1 Introduction

1.1 Motivation

Our goal is to understand Lazić's proof of the following theorem.

Theorem 1.1. *Let X be a smooth projective variety. Then the canonical ring*

$$R(X, K_X) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra ("is finitely generated", for short).

We want to give some reasons why this is an interesting result.

- When classifying projective varieties, one looks for a natural embedding in projective space. Such an embedding exists whenever we have a variety whose canonical divisor is ample. So we ask, given a variety X, is X birational to a variety with ample canonical divisor? For this to hold, X needs to be of general type, meaning that K_X is big. Conversely, given a smooth projective variety X of general type with finitely generated canonical ring, we can consider the variety

  $$X_{\text{can}} := \operatorname{Proj} R(X, K_X),$$

  called the *canonical model* of X. One can show that X_{can} is a normal variety with canonical singularities which is birational to X and whose canonical divisor $K_{X_{\text{can}}}$ is ample. It is also possible to describe explicitly a birational map $X \dashrightarrow X_{\text{can}}$: Choose a positive integer r such that the r-th Veronese subring $R(X, rK_X)$ is generated by $H^0(X, rK_X)$. Then the complete linear system $|rK_X|$ defines a rational map

  $$X \dashrightarrow \mathbb{P}(H^0(X, rK_X)^*)$$

  which is birational onto its image, and the image is precisely X_{can}.

- For any variety, the canonical ring is one of two rings naturally attached to it (the Cox ring is the other). Therefore it is natural to ask about its properties.

- In the minimal model program, when one encounters a contraction $f : X \to Y$ whose exceptional locus has codimension ≥ 2, it is not possible to contract further because Y is too singular (K_Y is not $\mathbb{Q}$-Cartier). Instead, one would like to replace f by its *flip*, which is a variety X^+ together with a morphism $f^+ : X^+ \to Y$ such that K_{X^+} is f^+-ample. The existence of flips was a big problem in minimal model theory, but it is easy to see that it is equivalent to the finite generation of the *relative canonical ring*

$$\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mK_X),$$

which in turn follows from Theorem 1.1.

1.2 Main statement

In fact, Lazić proves a theorem slightly different from Theorem 1.1:

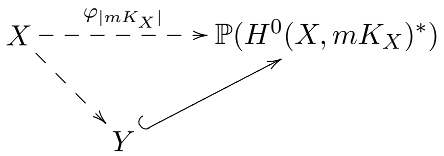
Theorem A. *Let X be a smooth projective variety of dimension n. Let $B_1, \dots, B_k$ be $\mathbb{Q}$-divisors on X such that $\lfloor B_i \rfloor = 0$ for all i, and such that the support of $\sum_{i=1}^k B_i$ has simple normal crossings. Let A be an ample $\mathbb{Q}$-divisor on X, and denote $D_i = K_X + A + B_i$ for every i.*

Then the adjoint ring

$$R(X; D_1, \dots, D_k) = \bigoplus_{(m_1,\dots,m_k)\in\mathbb{N}^k} H^0\big(X, \mathcal{O}_X\big(\lfloor \sum m_i D_i \rfloor\big)\big)$$

is finitely generated.

Let us indicate briefly how to deduce Theorem 1.1 from Theorem A. If X is a smooth projective variety with $\kappa(X, K_X) \geq 0$, we have the Iitaka fibration



defined by the sections of a suitable multiple of K_X. Now Fujino and Mori have shown that there exists an effective divisor Δ on Y such that (Y, Δ) is klt and $R(X, K_X)$ and $R(Y, K_Y + \Delta)$ have isomorphic Veronese subrings. So the former is finitely generated if and only if the latter is. But note that

$$\kappa(Y, K_Y + \Delta) = \kappa(X, K_X) = \dim Y,$$

which means that $K_Y + \Delta$ is big. Thus we may write $K_Y + \Delta \sim_{\mathbb{Q}} A + B$ with A ample and $B \geq 0$. For sufficiently small rational $\varepsilon > 0$, set $\Delta' = \varepsilon A + (\Delta + \varepsilon B)$. Then $K_Y + \Delta' \sim_{\mathbb{Q}} (1+\varepsilon)(K_Y + \Delta)$, so $R(Y, K_Y + \Delta)$ and $R(Y, K_Y + \Delta')$ have isomorphic Veronese subrings, hence it suffices to prove that $R(Y, K_Y + \Delta')$ is finitely generated. Since (Y, Δ') is klt, this follows from the $k = 1$ case of Theorem A (after passing to a log resolution).

