The discrete Pompeiu problem on the plane

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Abstract

We say that a finite subset E of the Euclidean plane $\mathbb{R}^2$ has the discrete Pompeiu property with respect to isometries (similarities), if, whenever $f : \mathbb{R}^2 \to \mathbb{C}$ is such that the sum of the values of f on any congruent (similar) copy of E is zero, then f is identically zero. We show that every parallelogram and every quadrangle with rational coordinates has the discrete Pompeiu property w.r.t. isometries. We also present a family of quadrangles depending on a continuous parameter having the same property. We investigate the weighted version of the discrete Pompeiu property as well, and show that every finite linear set with commensurable distances has the weighted discrete Pompeiu property w.r.t. isometries, and every finite set has the weighted discrete Pompeiu property w.r.t. similarities.

1 Introduction

Let K be a compact subset of the plane having positive Lebesgue measure. The set K is said to have the Pompeiu property if the following condition is satisfied: whenever f is a continuous function defined on the plane, and the integral of f over every congruent copy of K is zero, then $f \equiv 0$. It is known that the closed disc does not have the Pompeiu property, while all polygons have. (As for the history of the problem, see [10] and [12].)

Replacing the Lebesgue measure by the counting measure, and the isometry group by an arbitrary family $\mathcal{G}$ of bijections mapping a set X onto itself, we obtain the following

notion. Let K be a nonempty finite subset of X. We say that K *has the discrete Pompeiu property with respect to the family $\mathcal{G}$ if the following condition is satisfied: whenever $f: X \to \mathbb{C}$ is such that $\sum_{x \in K} f(\phi(x)) = 0$ for every $\phi \in \mathcal{G}$, then $f \equiv 0$.*

We also introduce the weighted version of the discrete Pompeiu property. We say that the n-tuple $K = (x_1, \ldots, x_n)$ *has the weighted discrete Pompeiu property with respect to the family $\mathcal{G}$ if the following condition is satisfied: whenever $\alpha_1, \ldots, \alpha_n$ are complex numbers with $\sum_{i=1}^{n} \alpha_i \neq 0$ and $f: X \to \mathbb{C}$ is such that $\sum_{j=1}^{n} a_j f(\phi(x_j)) = 0$ for every $\phi \in \mathcal{G}$, then $f \equiv 0$.*

Apparently, the first results concerning the discrete Pompeiu property appeared in [13], where the author considers the Pompeiu problem for finite subsets of $\mathbb{Z}^n$ w.r.t. translations. The interest in the topic revived shortly after the 70th William Lowell Putnam Mathematical Competition (2009), where the following problem was posed: Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f \equiv 0$? This is nothing but asking whether the set of vertices of a square has the discrete Pompeiu property with respect to the similarities of the plane. This problem motivated the paper [3] by C. de Groote and M. Duerinckx. They prove that every finite and nonempty subset of $\mathbb{R}^2$ has the discrete Pompeiu property w.r.t. direct similarities. Another generalization of the Putnam problem appeared in [4], where it is proved that the set of vertices of a square has the discrete Pompeiu property with respect to the group of isometries. Recently, M. J. Puls [9] considered the discrete Pompeiu problem in groups.

In this paper we improve the results of [3] and [4]. We show that every finite and nonempty subset of $\mathbb{R}^2$ has the weighted discrete Pompeiu property w.r.t. direct similarities (Theorem 3.3). We also show that the set of vertices of every parallelogram has the discrete Pompeiu property with respect to the group of rigid motions (Theorem 5.1). We show the same for quadrangles with rational coordinates (Theorem 5.2), and for a family of quadrangles depending on a continuous parameter (Theorem 5.3). We also prove that in $\mathbb{R}^2$ all linear sets with commensurable distances have the discrete Pompeiu property w.r.t. rigid motions (Theorem 4.5).

These results motivate the following questions: is it true that every four element subset of the plane has the discrete Pompeiu property with respect to the group of isometries? Is it true that every nonempty and finite subset of the plane has the same property? We do not know the answer.

