Equivalent operator preconditioning for elliptic finite element problems

János Karátson*, Owe Axelsson

* Dept. Appl. Anal. Comp. Math. & Numnet Research Group; ELTE Univ. & Dept. Anal., Technical Univ.; Budapest, Hungary

October 17, 2016

Outline of the talk

- Basic ideas
- Equivalent operators in Hilbert space
- Operator preconditioning for elliptic problems
  - Single equations
  - Systems (transport type, saddle-point)
  - Convection-dominated equations
  - Helmholtz problems

The problem of preconditioning

- Linear(ized) algebraic system: $\mathbf{Ac} = \mathbf{d}$.
- Preconditioning matrix: $\mathbf{B}$
  Preconditioned algebraic system: $\mathbf{B}^{-1}\mathbf{Ac} = \mathbf{B}^{-1}\mathbf{d}$.
  CG (CGN, GCG-LS, GMRES) iteration $\rightarrow$ auxiliary systems $\mathbf{Bz} = \mathbf{r}$
- Twofold goals:
  - Faster CG convergence $\rightarrow$ $\mathbf{B} \approx \mathbf{A}$
  - Low cost $\rightarrow$ $\mathbf{B} \approx \mathbf{I}$

  Conflicting goals $\rightarrow$ a compromise needed.
- Various strategies mainly use algebraic structure of $\mathbf{A}$.

The problem of preconditioning

- Linear(ized) algebraic system: $\mathbf{A}\mathbf{c} = \mathbf{d}.$
- Preconditioning matrix: $\mathbf{B}$

  Preconditioned algebraic system: $\mathbf{B}^{-1}\mathbf{A}\mathbf{c} = \mathbf{B}^{-1}\mathbf{d}.$

  CG (CGN, GCG-LS, GMRES) iteration $\to$ auxiliary systems $\mathbf{B}\mathbf{z} = \mathbf{r}$
- Twofold goals:
  - Faster CG convergence $\to$ $\mathbf{B} \approx \mathbf{A}$
  - Low cost $\to$ $\mathbf{B} \approx \mathbf{I}$

  Conflicting goals $\to$ a compromise needed.
- Various strategies mainly use algebraic structure of $\mathbf{A}$.

Equivalent operator preconditioning

The problem: discretized linear elliptic PDE, using FEM:

$$\mathbf{L}_h \mathbf{c} = \mathbf{d} \qquad (SLAE)$$

Disadvantage: $cond(\mathbf{L}_h) \to \infty$ as $h \to 0$.

Advantage: for certain PDEs, (SLAE) can be solved optimally or quasi-optimally,
i.e. with $O(n)$ or $O(n \log n)$ operations.

E.g.: such problems: symmetric elliptic equations,
equations with constant coefficients;
such methods: multigrid, FFT.

Equivalent operator preconditioning

The problem: discretized linear elliptic PDE, using FEM:

$$\mathbf{L}_h \mathbf{c} = \mathbf{d} \qquad (SLAE)$$

Disadvantage: $cond(\mathbf{L}_h) \to \infty$ as $h \to 0$.

Advantage: for certain PDEs, (SLAE) can be solved optimally or quasi-optimally,
i.e. with $O(n)$ or $O(n \log n)$ operations.

E.g.: such problems: symmetric elliptic equations,
equations with constant coefficients;
such methods: multigrid, FFT.

Equivalent operator preconditioning

The problem: discretized linear elliptic PDE, using FEM:

$$\mathbf{L}_h \mathbf{c} = \mathbf{d} \qquad (SLAE)$$

Disadvantage: $cond(\mathbf{L}_h) \to \infty$ as $h \to 0$.

Advantage: for certain PDEs, (SLAE) can be solved optimally or quasi-optimally,
i.e. with $O(n)$ or $O(n \log n)$ operations.

E.g.: such problems: symmetric elliptic equations,
equations with constant coefficients;
such methods: multigrid, FFT.

Equivalent operator preconditioning

The problem: discretized linear elliptic PDE, using FEM:

$$\mathbf{L}_h \mathbf{c} = \mathbf{d} \qquad (SLAE)$$

Disadvantage: $cond(\mathbf{L}_h) \to \infty$ as $h \to 0$.

Advantage: for certain PDEs, (SLAE) can be solved optimally or quasi-optimally,
i.e. with $O(n)$ or $O(n \log n)$ operations.

E.g.: such problems: symmetric elliptic equations,
equations with constant coefficients;
such methods: multigrid, FFT.

More general problems: nonsymmetric eqns; systems; parameter-dependent problems.

Equivalent operator preconditioning

Proposal: let S be *another elliptic operator*, such that systems $\mathbf{S}_h \mathbf{z} = \mathbf{r}$ can be solved (quasi-)optimally.