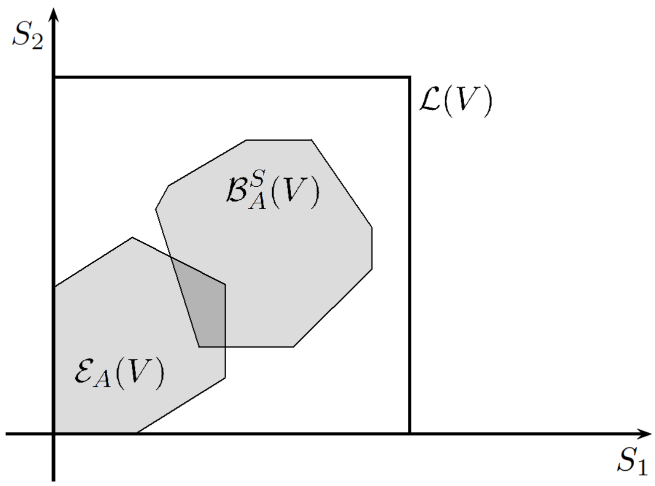


Figure 1: The polytopes of Definition 1.2 in the case $p = 2$.

1.3 Some definitions

Before we can give an outline of the proof of Theorem A, we need some definitions.

Definition 1.2. Let $(X, S+\sum_{i=1}^p S_i)$ be a log smooth projective pair, where S and all S_i are distinct prime divisors, let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let A be a $\mathbb{Q}$-divisor on X. We define

$$\begin{aligned} \mathcal{L}(V) &= \{B = \textstyle\sum b_i S_i \in V \mid 0 \le b_i \le 1 \text{ for all } i\}, \\ \mathcal{E}_A(V) &= \{B \in \mathcal{L}(V) \mid |K_X + A + B|_{\mathbb{R}} \neq \emptyset\}, \\ \mathcal{B}_A^S(V) &= \{B \in \mathcal{L}(V) \mid S \not\subseteq \mathbf{B}(K_X + S + A + B)\}. \end{aligned}$$

These definitions are best remembered by noting that:

$\mathcal{L}(V)$ is defined by the condition that a certain pair is log canonical,

$\mathcal{E}_A(V)$ by the condition that a certain divisor is effective, and

$\mathcal{B}_A^S(V)$ by the condition that a certain stable base locus does not contain S.

Also note that in the definition of $\mathcal{B}_A^S(V)$, one considers $K_X+S+A+B$ instead of $K_X + A + B$ because one would like to apply the adjunction formula. For example, if $B \in \mathcal{B}_A^S(V)$ then $B|_S \in \mathcal{E}_{A|_S}(W)$, where $W = \sum_{i=1}^p \mathbb{R}S_i|_S \subseteq \mathrm{Div}_{\mathbb{R}}(S)$.

It is clear that $\mathcal{L}(V)$ is a rational polytope (with respect to the canonical basis of V). Indeed, it is simply given by the "hypercube" $[0,1]^p$. On the other hand, $\mathcal{E}_A(V)$ and $\mathcal{B}_A^S(V)$ are a priori only bounded convex sets. A precise analysis of their structure is one of the main ingredients to the proof.

1.4 Outline of the proof

The proof of Theorem A is by induction on the dimension of X. As part of the induction, the following theorems are additionally proven.

Theorem B. *Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where $S_1, \dots, S_p$ are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_\mathbb{R}(X)$, and let A be an ample $\mathbb{Q}$-divisor on X.*

Then $\mathcal{E}_A(V)$ is a rational polytope.

Theorem 1.3. *Let $(X, S+\sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_\mathbb{R}(X)$ and let A be an ample $\mathbb{Q}$-divisor on X. Then $\mathcal{B}_A^S(V)$ is a rational polytope, and*

$$\mathcal{B}_A^S(V) = \{B \in \mathcal{L}(V) \mid \sigma_S(K_X + S + A + B) = 0\}.$$

The precise structure of the induction is the following.