We conclude this introduction with a remark concerning the family of translations in an Abelian group. As the following proposition shows, this family is 'too small': finite sets, in general, cannot have the discrete Pompeiu property w.r.t. this group.

Proposition 1.1. *Let G be a torsion free Abelian group. If E is a finite subset of G containing at least 2 elements, then E does not have the discrete Pompeiu property w.r.t. the family of all translations of G.*

Proof. Note that if the torsion free rank of G is less than continuum, then this is a special case of [9, Theorem 3.1]. In the general case let H be the subgroup of G generated by E. Then H is a finitely generated torsion free Abelian group, and thus H is isomorphic to $\mathbb{Z}^n$ for some finite n. By Zeilberger's theorem [13], E does not have the discrete Pompeiu property in H w.r.t. the family of translations; that is, there is a nonzero function $f \colon H \to \mathbb{C}$ such that the sum of the values of f taken on any translate of E is zero. It is clear that we can find such a function on every coset of H. The union of these functions has the same property on G, showing that E does not have the discrete Pompeiu property in G w.r.t. the family of translations on G. $\square$

In the proposition above we cannot omit the requirement that G be torsion free. E.g., if G is a finite group having $n \geq 3$ elements and E is a subset of G having $n-1$ elements, then E has the discrete Pompeiu property w.r.t. translations. Indeed, if the sum of the values of f is zero on each translate of E then f must be constant, and the constant must be zero.

2 Preliminaries: generalized polynomials and exponential functions on Abelian groups

Let G be an Abelian group. If $f \colon G \to \mathbb{C}$ and $h \in G$, then $\Delta_h f$ denotes the function defined by $\Delta_h f(x) = f(x + h) - f(x)$ $(x \in G)$. The function $f \colon G \to \mathbb{C}$ is said to be a *generalized polynomial* if there is an n such that $\Delta_{h_1} \ldots \Delta_{h_{n+1}} f \equiv 0$ for every $h_1, \ldots, h_{n+1} \in G$. The degree of f is the smallest such n. Thus the generalized polynomials of degree zero are the nonzero constant functions. The degree of the identically zero function is -1 by definition.

The function $g \colon G \to \mathbb{C}$ is an *exponential*, if $g \neq 0$ and $g(x + y) = g(x) \cdot g(y)$ for every $x, y \in G$. By a *monom* we mean a function of the form $p \cdot g$, where p is a generalized polynomial, and g is an exponential. Finite sums of monoms are called *polynomial-exponential functions*.

Let $\mathbb{C}^G$ denote the linear space of all complex valued functions defined on G equipped with the product topology. By a *variety on G* we mean a translation invariant, closed, linear subspace of $\mathbb{C}^G$. We say that spectral analysis holds in G, if every nonzero variety contains an exponential function.

We shall need the fact that spectral analysis holds in every finitely generated and

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torsion free Abelian group. In fact, this is true in every Abelian group whose torsion free rank is less than continuum [6]. However, for finitely generated and torsion free Abelian groups this also follows from Lefranc's theorem. Lefranc proved in [7] that if n is finite then *spectral synthesis* holds in $\mathbb{Z}^n$; that is, every variety on $\mathbb{Z}^n$ is spanned by polynomial-exponential functions. Therefore, if a variety V on $\mathbb{Z}^n$ contains nonzero functions, then it has to contain nonzero polynomial-exponential functions. It is easy to see that if a polynomial-exponential function $\sum_{i=1}^{n} p_i \cdot g_i$ is contained in a variety V, where $p_1, \ldots, p_n$ are nonzero generalized polynomials and $g_1, \ldots, g_n$ are distinct exponentials, then necessarily $g_i \in V$ holds for every $i = 1, \ldots, n$. Since every finitely generated and torsion free Abelian group is isomorphic to $\mathbb{Z}^n$ for some finite n, it follows that spectral analysis (and, in fact, spectral synthesis) holds in such groups. We shall need the following special case.