Preconditioning matrix: $\mathbf{S}_h$.

CG iteration for system $\mathbf{S}_h^{-1} \mathbf{L}_h \mathbf{c} = \mathbf{S}_h^{-1} \mathbf{d}$

↓

if the convergence is *mesh-independent*, then the original problem is solved also (quasi-)optimally (since $const. \cdot O(n) = O(n)$)

Equivalent operator preconditioning

Proposal: let S be *another elliptic operator*, such that systems $\mathbf{S}_h \mathbf{z} = \mathbf{r}$ can be solved (quasi-)optimally.

Preconditioning matrix: $\mathbf{S}_h$.

CG iteration for system $\mathbf{S}_h^{-1} \mathbf{L}_h \mathbf{c} = \mathbf{S}_h^{-1} \mathbf{d}$

$\downarrow$

if the convergence is *mesh-independent*, then the original problem is solved also (quasi-)optimally (since $const. \cdot O(n) = O(n)$)

Equivalent operator preconditioning

Proposal: let S be *another elliptic operator*, such that systems $\mathbf{S}_h\mathbf{z} = \mathbf{r}$ can be solved (quasi-)optimally.

Preconditioning matrix: $\mathbf{S}_h$.

CG iteration for system $\mathbf{S}_h^{-1}\mathbf{L}_h\mathbf{c} = \mathbf{S}_h^{-1}\mathbf{d}$

$\downarrow$

if the convergence is *mesh-independent*, then the original problem is solved also (quasi-)optimally (since $const. \cdot O(n) = O(n)$)

Equivalent operator preconditioning

Theory of mesh-independent *linear* convergence:

Various early and later works (1966 to present)

[Dyakonov, Gunn, Concus, Golub, Elman, Widlund, Cao, Hiptmair, Mardal, Winther...

T. Manteuffel, Goldstein, Faber, Parter, Otto]

$\rightarrow$ a solid theoretical framework:
theory of *equivalent operators in Hilbert space*

Under proper assumptions:

if $L \sim S \quad \Rightarrow \quad$ mesh independent linear convergence

Equivalent operator preconditioning

Theory of mesh-independent *linear* convergence:

Various early and later works (1966 to present)

[Dyakonov, Gunn, Concus, Golub, Elman, Widlund, Cao, Hiptmair, Mardal, Winther...
T. Manteuffel, Goldstein, Faber, Parter, Otto]

$\rightarrow$ a solid theoretical framework:
theory of *equivalent operators in Hilbert space*

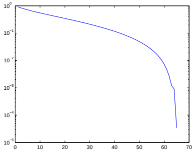
Under proper assumptions:

if $L \sim S \quad \Rightarrow \quad$ mesh independent linear convergence

Equivalent operator preconditioning

Compact-equivalent operators in Hilbert space

- Motivation: CG convergence history

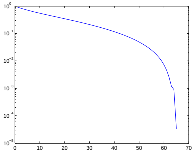


2 phases: linear conv. – superlin. conv.

Equivalent operator preconditioning

Compact-equivalent operators in Hilbert space

- Motivation: CG convergence history

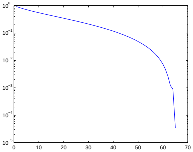


mesh independence: $\underbrace{\text{linear conv.}}_{\text{equivalent op.-s}}$ – superlin. conv.

Equivalent operator preconditioning

Compact-equivalent operators in Hilbert space

- Motivation: CG convergence history



mesh independence: $\underbrace{\text{linear conv.}}_{\text{equivalent op.-s}}$ – $\underbrace{\text{superlin. conv.}}_{?}$

Equivalent operator preconditioning

Compact-equivalent operators in Hilbert space

(theory of mesh-independent superlinear convergence, [Axelsson-Karátson, SIAM J. Numer. Anal. 2007]).

Let L and N be unbounded coercive operators in a Hilbert space, let L_S and N_S be their suitable weak forms in an energy space H_S.

Def. L and N are compact-equivalent if

$L_S = \mu N_S + Q_S$, where $\mu > 0$ and Q_S is compact.

Equivalent operator preconditioning

Compact-equivalent operators in Hilbert space

(theory of mesh-independent superlinear convergence, [Axelsson-Karátson, SIAM J. Numer. Anal. 2007]).

Let L and N be unbounded coercive operators in a Hilbert space, let L_S and N_S be their suitable weak forms in an energy space H_S.

Def. L and N are compact-equivalent if
$L_S = \mu N_S + Q_S$, where $\mu > 0$ and Q_S is compact.

Equivalent operator preconditioning

Compact-equivalent operators in Hilbert space

(theory of mesh-independent superlinear convergence,
[Axelsson-Karátson, SIAM J. Numer. Anal. 2007]).

Let L and N be unbounded coercive operators in a Hilbert space,
let L_S and N_S be their suitable weak forms in an energy space H_S.