- Theorems A_{n-1} and B_{n-1} imply Theorem 1.3_n (see Section 3).
- Theorems A_{n-1}, B_{n-1} and 1.3_n imply Theorem B_n (see Section 4).
- Theorems A_{n-1} and B_n imply Theorem A_n (see Section 5).

Here and elsewhere, "Theorem A_n" means "Theorem A in the case $\dim X = n$", and so on.

2 The lifting lemma

The following theorem, which is known as the "lifting lemma", is crucial to the proof. It was originally proved by Hacon and McKernan. In a nutshell, it gives a sufficient condition for sections of an adjoint line bundle $K_S + \cdots$ on a divisor $S \subset X$ to be liftable to sections of $K_X + S + \cdots$. The condition says that the sections we would like to lift need to vanish along certain divisors (to certain orders).

Theorem 2.1. *Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_\mathbb{R}(X)$, and let $B \in \mathcal{L}(V)$ be a $\mathbb{Q}$-divisor such that $(S, B_{|S})$ is canonical. Let A be an ample $\mathbb{Q}$-divisor on X and denote $\Delta = S + A + B$. Let $C \geq 0$ be a $\mathbb{Q}$-divisor on S, and let m be a positive integer such that mA, mB and mC are integral.*

Assume that there exists a positive integer $q \gg 0$ such that qA is very ample, $S \not\subseteq \mathrm{Bs}|qm(K_X + \Delta + \frac{1}{m}A)|$ and

$$C \leq \max\left\{B_{|S} - \frac{1}{qm}\,\mathrm{Fix}\,|qm(K_X + \Delta + \tfrac{1}{m}A)|_S,\ 0\right\}.$$

(The maximum is taken component-wise.) Then

$$|m(K_S + A_{|S} + C)| + m(B_{|S} - C) \subseteq |m(K_X + \Delta)|_S.$$

In particular, if $|m(K_S + A_{|S} + C)| \neq \emptyset$*, then* $|m(K_X + \Delta)|_S \neq \emptyset$*, and*

$$\operatorname{Fix}|m(K_S+A_{|S}+C)|+m(B_{|S}-C) \geq \operatorname{Fix}|m(K_X+\Delta)|_S \geq m\,\mathbf{Fix}_S(K_X+\Delta).$$

The following is an immediate consequence of the lifting lemma. Here we see very clearly what is happening: If a divisor $D \in |m(K_S + A|_S + B|_S)|$ is liftable to $|m(K_X + S + A + B)|$, then $D \geq \operatorname{Fix}|m(K_X + S + A + B)|_S$ by definition. In particular, $D \geq m(B|_S - \Phi_m)$. The lifting lemma says this necessary condition is also sufficient.

Corollary 2.2. *Let* $(X, S + \sum_{i=1}^p S_i)$ *be a log smooth projective pair, where* S *and all* S_i *are distinct prime divisors. Let* $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$*, and let* $B \in \mathcal{L}(V)$ *be a* $\mathbb{Q}$*-divisor such that* $(S, B_{|S})$ *is canonical. Let* A *be an ample* $\mathbb{Q}$*-divisor on* X *and denote* $\Delta = S+A+B$*. Let* m *be a positive integer such that* mA *and* mB *are integral, and such that* $S \not\subseteq \operatorname{Bs}|m(K_X + \Delta)|$*. Denote* $\Phi_m = \max\left\{B_{|S} - \frac{1}{m}\operatorname{Fix}|m(K_X + \Delta)|_S, 0\right\}$*.*

Then

$$|m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) = |m(K_X + \Delta)|_S.$$

3 Proof of Theorem 1.3_n

Let $||\cdot||$ be any norm on $\mathbb{R}^n$. In the applications, $||\cdot||$ will mostly be the sup-norm. This has the advantage that a closed ball of rational radius around a rational point is a rational polytope.

The following result is a very simple example of Diophantine approximation.

Lemma 3.1. *Let* $x \in \mathbb{R}^n$ *be a point, and fix a real number* $\varepsilon > 0$*. Then there are finitely many points* $x_i \in \mathbb{R}^n$ *and positive integers* k_i *such that* $k_i x_i$ *are integral,* $||x - x_i|| < \varepsilon/k_i$*, and* x *is a convex combination of the* x_i*.*

The next lemma gives a criterion for a set to be a rational polytope.