Lemma 2.1. *Let G be a finitely generated subgroup of the additive group of $\mathbb{C}$, let $\alpha_{j,k}, b_{j,k}$ $(j = 1, \ldots, n, \ k = 1, \ldots, m)$ be complex numbers, and let Y be a subset of $\mathbb{C}$. Let V denote the set of functions $f \colon G \to \mathbb{C}$ such that*

$$\sum_{j=1}^{n} \alpha_{j,k} f(x + b_{j,k}y) = 0$$

for every $k = 1, \ldots, m$, $x \in G$ and $y \in Y$ satisfying $b_{j,k}y \in G$ for every $j = 1, \ldots, n$ and $k = 1, \ldots, m$. Then V is a variety on G. Consequently, if V contains a non-identically zero function, then V contains an exponential function defined on G.

Proof. It is clear that V is a translation invariant linear subspace of $\mathbb{C}^G$. Since G is countable, the topology of $\mathbb{C}^G$ is the topology determined by pointwise convergence. Obviously, if $f_i \in V$ and $f_i \to f$ pointwise on G, then $f \in V$. Thus V is closed. $\square$

3 Similarities

It was shown by C. de Groote and M. Duerinckx in [3] that *every finite and nonempty subset of $\mathbb{R}^2$ has the discrete Pompeiu property w.r.t. direct similarities.* By a direct similarity we mean a transformation that is a composition of translations, rotations and homothetic transformations. The authors also discuss the possible generalizations when $\mathbb{R}^2$ is replaced by K^p where K is a field, and the transformation group is a subgroup of $AGL(p, K)$. We note that the argument given by C. de Groote and M. Duerinckx also proves the following generalization.

Proposition 3.1. *Let $\mathcal{G}$ be a transitive and locally commutative transformation group acting on X such that for every $x, y, z \in X$ with $y \neq x \neq z$ there exists a map $f \in \mathcal{G}$ such*

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that $f(x) = x$ and $f(y) = z$. Then every finite and nonempty proper subset of X has the discrete Pompeiu property w.r.t. $\mathcal{G}$.

We say that a transformation $g\colon \mathbb{R} \to \mathbb{R}$ is an order preserving similarity, if $g(x) = a + cx$ for every $x \in \mathbb{R}$, where $a \in \mathbb{R}$ and $c > 0$.

Proposition 3.2. *Every finite and nonempty subset of $\mathbb{R}$ has the discrete Pompeiu property w.r.t. the group of order preserving similarities.*

Proof. Although Proposition 3.1 cannot be applied directly, a variant of the argument given by C. de Groote and M. Duerinckx in [3] works. Let $E = \{x_1, \ldots, x_n\}$. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is such that $\sum_{i=1}^{n} f(a + cx_i) = 0$ for every $a \in \mathbb{R}$ and $c > 0$. Replacing E by a translated copy we may assume that $0 = x_1 < x_2 < \ldots < x_n$. We put $A_i = \{x_i + x_i x_j \colon j = 2, \ldots, n\}$ and $B_j = \{x_i + x_i x_j \colon i = 2, \ldots, n\}$. Then $A_i \cup \{x_i\}$ is the image of E under an order preserving similarity for every $i \geq 2$, and thus $\sum_{j=2}^{n} f(x_i + x_i x_j) = -f(x_i)$ $(i = 2, \ldots, n)$. Similarly, $B_j \cup \{0\}$ is the image of E under an order preserving similarity for every $j \geq 2$, and thus $\sum_{i=2}^{n} f(x_i + x_i x_j) = -f(0)$ $(j = 2, \ldots, n)$. Therefore,

$$f(0) = -\sum_{i=2}^{n} f(x_i) = \sum_{i=2}^{n}\sum_{j=2}^{n} f(x_i + x_i x_j) = \sum_{j=2}^{n}\sum_{i=2}^{n} f(x_i + x_i x_j) =$$
$$= \sum_{j=2}^{n}(-f(0)) = -(n-1)f(0).$$

