Topology A (TOPA) / Alex Küronya / Spring 2010

Final exam

All problems are worth 10 points.

1. Let X be a regular topological space, $x, y \in X$ two different points. Prove that there exist open subsets $U, V \subseteq X$ such that $x \in U$,$y \in V$, and $\overline{U} \cap \overline{V} = \emptyset$.

2. Show that if for a continuous map $f : \mathbb{S}^2 \to \mathbb{S}^2$, $f(x) \neq f(-x)$ holds for every $x \in \mathbb{S}^2$, then f is surjective.

3. Let X be a compact topological space, $\{ A_\alpha \mid \alpha \in I \}$ an arbitrary collection of closed sets, which is closed with respect to finite intersections. If for an open set $U \subseteq X$ one has $\bigcap_\alpha A_\alpha \subseteq U$, then there exists $\alpha \in I$ for which $A_\alpha \subseteq U$.

4. Prove that if $p : X \to Y$ and $q : Y \to Z$ are covering maps, and $q^{-1}(z)$ is finite for every $z \in Z$, then $q \circ p$ is also a covering map.

5. Show that if $A \subseteq \mathbb{D}^2$ is a retract of $\mathbb{D}^2$, then every continuous map $f : A \to A$ has a fixed point.

6. Let $f : X \to Y$ be a map between topological spaces, $I \stackrel{\text{def}}{=} [0, 1]$. We define

$$\begin{aligned} f_0 : X \times \{0\} &\longrightarrow Y \\ (x, 0) &\mapsto f(x) \ . \end{aligned}$$

The *mapping cylinder of f* is the space

$$M_f \stackrel{\text{def}}{=} (X \times I) \cup_{f_0} Y \ .$$

Show that M_f is Hausdorff whenever X and Y are.

Bonus problem:

7. Prove that an arbitrary product of connected topological spaces is again connected.

Have fun!