Def. L and N are compact-equivalent if
$L_S = \mu N_S + Q_S$, where $\mu > 0$ and Q_S is compact.

Equivalent operator preconditioning

Compact-equivalent operators in Hilbert space

(theory of mesh-independent superlinear convergence,
[Axelsson-Karátson, SIAM J. Numer. Anal. 2007]).

Let L and N be unbounded coercive operators in a Hilbert space,
let L_S and N_S be their suitable weak forms in an energy space H_S.

Def. L and N are compact-equivalent if
$L_S = \mu N_S + Q_S$, where $\mu > 0$ and Q_S is compact.

Special case: if $N = S$ is symmetric then $L_S = \mu I + Q_S$.

Equivalent operator preconditioning

Compact-equivalent operators in Hilbert space

(theory of mesh-independent superlinear convergence,
[Axelsson-Karátson, SIAM J. Numer. Anal. 2007]).

Let L and N be unbounded coercive operators in a Hilbert space,
let L_S and N_S be their suitable weak forms in an energy space H_S.

Def. L and N are compact-equivalent if
$L_S = \mu N_S + Q_S$, where $\mu > 0$ and Q_S is compact.

Special case: if $N = S$ is symmetric then $L_S = I + Q_S$.
We may let $\mu = 1 \;\rightarrow\;$ compact perturbation of the identity.

Equivalent operator preconditioning

Theorem

If $L_S = I + Q_S$, then for any Galerkin subspace the CGN iteration for system $\mathbf{S}_h^{-1}\mathbf{L}_h\mathbf{c} = \mathbf{S}_h^{-1}\mathbf{d}$ satisfies

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \leq \ \varepsilon_k \qquad (k = 1, 2, ..., n),$$

where $\varepsilon_k \to 0$ is a sequence independent of V_h. In fact,

$$\varepsilon_k := \frac{2}{km^2} \sum_{i=1}^{k} \Big(\big|\lambda_i(Q_S^* + Q_S)\big| + \lambda_i(Q_S^* Q_S) \Big).$$

→ Mesh-independent superlinear convergence.

Elliptic equations

Case 1: *scalar elliptic operators.*

We consider elliptic operators

$$Lu \equiv -\operatorname{div}(A\,\nabla u) + \mathbf{b}\cdot\nabla u + cu$$

for $u_{|\Gamma_D} = 0, \quad \frac{\partial u}{\partial \nu_A} + \alpha u_{|\Gamma_N} = 0.$

Elliptic equations

Case 1: *scalar elliptic operators.*

We consider elliptic operators

$$Lu \equiv -\mathrm{div}(A\,\nabla u) + \mathbf{b}\cdot\nabla u + cu$$

$$\text{for} \quad u_{|\Gamma_D} = 0, \qquad \frac{\partial u}{\partial \nu_A} + \alpha u_{|\Gamma_N} = 0.$$

Elliptic equations

Assumptions 1. (standard for having H^1-*coercivity*)

(i) $\Omega \subset \mathbf{R}^d$ is a bounded piecewise C^1 domain; Γ_D, Γ_N are disjoint open measurable subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$;

(ii) $A \in L^\infty(\overline{\Omega}, \mathbf{R}^{d\times d})$ and for all $x \in \overline{\Omega}$ the matrix $A(x)$ is symmetric; further, $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$, $\alpha \in L^\infty(\Gamma_N)$;

(iii) we have the following properties which will imply coercivity:
$\exists p > 0$: $\quad A(x)\xi \cdot \xi \geq p\,|\xi|^2 \qquad (\forall x \in \overline{\Omega},\ \xi \in \mathbf{R}^d)$;
$\hat{c} := c - \frac{1}{2}\,\mathrm{div}\mathbf{b} \geq 0$ in Ω, $\quad \hat{\alpha} := \alpha + \frac{1}{2}\,(\mathbf{b}\cdot\nu) \geq 0$ on Γ_N;

(iv) either $\Gamma_D \neq \emptyset$, or $\hat{c}$ or $\hat{\alpha}$ has a positive lower bound.

Elliptic equations

Characterization of compact-equivalence:

Theorem

Let the elliptic operators L_1 and L_2 satisfy Assumptions 1. Then L_1 and L_2 are compact-equivalent in $H^1_D(\Omega)$ if and only if their principal parts coincide up to some constant $\mu > 0$, i.e. $A_1 = \mu A_2$.

Elliptic equations

Freedom: choice of lower order coefficients.

Example: convection-diffusion operator

$$Lu \equiv -\Delta u + \mathbf{b}(x) \cdot \nabla u + c(x) u$$

$$\text{for} \quad u_{|\Gamma_D} = 0, \quad \frac{\partial u}{\partial \nu} + \alpha(x) u_{|\Gamma_N} = 0.$$

Elliptic equations

Freedom: choice of lower order coefficients.