Thus we have $f(0) = 0$. For every $b \in \mathbb{R}$, the function $T_b f$ defined by $T_b f(x) = f(x + b)$ also satisfies the condition $\sum_{i=1}^{n} T_b f(a + cx_i) = 0$ for every $a \in \mathbb{R}$ and $c > 0$. Therefore, $T_b f(0) = f(b) = 0$ for every $b \in \mathbb{R}$. $\square$

De Groote and M. Duerinckx ask in [3] if the finite subsets of the plane have the weighted discrete Pompeiu property w.r.t. direct similarities. In the next theorem we show that the answer is affirmative.

Theorem 3.3. *Every n-tuple of distinct points of $\mathbb{R}^2$ has the weighted discrete Pompeiu property w.r.t. direct similarities.*

Proof. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We put $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Let $(b_1, \ldots, b_n)$ be an n-tuple of distinct complex numbers. Let $\alpha_1, \ldots, \alpha_n$ be complex numbers such that $\sum_{i=1}^{n} \alpha_i \neq 0$, and let $f\colon \mathbb{C} \to \mathbb{C}$ be such that

$$\sum_{i=1}^{n} \alpha_i f(x + b_i y) = 0 \tag{1}$$

for every $x \in \mathbb{C}$ and $y \in \mathbb{C}^*$. We have to prove that $f \equiv 0$.

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If (1) holds for every $x, y \in \mathbb{C}$, then $f \equiv 0$ is one of the statements of [5, Theorem 2.4]. Therefore, it is enough to show that if (1) holds for every $x \in \mathbb{C}$ and $y \in \mathbb{C}^*$, then it holds for every $x, y \in \mathbb{C}$. In the following theorem we shall prove more.

We say that a family I of subsets of $\mathbb{C}$ is a proper and translation invariant ideal, if $A, B \in I$ implies $A \cup B \in I$, $A \in I$ and $B \subset A$ implies $B \in I$, $\mathbb{C} \notin I$, and if $A \in I$ then $A + c = \{x + c \colon x \in A\} \in I$ for every $c \in \mathbb{C}$. It is clear that the family of finite subsets of $\mathbb{R}$ is a proper and translation invariant ideal.

Theorem 3.4. *Let I be a proper and translation invariant ideal of subsets of $\mathbb{C}$. Let $b_1, \ldots, b_n$ be distinct complex numbers, and suppose that the functions $f_1, \ldots, f_n \colon \mathbb{C} \to \mathbb{C}$ satisfy*

$$\sum_{i=1}^{n} f_i(x + b_i y) = 0 \tag{2}$$

for every $x \in \mathbb{C}$ and $y \in \mathbb{C} \setminus A$, where $A \in I$. Then each f_i is a generalized polynomial of degree $\leq n - 2$, and (2) holds for every $x, y \in \mathbb{C}$.

Proof. First we prove that each f_i is a generalized polynomial of degree $\leq n - 2$. We prove by induction on n. The case of $n = 1$ is obvious.

Now let $n \geq 2$, and suppose that the statement is true for $n - 1$. Let $f_1, \ldots, f_n$ satisfy (2) for every x and $y \notin A$, where $A \in I$. Since the role of the functions f_i is symmetric, it is enough to prove that f_1 is a generalized polynomial of degree $\leq n - 2$. Note that $b_1 \neq b_n$ by assumption. Let $h \in \mathbb{C}$ be fixed. Then we have

$$\sum_{i=1}^{n} f_i(x + h + b_i y) = 0 \tag{3}$$

for every x and $y \in \mathbb{C} \setminus A$, and

$$\sum_{i=1}^{n} f_i(u + b_i v) = 0 \tag{4}$$

for every u and $v \notin A$. Substituting $u = x - b_1 h/(b_n - b_1)$ and $v = y + h/(b_n - b_1)$ into (4) and subtracting from (3) we obtain