Lemma 3.2. *Let* $\mathcal{P} \subset \mathbb{R}^n$ *be a bounded convex set. Then* $\mathcal{P}$ *is a rational polytope if and only if there is a constant* $\varepsilon > 0$ *such that for all* $w \in \mathcal{P}$*,* $v \in \mathbb{Q}^n$*, and* $\ell \in \mathbb{N}^+$ *with* ℓv *integral and* $||v - w|| < \varepsilon/\ell$*, we have* $v \in \mathcal{P}$*.*

Setup 3.3. Let $(X, S+\sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, let A be an ample $\mathbb{Q}$-divisor on X, and let $W \subseteq \operatorname{Div}_{\mathbb{R}}(S)$ be the subspace spanned by the components of $\sum S_{i|S}$. For $\mathbb{Q}$-divisors $E \in \mathcal{E}_{A_{|S}}(W)$ and $B \in \mathcal{B}_A^S(V)$, let

$$\mathbf{F}(E) = \mathbf{Fix}(K_S + A_{|S} + E) \quad \text{and} \quad \mathbf{F}_S(B) = \mathbf{Fix}_S(K_X + S + A + B).$$

Denote

$$\Phi_m(B) = \max\left\{B_{|S} - \tfrac{1}{m}\operatorname{Fix}|m(K_X + S + A + B)|_S,\, 0\right\}$$

for every sufficiently divisible positive integer m, and let

$$\mathbf{\Phi}(B) = \max\{B_{|S} - \mathbf{F}_S(B),\, 0\}.$$

Note that $\mathbf{\Phi}(B) = \limsup \Phi_m(B)$.

Theorem 3.4. *Assume Theorem A_{n-1} and Theorem B_{n-1}, and let the assumptions of Setup 3.3$_n$ hold. Let $\mathcal{G}$ be a rational polytope contained in the interior of $\mathcal{L}(V)$, and assume that $(S, G_{|S})$ is terminal for every $G \in \mathcal{G}$. Denote $\mathcal{P} = \mathcal{G} \cap \mathcal{B}_A^S(V)$. Then:*

1. *$\mathcal{P}$ is a rational polytope,*
2. *$\mathbf{\Phi}$ extends to a rational piecewise affine function on $\mathcal{P}$, and there exists a positive integer ℓ such that $\mathbf{\Phi}(P) = \Phi_m(P)$ for every $P \in \mathcal{P}$ and every positive integer m such that mP/ℓ is integral.*

Corollary 3.5. *Theorem A_{n-1} and Theorem B_{n-1} imply Theorem 1.3$_n$.*

4 Proof of Theorem B$_n$

Lemma 4.1. *Let (X, B) be a log smooth projective pair, where B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$, and let A be a nef and big $\mathbb{Q}$-divisor.*

If $K_X + A + B$ is numerically equivalent to an effective $\mathbb{R}$-divisor, then it is also linearly equivalent to an effective $\mathbb{Q}$-divisor.

Sketch of proof. An application of Kawamata-Viehweg vanishing tells us that h^0 of certain divisors equals their Euler characteristic. Then use the fact that the Euler characteristic is a numerical invariant. □

The next lemma is in some sense complementary to Lemma 4.1. Both lemmas put together say that if (X, Δ) is klt, Δ is big, and $K_X + \Delta$ is pseudoeffective, then $K_X + \Delta$ is ($\mathbb{R}$-linearly) effective.

Lemma 4.2. *Assume Theorems A_{n-1} and B_{n-1}.*

Let (X, B) be a log smooth projective pair, where B is an $\mathbb{R}$-divisor such that $\lfloor B \rfloor = 0$. Let A be an ample $\mathbb{Q}$-divisor on X, and assume that $K_X + A + B$ is a pseudo-effective divisor such that $K_X + A + B \not\equiv N_\sigma(K_X + A + B)$.

Then there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X + A + B \sim_{\mathbb{R}} F$.