Example: convection-diffusion operator

$$Lu \equiv -\Delta u + \mathbf{b}(x) \cdot \nabla u + c(x)u$$

$$\text{for} \quad u_{|\Gamma_D} = 0, \quad \frac{\partial u}{\partial \nu} + \alpha(x) u_{|\Gamma_N} = 0.$$

Preconditioning operator:

$$Su \equiv -\Delta u + \mathbf{w}(x) \cdot \nabla u + \sigma(x)u$$

$$\text{for} \quad u_{|\Gamma_D} = 0, \quad \frac{\partial u}{\partial \nu} + \beta(x) u_{|\Gamma_N} = 0.$$

Elliptic equations

Freedom: choice of lower order coefficients.

Example: convection-diffusion operator

$$Lu \equiv -\Delta u + \mathbf{b}(x) \cdot \nabla u + c(x)u$$

$$\text{for} \quad u_{|\Gamma_D} = 0, \quad \frac{\partial u}{\partial \nu} + \alpha(x) u_{|\Gamma_N} = 0.$$

Preconditioning operator: e.g.

$$Su \equiv -\Delta u + \qquad\qquad + \sigma \quad u$$

$$\text{for} \quad u_{|\Gamma_D} = 0, \quad \frac{\partial u}{\partial \nu} + \beta \quad u_{|\Gamma_N} = 0.$$

Elliptic equations

Freedom: choice of lower order coefficients.

Example: convection-diffusion operator

$$Lu \equiv -\Delta u + \mathbf{b}(x) \cdot \nabla u + c(x) u$$

$$\text{for} \quad u_{|\Gamma_D} = 0, \quad \frac{\partial u}{\partial \nu} + \alpha(x) u_{|\Gamma_N} = 0.$$

Preconditioning operator: e.g.

$$Su \equiv -\Delta u + \sigma u$$

$$\text{for} \quad u_{|\Gamma_D} = 0, \quad \frac{\partial u}{\partial \nu} + \beta u_{|\Gamma_N} = 0$$

$\rightarrow$ symmetric operator with constant coefficients.

Elliptic systems: saddle-point

Case 2: *Stokes problem*

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \end{cases}$$

with b.c. $\mathbf{u}_{|\partial\Omega} = 0.$

Elliptic systems: saddle-point

Case 2: *Stokes problem*

Regularized form:

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} \\ \operatorname{div} \mathbf{u} - \sigma \Delta p = \sigma \operatorname{div} \mathbf{f} \end{cases}$$

with b.c. $\mathbf{u}_{|\partial\Omega} = 0$, $\partial_\nu p_{|\partial\Omega} = 0$.

Elliptic systems: saddle-point

Case 2: *Stokes problem*

Preconditioning operator: auxiliary problems

$$\begin{cases} -\Delta \mathbf{u} \qquad\qquad = \dots \\ \qquad\quad -\sigma \Delta p = \dots \end{cases}$$

with b.c. $\mathbf{u}_{|\partial\Omega} = 0,\ \partial_\nu p_{|\partial\Omega} = 0.$

Independent Poisson equations.

FEM solution: mesh independent superlinear convergence.

Elliptic systems: saddle-point

Case 2: *Stokes problem*

Preconditioning operator: auxiliary problems

$$\begin{cases} -\Delta \mathbf{u} & = \dots \\ & -\sigma \Delta p = \dots \end{cases}$$

with b.c. $\mathbf{u}_{|\partial\Omega} = 0$, $\partial_\nu p_{|\partial\Omega} = 0$.

Independent Poisson equations.

FEM solution: mesh independent superlinear convergence.

Elliptic systems: saddle-point

Case 2: *Stokes problem*

Preconditioning operator: auxiliary problems

$$\begin{cases} -\Delta \mathbf{u} \qquad\qquad = \dots \\ \qquad\qquad -\sigma \Delta p = \dots \end{cases}$$

with b.c. $\mathbf{u}_{|\partial\Omega} = 0$, $\partial_\nu p_{|\partial\Omega} = 0$.

Independent Poisson equations.

FEM solution: mesh independent superlinear convergence.