$$\Delta_h f_1(x + b_1 y) + \sum_{i=2}^{n-1} \left[f_i(x + h + b_i y) - f_i\left(x + \frac{b_i - b_1}{b_n - b_i} h + b_i y\right) \right] = 0$$

for every y such that $y \notin A$ and $v = y + h/(b_n - b_1) \notin A$. (If $n = 2$ then the sum on the left hand side is empty.) Putting $g_i(z) = f_i(z + h) - f_i\left(z + \frac{b_i - b_1}{b_n - b_i} h\right)$ $(z \in \mathbb{C})$, we obtain that

$$\Delta_h f_1(x + b_1 y) + \sum_{i=2}^{n-1} g_i(x + b_i y) = 0$$

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for every x and for every $y \notin A \cup (A - h/(b_n - b_1))$. Since $A \cup (A - h/(b_n - b_1)) \in I$, it follows from the induction hypothesis that $\Delta_h f_1$ is a generalized polynomial of degree $\leq n-3$. As this is true for every h, we obtain that f_1 is a generalized polynomial of degree $\leq n - 2$.

We still have to prove that (2) holds for every $x, y \in \mathbb{C}$. Let $x \in \mathbb{C}$ be fixed, and put $G(y) = \sum_{i=1}^{n} f_i(x + b_i y)$ for every $y \in \mathbb{C}$. We have to prove that $G(y) = 0$ for every $y \in \mathbb{C}$.

It is easy to see that if f is a generalized polynomial, then so is $y \mapsto f(x + by) = g(y)$. This can be proved by induction on the degree of f, using $\Delta_h g(y) = \Delta_{bh} f(x + by)$. Since each f_i is a generalized polynomial, it follows that so is $g_i(y) = f_i(x + b_i y)$ for every i, and thus so is $G = g_1 + \ldots + g_n$.

We know that $G(y) = 0$ for every $y \notin A$. Therefore, in order to prove $G \equiv 0$, it is enough to show that if $f \colon \mathbb{C} \to \mathbb{C}$ is a generalized polynomial and $f(x) = 0$ for every $x \in \mathbb{C} \setminus A$ where $A \in I$, then $f \equiv 0$. We prove by induction on the degree of f. The statement is obvious if the degree is ≤ 0. Indeed, in this case f is constant, and has a value equal to zero, since I is a proper ideal. Suppose the degree of f is $n > 0$, and the statement is true for generalized polynomials of degree $< n$. For every h, we have $\Delta_h f(x) = 0$ for every $x \in \mathbb{C} \setminus (A \cup (A - h))$. Since $A \cup (A - h) \in I$, it follows from the induction hypothesis that $\Delta_h f(x) = 0$ for every x. This is true for every h, which shows that f is constant. As we saw above, the constant must be zero. This completes the proof. $\square$

4   Isometries and rigid motions: some general remarks

By a rigid motion we mean an isometry that preserves orientation. An isometry of $\mathbb{R}^2$ is a rigid motion if it is a translation or a rotation.

Proposition 4.1. *Every subset of the plane containing* 1, 2 *or* 3 *points has the discrete Pompeiu property w.r.t. rigid motions.*

Proof. The case of the singletons is obvious. Let $E = \{a, b\}$ and $r = |a - b| > 0$. Suppose that $f \colon \mathbb{R}^2 \to \mathbb{C}$ is such that $f(\sigma(a)) + f(\sigma(b)) = 0$ for every rigid motion σ. Then f has the same value at every pair of points a_1, a_2 with distance $\leq 2r$. Indeed, there is a point b such that $|b - a_i| = r$ $(i = 1, 2)$, and thus $f(a_1) = -f(b) = f(a_2)$. Now, any two points $a, b \in \mathbb{R}^2$ can be joined by a sequence of points $a = a_0, \ldots, a_n = b$ such that $|a_i - a_{i-1}| \leq 2r$, and thus $f(a) = f(b)$. Therefore, f must be constant, and the value of the constant must be zero.