Proof. Set $\Delta = A + B$. By the assumption $K_X + \Delta \not\equiv N_\sigma(K_X + \Delta)$, there is a number $k \in \mathbb{N}^+$ such that kA is integral and

$$h^0(X, \lfloor m(K_X + \Delta)\rfloor + kA) \to \infty$$

as $m \to \infty$. In particular, we find an $m \in \mathbb{N}^+$ with

$$h^0(X, \lfloor mk(K_X + \Delta)\rfloor + kA) > \binom{n + nk}{n}.$$

Since the right-hand side is the number of conditions that a given divisor has multiplicity $> nk$ at a certain point fixed in advance, we obtain an effective divisor $G \sim_{\mathbb{R}} mk(K_X + \Delta) + kA$ with $\mathrm{mult}_x G > nk$, for some $x \notin \mathrm{Supp}\, N_\sigma(K_X + \Delta)$.

Set $D := \frac{1}{mk}G$ and consider a log resolution $f : Y \to X$ of $(X, B + D)$ constructed by first blowing up x, giving an exceptional divisor $P \subset Y$. For any $0 \le t \le m$, the ramification formula reads

$$K_Y + C_t = f^*(K_X + B + tD) + E_t,$$

where C_t, E_t are effective and do not have any common components. Now define

$$B_t := \max\{C_t - N_\sigma(K_Y + f^*A_t + C_t), 0\},$$

where $A_t := \left(1 - \frac{t}{m}\right)A$. We make a few observations:

1) $N_\sigma(K_Y + f^*A_t + C_t) = (1+t)N_\sigma(f^*(K_X + \Delta)) + E_t$, so B_t is continuous as a function of t.

2) $\min\{B_t, N_\sigma(K_Y + f^*A_t + B_t)\} = 0$.

3) $\lfloor B_0\rfloor = 0$, but $\mathrm{mult}_P B_m > 1$.

By 1) and 3), there is a minimal $0 < \lambda < m$ such that $\mathrm{mult}_S B_\lambda = 1$ for some prime divisor S. Then by 2), $\sigma_S(K_Y + f^*A_\lambda + B_\lambda) = 0$. Now Theorem 1.3_n tells us that

$$S \not\subset \mathbf{B}(K_Y + S + f^*A_\lambda + (B_\lambda - S)),$$

in particular,

$$K_Y + f^*A_\lambda + B_\lambda \sim_{\mathbb{R}} F' \ge 0$$

for some effective divisor F'. (Here we are actually cheating a little because f^*A_λ is not ample, as required in Theorem 1.3_n, but only big and nef. In reality one needs to subtract a small effective exceptional divisor from f^*A_λ to make things work.) The latter linear equivalence may be pushed down to X, giving

$$K_X + \Delta \sim_{\mathbb{R}} \frac{1}{1+\lambda} f_*(F' + C_\lambda - B_\lambda) \ge 0.$$

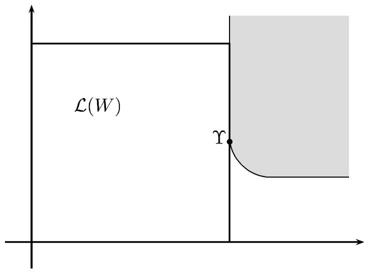
This is what we wanted to show. □

The following is an easy property of rational polyhedral cones.

Lemma 4.3. *Let $\mathscr{C} \subset \mathbb{R}^n$ be a rational polyhedral cone. If $(x_m) \subset \mathscr{C}$ is a sequence converging to $x \in \mathscr{C}$, then there exists an $\varepsilon > 0$ such that for all $m \gg 0$,*

$$x_m + \varepsilon(x_m - x) \in \mathscr{C}.$$

The next lemma says that in a certain situation, the pseudo-effective cone has a similar property. In particular, the (closed, convex) set $\{\Upsilon' \in W \mid K_X + A + \Upsilon'$ is pseudo-effective$\}$ cannot have a "circular part" like the shaded region in the following picture.



Lemma 4.4. *Assume Theorems A_{n-1} and B_{n-1}.*

Let $(X, S + S_1 + \cdots + S_p)$ be a log smooth projective pair of dimension n, A an ample $\mathbb{Q}$-divisor on X, $W = \langle S, S_1, \ldots, S_p\rangle_{\mathbb{R}}$, and assume $\Upsilon \in \mathcal{L}(W), \Upsilon_m \in W$ are divisors such that

- $\Upsilon_m \to \Upsilon$,
- *there is* $0 \leq \Sigma \in W$ *such that* $K_X + A + \Upsilon \sim_{\mathbb{R}} \Sigma$,
- *all* $K_X + A + \Upsilon_m$ *are pseudo-effective.*

Assume furthermore the following technical conditions:

- $\mathrm{mult}_S \Upsilon = 1$,
- $\mathrm{mult}_S \Sigma > 0$,
- $\sigma_S(K_X + A + \Upsilon) = 0$.