Elliptic systems: transport problems

Case 3: *elliptic systems of transport type*

We consider ℓ-tuples of operators

$$L_i u \equiv -\operatorname{div}(A_i \nabla u_i) + \mathbf{b}_i \cdot \nabla u_i + \sum_{j=1}^{\ell} V_{ij} u_j \qquad (i = 1, \ldots, \ell)$$

for $\quad u_{i\,|\Gamma_D} = 0,$

$\frac{\partial u_i}{\partial \nu_A} + \alpha_i u_{i\,|\Gamma_N} = 0.$

Elliptic systems: transport problems

Case 3: *elliptic systems of transport type*

We consider ℓ-tuples of operators

$$L_i u \equiv -\mathrm{div}(A_i \nabla u_i) + \mathbf{b}_i \cdot \nabla u_i + \sum_{j=1}^{\ell} V_{ij} u_j \qquad (i = 1, \ldots, \ell)$$

for $\quad u_{i\,|\Gamma_D} = 0,$

$\frac{\partial u_i}{\partial \nu_A} + \alpha_i u_{i\,|\Gamma_N} = 0.$

Elliptic systems: transport problems

Case 3: *elliptic systems of transport type*

We consider ℓ-tuples of operators

$$L_i u \equiv -\mathrm{div}(A_i \nabla u_i) + \mathbf{b}_i \cdot \nabla u_i + \underbrace{\sum_{j=1}^{\ell} V_{ij} u_j}_{\text{coupling}} \qquad (i = 1, \ldots, \ell)$$

for $\quad u_{i\,|\Gamma_D} = 0,$

$\frac{\partial u_i}{\partial \nu_A} + \alpha_i u_{i\,|\Gamma_N} = 0.$ (matrix V)

Elliptic systems: transport problems

Assumptions:

(i) Ω, A_i, α_i as before

(ii) Smoothness: as before

(iii) Coercivity:

$$\lambda_{min}(V+V^T) - \max_i \operatorname{div}\mathbf{b}_i \geq 0.$$

For example: $\operatorname{div}\mathbf{b}_i = 0 \quad (\forall i)$, and
V is positive semidefinite.

Elliptic systems: transport problems

Assumptions:

(i) Ω, A_i, α_i as before

(ii) Smoothness: as before

(iii) Coercivity:

$$\lambda_{min}(V + V^T) - \max_i \operatorname{div} \mathbf{b}_i \geq 0.$$

For example: $\operatorname{div} \mathbf{b}_i = 0 \;\; (\forall i)$, and
V is positive semidefinite.

Elliptic systems: transport problems

Main idea of equivalent preconditioning:

define an ℓ-tuple of separate (i.e. independent) symmetric preconditioning operators

$$S_i u_i := -\operatorname{div}(A_i \nabla u_i) + \sigma_i u_i \qquad (i = 1, \dots, \ell)$$

$$\text{for} \quad u_{i\,|\Gamma_D} = 0, \quad \frac{\partial u_i}{\partial \nu_A} + \beta_i u_{i\,|\Gamma_N} = 0.$$

Elliptic systems: transport problems

For example: let the original operators be of the form

$$L_i u \equiv -\Delta u_i + \mathbf{b}_i \cdot \nabla u_i + \sum_{j=1}^{\ell} V_{ij} u_j$$

$(i = 1, \dots, \ell)$.

Elliptic systems: transport problems

For example: let the original operators be of the form

$$L_i u \equiv \underbrace{-\Delta u_i + \mathbf{b}_i \cdot \nabla u_i}_{N_i u_i} + \sum_{j=1}^{\ell} V_{ij} u_j = \quad N_i u_i + \sum_{j=1}^{\ell} V_{ij} u_j$$

$(i = 1, \ldots, \ell)$.

Elliptic systems: transport problems

Original PDE system:

$$\begin{cases} (N_1 + V_{11})u_1 + \quad V_{12}\ u_2 + \cdots + \quad V_{1\ell}\ u_\ell = g_1 \\ V_{21}\ u_1 + (N_2 + V_{22})u_2 + \cdots + \quad V_{2\ell}\ u_\ell = g_2 \\ \qquad \cdots\cdots \\ V_{\ell 1}\ u_1 + \quad V_{\ell 2}\ u_2 + \cdots + (N_\ell + V_{\ell\ell})u_\ell = g_\ell \end{cases}$$

$+$ b.c.

Elliptic systems: transport problems

Preconditioning $\rightarrow$ auxiliary PDE systems

$$\begin{cases} (-\Delta+\sigma_1)u_1 & = r_1 \\ \quad (-\Delta+\sigma_2)u_2 & = r_2 \\ \quad\quad \ldots\ldots & \\ \quad\quad\quad (-\Delta+\sigma_\ell)u_\ell & = r_\ell \end{cases}$$

$+$ b.c.

Elliptic systems: transport problems

According block structure of the stiffness matrices.

Original system:

$$\mathbf{L}_h = \begin{pmatrix} \mathbf{L}_h^{11} & \mathbf{L}_h^{12} & \dots & \dots & \mathbf{L}_h^{1\ell} \\ \mathbf{L}_h^{21} & \mathbf{L}_h^{22} & \dots & \dots & \mathbf{L}_h^{2\ell} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{L}_h^{\ell 1} & \mathbf{L}_h^{\ell 2} & \dots & \dots & \mathbf{L}_h^{\ell\ell} \end{pmatrix}$$

Elliptic systems: transport problems

According block structure of the stiffness matrices.