Let $H = \{a, b, c\}$, where a, b, c are distinct, and let $f \colon \mathbb{R}^2 \to \mathbb{C}$ be such that $f(\sigma(a)) + f(\sigma(b)) + f(\sigma(c)) = 0$ for every rigid motion σ. By changing the notation of the points a, b, c

if necessary, we may assume that $c \neq (a+b)/2$. Let $c' = a+b-c$. Then c' is the reflection of c about middle point of the segment $[a, b]$, and thus $f(\sigma(b)) + f(\sigma(a)) + f(\sigma(c')) = 0$ for every rigid motion σ. Thus $f(\sigma(c')) = f(\sigma(c))$ for every rigid motion σ, which implies that $f(x) = f(y)$ whenever $|x - y| = |c' - c|$. The argument above shows that f is constant, and, in fact, $f \equiv 0$. $\square$

Remark 4.2. It is easy to see that if $n \leq 2$, then every n-tuple has the weighted discrete Pompeiu property w.r.t. isometries. The same is true for those triplets (a, b, c) whose points are not collinear. In this case we have to modify the proof above by choosing the point c' to be the reflection of c about the line going through a and b instead of the point $a + b - c$ in order to avoid changing the weights of a and b.

Proposition 4.3. *Let E be a finite set in the plane. If there exists an isometry σ such that $|E \cap \sigma(E)| = |E| - 1$, then E has the discrete Pompeiu property w.r.t. isometries.*

Proof. Let $E \backslash \sigma(E) = \{a\}$ and $\sigma(E) \backslash E = \{b\}$. If $f : X \to \mathbb{C}$ is such that $\sum_{x \in \phi(E)} f(x) = 0$ for every isometry ϕ, then taking the difference of the equations $\sum_{x \in (\phi\sigma)(E)} f(x) = 0$ and $\sum_{x \in \phi(E)} f(x) = 0$, we obtain $f(\phi(a)) = f(\phi(b))$ for every isometry ϕ. Thus $f(x) = f(y)$ whenever $|x - y| = |a - b|$. As we saw before, this implies that f is identically zero. $\square$

Remark 4.4. Concerning the discrete Pompeiu property in higher dimensions, we note that Proposition 4.3 holds without any essential modification in $\mathbb{R}^n$ for every $n \geq 2$. As for Proposition 4.1, it is easy to see that every subset of $\mathbb{R}^n$ ($n \geq 2$) containing affinely independent points has the discrete Pompeiu property w. r. t. isometries. Using an inductive argument it is enough to consider the case of $n + 1$ points in general position. Such a set satisfies the condition of Proposition 4.3: let σ be the reflection about a facet.

By Proposition 4.3, if a set E consists of consecutive vertices of a regular n-gon R, and $E \neq R$, then E has the discrete Pompeiu property w.r.t. isometries. Also, if E is a finite set of collinear points forming an arithmetic progression, then E has the discrete Pompeiu property w.r.t. isometries. Our following theorem is the generalization of this fact.

Theorem 4.5. *Let E be an n-tuple of collinear points in $\mathbb{R}^2$ with pairwise commensurable distances. Then E has the weighted discrete Pompeiu property w.r.t. rigid motions of the plane.*

Lemma 4.6. *Let $x_1, \ldots, x_n, y_1, \ldots, y_k \in \mathbb{R}^2$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_k \in \mathbb{C}$ be such that*

(i) *$y_1, \ldots, y_k$ are collinear with commensurable distances,*

(ii) *$\sum_{i=1}^{n} \alpha_i \neq 0$, and*

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(iii) *at least one of the numbers $\beta_1, \ldots, \beta_k$ is nonzero.*

If $f\colon \mathbb{R}^2 \to \mathbb{R}$ *is such that*

$$\sum_{i=1}^{n} \alpha_i f(\sigma(x_i)) = \sum_{j=1}^{k} \beta_j f(\sigma(y_j)) = 0 \tag{5}$$

for every rigid motion σ, then f is identically zero.

