathcal{X}_i$ $(i = 1, 2, 3)$ is the i^{th} chemical component (or species) and the positive real number k is the reaction rate constant. We are interested in the time evolution of the quantities of chemical components. If we assume ideal physical circumstances, i.e. the temperature, the pressure, and the volume of the vessel are constant in the given reaction, we can build up the mass action-type model of the reaction (15):

$$\begin{aligned} \dot{x}_1(t) &= -kx_1(t) + kx_2(t) \\ \dot{x}_2(t) &= kx_1(t) - 2kx_2(t) + kx_3(t) \\ \dot{x}_3(t) &= kx_2(t) - kx_3(t) \end{aligned} \tag{16}$$

where $x_i(t)$ is to be interpreted as the concentration of the species $\mathcal{X}_i$ at time t. Equation (16) is said to be the **induced kinetic differential equation** of (15).

To construct the lumping matrix M we use the following two facts.

It can be proved [4] that there exists a matrix $\hat{A} \in \mathbb{R}^{k \times k}$ such that $\hat{A}M = MA$ if

$$M = N \begin{bmatrix} s_1^\top \\ \vdots \\ s_k^\top \end{bmatrix},$$

where s_i $i = 1, \ldots, k$ are any independent, real eigenvectors of the matrix $A^\top$ and N is a non-singular $k \times k$ matrix.

Furthermore, if every element of the matrix $M = N \begin{bmatrix} s_1^\top \\ \vdots \\ s_k^\top \end{bmatrix}$ is nonnegative, moreover for every row of M there exists an element from that row which is the only nonzero element of its column, then the lumped system of the induced kinetic differential equation of a reaction is also an induced kinetic differential equation of a reaction [2].

According to these results and using the fact that eigenvectors of

$$A^\top = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}$$

are

$$s_1^\top = [1, 0, -1], \quad s_2^\top = [1, 1, 1], \quad s_3^\top = [-1, 2, -1],$$

we set

$$M := \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

in all the examples of this section. In this case the new variables become

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_3 \\ 2x_1 + x_2 \end{bmatrix}$$

and the lumped system is

$$\begin{aligned} \dot{\hat{x}}_1 &= -\frac{k}{2}\hat{x}_1 + \frac{k}{2}\hat{x}_2 \\ \dot{\hat{x}}_2 &= \frac{k}{2}\hat{x}_1 - \frac{k}{2}\hat{x}_2 \end{aligned}$$

which is the induced kinetic differential equation of the reaction

$$\hat{\mathcal{X}}_1 \stackrel{k/2}{\rightarrow} \hat{\mathcal{X}}_2 \quad \hat{\mathcal{X}}_2 \stackrel{k/2}{\rightarrow} \hat{\mathcal{X}}_1.$$

Let us remark that the new variables can be considered as groups of the old ones measured together, since they are nonnegative linear combinations of the old ones.

5.1 Examples of the the effect of lumping on controllability

First, let us consider the differential equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

then

$$W_{AB} = [B \mid AB \mid A^2B] = \begin{bmatrix} 1 & 0 & 0 & -k & k & 0 & 2k^2 & -3k^2 & k^2 \\ 0 & 1 & 0 & k & -2k & k & -3k^2 & 6k^2 & -3k^2 \\ 0 & 0 & 1 & 0 & k & -k & k^2 & -3k^2 & 2k^2 \end{bmatrix}.$$

This matrix has rank 3, therefore this system is completely controllable. The lumped system is

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & -\frac{k}{2} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

So $\hat{W}_{AB} = M[B \mid AB] = \begin{bmatrix} 0 & 1 & 2 & k & 0 & -k \\ 2 & 1 & 0 & -k & 0 & k \end{bmatrix}$ which has rank 2, therefore in this case the lumped system is also completely controllable.

Second, let us consider the control differential equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since in this case the matrix $W_{AB} = \begin{bmatrix} 1 & 1 & -k & 0 & k^2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & k & 0 & -k^2 & 0 \end{bmatrix}$ has only rank 2, this system is not completely controllable. At the same time the lumped system reads

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & -\frac{k}{2} \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Therefore $\hat{W}_{AB} = \begin{bmatrix} -2 & 3 & 2k & 0 \\ 2 & 3 & -2k & 0 \end{bmatrix}$ and this matrix has rank 2, so the lumped system is still completely controllable in this case.

5.2 Examples for the effect of lumping on observability

First, let us consider the observation system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

with observation matrix $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. So $V_{CA} = [C^\top \mid A^\top C^\top \mid (A^\top)^2 C^\top] = \begin{bmatrix} 1 & 0 & 0 & k & -k^2 & -2k^2 \\ 1 & 1 & -k & -k & 3k^2 & 3k^2 \\ 0 & 1 & k & 0 & -2k^2 & -k^2 \end{bmatrix}$ has rank 3, hence this system is completely observable. Since now $A^\top = A$, so $\tilde{M}$ can be chosen as M and $\tilde{A}$ as $\hat{A}$. Therefore $\hat{C} = C\tilde{M}^\top = CM^\top = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and the lumped system is

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & -\frac{k}{2} \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \quad \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}.$$

So $\hat{V}_{CA} = \begin{bmatrix} 1 & 3 & k & -k \\ 3 & 1 & -k & k \end{bmatrix}$ is of rank 2, therefore in this case the lumped system is also completely observable.

Second, let us cosider the observation system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e. now $C = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ and from this we get $\hat{C} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$. Since in this case the matrix $V_{CA} = \begin{bmatrix} 0 & 2 & k & -k & -k^2 & k^2 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -k & k & k^2 & -k^2 \end{bmatrix}$ has rank only 2, this system is not completely observable. At the same time the lumped system is

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & -\frac{k}{2} \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \quad \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}.$$

Therefore $\begin{bmatrix} 5 & 1 & -2k & 2k \\ 1 & 5 & 2k & -2k \end{bmatrix}$ and this matrix has rank 2, so the lumped system is still completely observable in this case.

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