Auxiliary systems:

$$\mathbf{S}_h = \begin{pmatrix} \mathbf{S}_h^1 & 0 & \dots & \dots & 0 \\ 0 & \mathbf{S}_h^2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \mathbf{S}_h^l \end{pmatrix}$$

Elliptic systems: transport problems

According block structure of the stiffness matrices.

Auxiliary systems:

$\downarrow$

- Parallelizability
- Cost of solution $\sim$ cost of a single equation

Elliptic systems: transport problems

Example: a parabolic system in modeling air pollution.

- FEM + time discretization + Newton linearization: $\rightarrow$

  FEM solution of linear elliptic systems.

- preconditioning operators (independent, symmetric): incorporate the time-step:

$$S_i p_i := -K\,\Delta p_i + \frac{1}{\tau}\, p_i \qquad (i = 1, \dots, \ell).$$

Numerical results: mesh-independent superlinear convergence

Elliptic systems: transport problems

Example: a parabolic system in modeling air pollution.

- FEM + time discretization + Newton linearization: $\rightarrow$

  FEM solution of linear elliptic systems.

- preconditioning operators (independent, symmetric): incorporate the time-step:

$$S_i p_i := -K\,\Delta p_i + \frac{1}{\tau}\,p_i \qquad (i = 1, \ldots, \ell).$$

Numerical results: mesh-independent superlinear convergence

Convection-dominated equations

Case 4: *Convection-diffusion equations, convection-dominated case:* $\varepsilon \ll 1$,

$$\begin{cases} -\varepsilon \Delta u + \mathbf{w} \cdot \nabla u = g \\ u_{|\partial\Omega} = 0. \end{cases}$$

Assumptions (a simple model case):

(i) $\Omega \subset \mathbf{R}^n$ is a polyhedral domain.

(ii) $\mathbf{w} \in C^1(\overline{\Omega}, \mathbf{R}^n)$, $\operatorname{div} \mathbf{w} = 0$.

(iii) $g \in L^2(\Omega)$.

Convection-dominated equations

Case 4: *Convection-diffusion equations, convection-dominated case:* $\varepsilon \ll 1$,

$$\begin{cases} -\varepsilon \Delta u + \mathbf{w} \cdot \nabla u = g \\ u_{|\partial\Omega} = 0. \end{cases}$$

Assumptions (a simple model case):

(i) $\Omega \subset \mathbf{R}^n$ is a polyhedral domain.

(ii) $\mathbf{w} \in C^1(\overline{\Omega}, \mathbf{R}^n)$, $\operatorname{div} \mathbf{w} = 0$.

(iii) $g \in L^2(\Omega)$.

Convection-dominated equations

Streamline diffusion FEM (SDFEM):

- $V_h \subset H_0^1(\Omega)$ piecewise linear subspace;
- choose paramaters $\delta_k > 0$ on elements $T_k \in \mathcal{T}$;
- replace test functions: $v_h \;\to\; v_h + \delta_k \, \mathbf{w} \cdot \nabla v_h \qquad \text{on } T_k$

$\Rightarrow$ stabilized bilinear form

$$a_{SD}(u_h, v_h) := \int_\Omega \big(\varepsilon \nabla u_h \cdot \nabla v_h + (\mathbf{w} \cdot \nabla u_h) v_h\big) + \sum_{k=1}^{N} \delta_k \int_{T_k} (\mathbf{w} \cdot \nabla u_h)\,(\mathbf{w} \cdot \nabla v_h)$$

on $V_h \times V_h$.

Convection-dominated equations

Stabilized inner product: SD-inner product

$$\langle u_h, v_h \rangle_{SD} := \int_\Omega \varepsilon \nabla u_h \cdot \nabla v_h + \sum_{k=1}^{N} \delta_k \int_{T_k} (\mathbf{w} \cdot \nabla u_h)(\mathbf{w} \cdot \nabla v_h) .$$

$\Rightarrow$ stable lower coercivity bound:

$$a_{SD}(u_h, u_h) \geq \|u_h\|_{SD}^2 \qquad \text{(i.e. } m = 1 \text{).}$$

Convection-dominated equations

Preconditioned CG iteration for the SLAE:
apply operator preconditioning.

Preconditioner = stiffness matrix for the SD-inner product:

$$(\mathbf{S}_h)_{ij} = \langle \varphi_i, \varphi_j \rangle_{SD}$$

$\Rightarrow$ here $\langle u_h, v_h \rangle_{SD} = \int_\Omega (S_\varepsilon u_h)\, v_h$,

where $S_\varepsilon u := -\mathrm{div}\,(A_\varepsilon \nabla u)$ with $A_\varepsilon = \varepsilon I + \delta\, \mathbf{w} \cdot \mathbf{w}^T$

$\Rightarrow$ $\mathbf{S}_h$ comes from a discretized symmetric elliptic operator

$\Rightarrow$ optimal $O(N)$ solvers available (multigrid, multilevel)

Convection-dominated equations

Preconditioned CG iteration for the SLAE:
apply operator preconditioning.