Then there exists an $\varepsilon > 0$ such that for infinitely many m,

$$K_X + A + \Upsilon_m + \varepsilon(\Upsilon_m - \Upsilon)$$

is pseudo-effective.

Sketch of proof. Set $V = \langle S_1, \dots, S_p \rangle_{\mathbb{R}} \subset W$ and $\Sigma_m = \Sigma + Y_m - Y$. Then $\Sigma_m \to \Sigma$ and the Σ_m are pseudo-effective. Let $Z \in V$ and $0 < \varepsilon \ll 1$ be such that $Y - \varepsilon Z - S$ is in the interior of $\mathcal{L}(V)$ and $A' = A + \varepsilon Z$ is still ample. Define

$$\mathcal{P} = \Sigma - (Y - \varepsilon Z - S) + \mathcal{B}^S_{A'}(V) \subset W,$$

and let $\mathcal{D} = \mathbb{R}_+ \cdot \mathcal{P} \subset W$ be the cone over $\mathcal{P}$. By Theorem 1.3_n, $\mathcal{D}$ is a rational polyhedral cone. And since $\Sigma - (Y - \varepsilon Z - S) \sim_{\mathbb{R}} K_X + S + A'$, all divisors in $\mathcal{D}$ are in particular pseudo-effective by the definition of $\mathcal{B}^S_{A'}(V)$.

We claim that setting $\Gamma_m = \Sigma_m - \sigma_S(\Sigma_m)S$, after passing to a subsequence we have $\Gamma_m \in \mathcal{D}$ for all m, and $\Gamma_m \to \Sigma$. We will not prove the claim here, but let us say how to finish the proof assuming the claim. By Lemma 4.3, there is an $\varepsilon > 0$ such that for $m \gg 0$, $\Psi_m = \Gamma_m + \varepsilon(\Gamma_m - \Sigma)$ is pseudo-effective (since it is in $\mathcal{D}$). Then

$$\Sigma'_m := \Sigma_m + \varepsilon(\Sigma_m - \Sigma) = \Psi_m + (1 + \varepsilon)(\Sigma_m - \Gamma_m)$$

is pseudo-effective too. Hence so is $K_X + A + \Upsilon_m + \varepsilon(\Upsilon_m - \Upsilon) \sim_{\mathbb{R}} \Sigma'_m$, which proves the lemma. □

Theorem 4.5. *Theorem A_{n-1} and Theorem B_{n-1} imply Theorem B_n.*

Sketch of proof. The proof is divided into five steps. Remember that the goal is to show that $\mathcal{E}_A(V)$ is a rational polytope.

1. Note that by Lemma 4.1, in the definition of $\mathcal{E}_A(V)$ we may replace linear equivalence by numerical equivalence without changing the set.

2. Show that $\mathcal{E}_A(V)$ is closed (apply Lemma 4.2).

3. Show that $\mathcal{E}_A(V)$ is locally a polytope, i.e. extreme points don't accumulate (apply Lemma 4.4).

4. By compactness, $\mathcal{E}_A(V)$ is then a polytope.

5. Show that $\mathcal{E}_A(V)$ is a *rational* polytope (because the set of numerically trivial divisors in V is a rational subspace of V).

The second step is by far the most difficult one. □

5 Proof of Theorem $\mathbf{A}_n$

The idea of Lazić's proof that Theorem A_{n-1} and Theorem B_n imply Theorem A_n goes back to Shokurov's proof of the existence of "pl-flips". This may roughly be described as follows: Start with a log smooth plt pair $(X, S+B)$, where S is a prime divisor, $\lfloor B \rfloor = 0$, and $S \sim_{\mathbb{Q}} r(K_X + S + B)$ for some rational $r > 0$. Then:

- Show that the restricted algebra $\operatorname{Res}_S R(X, K_X + S + B)$ is finitely generated. This is accomplished by finding a suitable divisor C on S such that

$$\operatorname{Res}_S R(X, K_X + S + B) \cong R(S, K_S + C).$$

By induction on the dimension, the latter ring is finitely generated, hence so is the former. — One might hope naively that taking $C = B|_S$ would do. However, it doesn't. This is where lifting lemmas come in.