Preconditioner = stiffness matrix for the SD-inner product:

$$(\mathbf{S}_h)_{ij} = \langle \varphi_i, \varphi_j \rangle_{SD}$$

$\Rightarrow$ here $\langle u_h, v_h \rangle_{SD} = \int_\Omega (S_\varepsilon u_h)\, v_h$,

where $S_\varepsilon u := -\mathrm{div}\left(A_\varepsilon \nabla u\right)$ with $A_\varepsilon = \varepsilon I + \delta\, \mathbf{w} \cdot \mathbf{w}^T$

$\Rightarrow$ $\mathbf{S}_h$ comes from a discretized symmetric elliptic operator

$\Rightarrow$ optimal $O(N)$ solvers available (multigrid, multilevel)

Convection-dominated equations

Preconditioned CG iteration for the SLAE:
apply operator preconditioning.

Preconditioner = stiffness matrix for the SD-inner product:

$$(\mathbf{S}_h)_{ij} = \langle \varphi_i, \varphi_j \rangle_{SD}$$

$\Rightarrow$ here $\langle u_h, v_h \rangle_{SD} = \int_\Omega (S_\varepsilon u_h)\, v_h$,

where $S_\varepsilon u := -\mathrm{div}\left(A_\varepsilon \nabla u\right)$ with $A_\varepsilon = \varepsilon I + \delta\, \mathbf{w} \cdot \mathbf{w}^T$

$\Rightarrow$ $\mathbf{S}_h$ comes from a discretized symmetric elliptic operator

$\Rightarrow$ optimal $O(N)$ solvers available (multigrid, multilevel)

Convection-dominated equations

Linear convergence estimate $\rightarrow$ we need bounds m and M.

Seen above: $m = 1$.

$M = ?$

Upper bound needed:

$$|a_{SD}(u_h, v_h)| \leq M \, \|u_h\|_{SD} \|v_h\|_{SD} \qquad (\forall u_h, v_h \in V_h),$$

where

$$a_{SD}(u_h, v_h) = \langle u_h, v_h \rangle_{SD} + \int_\Omega (\mathbf{w} \cdot \nabla u_h) v_h \, .$$

Convection-dominated equations

Linear convergence estimate $\rightarrow$ we need bounds m and M.

Seen above: $m = 1$.

$M = ?$

Upper bound needed:

$$|a_{SD}(u_h, v_h)| \leq M \, \|u_h\|_{SD} \|v_h\|_{SD} \qquad (\forall u_h, v_h \in V_h),$$

where

$$a_{SD}(u_h, v_h) = \langle u_h, v_h \rangle_{SD} + \int_\Omega (\mathbf{w} \cdot \nabla u_h) v_h \, .$$

Streamline Poincaré-Friedrichs inequality

Theorem. (Streamline Poincaré-Friedrichs inequality). Let $\mathbf{w} \in C^1(\overline{\Omega}, \mathbf{R}^n)$ be a globally rectifiable vector field on $\overline{\Omega}$. Then there exists a constant $C_{\mathbf{w}} > 0$ (depending on $\mathbf{w}$ but independent of v) such that

$$\|v\|_{L^2(\Omega)} \leq C_{\mathbf{w}} \|\mathbf{w} \cdot \nabla v\|_{L^2(\Omega)} \qquad (v \in H_0^1(\Omega)).$$

Streamline Poincaré-Friedrichs inequality

Then one can derive

$$|a_{SD}(u_h, v_h)| \le \left(1 + \frac{C_{\mathbf{w}}}{\delta_0}\right) \|u_h\|_{SD}\|v_h\|_{SD}$$

(where $\delta_0 := \min \delta_k$).
That is: the upper bound of a_{SD} satisfies

$$M \le 1 + \frac{C_{\mathbf{w}}}{\delta_0}$$

independently of ε.

Consequence: the PCG iterations converge with rate independently of ε $\rightarrow$ robustness.

[Axelsson–Karátson–Kovács, SIAM J. Numer. Anal. 2014]

Streamline Poincaré-Friedrichs inequality

Then one can derive

$$|a_{SD}(u_h, v_h)| \leq \left(1 + \frac{C_{\mathbf{w}}}{\delta_0}\right) \|u_h\|_{SD} \|v_h\|_{SD}$$

(where $\delta_0 := \min \delta_k$).
That is: the upper bound of a_{SD} satisfies

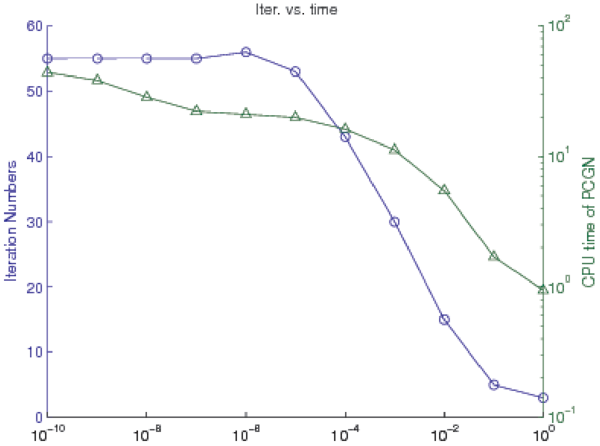
$$M \leq 1 + \frac{C_{\mathbf{w}}}{\delta_0}$$

independently of ε.

Consequence: the PCG iterations converge with rate independently of ε $\quad \to$ robustness.

[Axelsson–Karátson–Kovács, SIAM J. Numer. Anal. 2014]

Numerical experiments



Numerical experiments

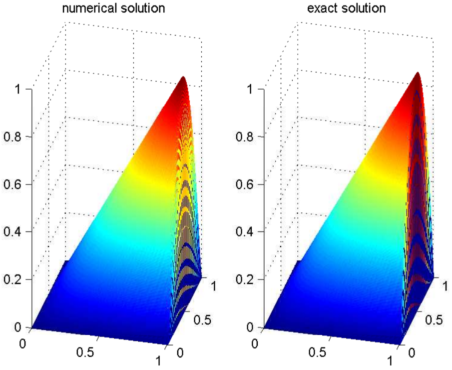


Figure : result for $\varepsilon = 10^{-10}$ – no unphysical oscillations.

Helmholtz equations

Case 5: *Helmholtz equations and shifted Laplace preconditioners.*

The Helmholtz equation:

$$\begin{cases} -\Delta u - \kappa^2 u & = & g \\ \left(\frac{\partial u}{\partial n} - i\kappa u\right)_{|\partial\Omega} & = & 0 \end{cases} \qquad (1)$$

(a model problem for high-frequency wave scattering).

Preconditioner : the stiffness matrix of the "complex shifted Laplace" problem (using a proper "absorption" parameter)

$$\begin{cases} -\Delta u - (\kappa^2 + i\varepsilon)u & = & g \\ \left(\frac{\partial u}{\partial n} - i\mu u\right)_{|\partial\Omega} & = & 0 \end{cases} \qquad (2)$$

[Erlangga, Gander, Magoules, Graham, Enquist, Ying, Shanks...]

Helmholtz equations

Case 5: *Helmholtz equations and shifted Laplace preconditioners.*

The Helmholtz equation:

$$\begin{cases} -\Delta u - \kappa^2 u = g \\ \left(\frac{\partial u}{\partial n} - i\kappa u\right)_{|\partial\Omega} = 0 \end{cases} \qquad (1)$$

(a model problem for high-frequency wave scattering).

Preconditioner : the stiffness matrix of the "complex shifted Laplace" problem (using a proper "absorption" parameter)

$$\begin{cases} -\Delta u - (\kappa^2 + i\varepsilon) u = g \\ \left(\frac{\partial u}{\partial n} - i\mu u\right)_{|\partial\Omega} = 0 \end{cases} \qquad (2)$$

[Erlangga, Gander, Magoules, Graham, Enquist, Ying, Shanks...]

Helmholtz equations

Assumptions:

discrete inf-sup-condition for both the original and auxiliary problems

Theorem. Mesh-independent superlinear convergence:

$$\left(\frac{\|r_k\|_{S_h}}{\|r_0\|_{S_h}}\right)^{1/k} \leq \varepsilon_k \to 0$$

where $\varepsilon_k := \frac{M}{m_0 m_1} \cdot \frac{1}{k} \sum_{i=1}^{k} s_i(Q_S)$ for the GMRES, and an analogous formula holds for the CGN.

(Here $s_i(Q_S)$ = the singular values of a compact operator, arising from the weak formulations.)

Helmholtz equations

Assumptions:

discrete inf-sup-condition for both the original and auxiliary problems

Theorem. Mesh-independent superlinear convergence:

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \leq \ \varepsilon_k \to 0$$

where $\varepsilon_k := \frac{M}{m_0 m_1} \cdot \frac{1}{k} \sum_{i=1}^{k} s_i(Q_S)$ for the GMRES, and an analogous formula holds for the CGN.

(Here $s_i(Q_S) =$ the singular values of a compact operator, arising from the weak formulations.)

Thank you for your attention!