- Conclude finite generation of $R(X, K_X+S+B)$ from that of $\operatorname{Res}_S R(X, K_X + S + B)$. This step is actually much easier: By the assumption $S \sim_{\mathbb{Q}} r(K_X + S + B)$, we only need to conclude finite generation of $R(X, S)$ from that of $\operatorname{Res}_S R(X, S)$. But it is easy to see that if

$$\sigma_1, \dots, \sigma_\ell \in R(X, S)$$

are sections such that $\sigma_1|_S, \dots, \sigma_\ell|_S$ generate $\operatorname{Res}_S R(X, S)$, and $\sigma \in H^0(X, \mathcal{O}_X(S))$ is a section whose zero divisor is exactly S, then the set $\{\sigma, \sigma_1, \dots, \sigma_\ell\}$ generates $R(X, S)$.

Comparing Lazić's proof to the special case done by Shokurov, the first step is very similar: apply a lifting lemma and do induction on the dimension. However, in the second step we cannot use an assumption like $S \sim_{\mathbb{Q}} r(K_X + S + B)$, because this held only by a "relative Picard number one" argument, but in general the Picard number may be arbitrarily large. So $H^0(X, \mathcal{O}_X(S)) \not\subset R(X, K_X + S + B)$ and we cannot take an extra generator σ like Shokurov did. In order to remedy this situation, we should consider a bigger ring, like $R(X; K_X + S + B, S)$. Note that this ring is graded over $\mathbb{N}^2$ instead of $\mathbb{N}$. This was Corti's original idea: that sometimes higher rank grading is better than rank one.

The first step in the strategy outlined above is carried out by the following lemma.

Lemma 5.1. *Assume Theorems A_{n-1} and Theorem B_{n-1}.*

Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, let A be an ample $\mathbb{Q}$-divisor on X, let $B_1, \dots, B_m \in \mathcal{E}_{S+A}(V)$ be $\mathbb{Q}$-divisors, and denote $D_i = K_X + S + A + B_i$.

Then the ring $\operatorname{Res}_S R(X; D_1, \dots, D_m)$ is finitely generated.

A special case of the second step is contained in the next theorem.

Theorem 5.2. *Assume Theorems A_{n-1} and Theorem B_{n-1}.*

Let $(X, S_1 + S_2)$ be a log smooth projective pair of dimension n, where S_1 and S_2 are distinct prime divisors. Let B be a $\mathbb{Q}$-divisor with $\lfloor B \rfloor = 0$ which is supported on $S_1 + S_2$, and let A be an ample $\mathbb{Q}$-divisor. Assume that $K_X + A + B \sim_{\mathbb{Q}} D$ for some effective $\mathbb{Q}$-divisor D supported on $S_1 + S_2$.

Then the ring $R(X, K_X + A + B)$ is finitely generated.

Obviously, Theorem 5.2 stops short of proving Theorem A_n in full generality. We indicate how to get rid of the remaining extra assumptions.

- If we have an arbitrary number of components $S_1, \ldots, S_p$ instead of two, the proof goes exactly the same way. It is just more difficult to draw pictures.

- If $K_X + A + B$ is not $\mathbb{Q}$-linearly equivalent to any effective divisor, its section ring is trivial, so we are done anyway. If $K_X + A + B \sim_{\mathbb{Q}} D \geq 0$, we pass to a log resolution of $(X, S_1 + \cdots + S_p + D)$, where we may assume that the support of D is snc.

- If we have an arbitrary number of divisors $B_1, \ldots, B_k$ instead of just one, the proof is again similar to the one above, but a bit more complicated. In particular, the second point of this list (showing that we may assume all $K_X + A + B_i$ to be effective) is not so easy, and we have to apply Theorem B_n.

Altogether, this shows the following.

Theorem 5.3. *Theorem A_{n-1} and Theorem B_n imply Theorem A_n.*