Proof. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We put $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $S^1 = \{x \in \mathbb{C} \colon |x| = 1\}$. Then every rigid motion is of the form $x \mapsto a + ux$ ($x \in \mathbb{C}$), where $a \in \mathbb{C}$ and $u \in S^1$.

Let $a, c \in \mathbb{C}$ and $c \neq 0$. If we replace x_i by $a + cx_i$, y_j by $a + cy_j$ for every i and j, and replace f by $f_1(x) = (x/c)$, then (5) remains valid for every rigid motion σ. Indeed, for every σ, the map $x \mapsto \sigma(a + cx)/c$ is a rigid motion if and only if σ is. Note that if f_1 is identically zero, then so is f.

Therefore, replacing x_i by $a + cx_i$, y_j by $a + cy_j$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, k$ with a suitable $a \in \mathbb{C}$ and $c \in \mathbb{C}^*$, we may assume that $y_1, \ldots, y_k$ are positive integers. By supplementing the system if necessary, we may assume that $y_j = j$ ($j = 1, \ldots, m$). We put $\beta_j = 0$ for every added j. Then we have

$$\sum_{i=1}^{n} \alpha_i f(x + ux_i) = \sum_{j=1}^{m} \beta_j f(x + ju) = 0 \tag{6}$$

for every $x \in \mathbb{C}$ and $u \in S^1$. We show that this implies $f \equiv 0$. Suppose that f is not identically zero, and let $z_0 \in \mathbb{C}$ be such that $f(z_0) \neq 0$.

Let K be an integer greater than $\max_{1 \leq i \leq n} |x_i|$. It is clear that every $z \in \mathbb{C}$ with $|z| < K$ is the sum of K elements of S^1. Let U be a finite subset of S^1 such that $1 \in U$, and x_i/ν is the sum of K elements of U for every $i = 1, \ldots, n$ and $\nu = 1, \ldots, N$, where $N = m^{K \cdot m^K}$.

Let G denote the additive subgroup of $\mathbb{C}$ generated by the elements z_0, $u \in U$ and ux_i ($u \in U$, $i = 1, \ldots, n$). Then G is a finitely generated subgroup of $\mathbb{C}$. Let V denote the set of functions defined on G and satisfying (6) for every $x \in G$ and $u \in U$. The set of functions V contains the restriction of f to G which is not identically zero, as $z_0 \in G$. Therefore, by Lemma 2.1, V contains an exponential function $g\colon G \to \mathbb{C}$.

If $u \in U$, then (6) gives $\sum_{j=1}^{m} \beta_j g(u)^j = 0$. Therefore, $g(u)$ is a root of the polynomial $p(x) = \sum_{j=1}^{m} \beta_j x^{j-1}$. Let Λ denote the set of the nonzero roots of p. Then Λ has at most $m - 1$ elements, and $g(u) \in \Lambda$ for every $u \in U$. For every $i = 1, \ldots, n$ and $\nu = 1, \ldots, N$, x_i/ν is the sum of K elements of U. Thus $g(x_i/\nu)$ is the product of K elements of

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$g(U) \subset \Lambda$. Therefore, the set $F = \{g(x_i/\nu) : i = 1, \ldots, n, \ \nu = 1, \ldots, N\}$ has less than m^K elements.

Let $1 \le i \le n$ be fixed, and put $g(x_i) = c$. We prove $c = 1$. Since $g(x_i/\nu) \in F$ for every $\nu = 1, \ldots, m^K$, there are integers $1 \le \nu < \mu \le m^K$ such that $g(x_i/\nu) = g(x_i/\mu)$. Then

$$c^\mu = g(x_i/\nu)^{\nu\mu} = g(x_i/\mu)^{\nu\mu} = c^\nu,$$

and thus $c^{\mu-\nu} = 1$. Let $\mu - \nu = s$, then $s < m^K$ and $c^s = 1$. If $s = 1$, then $c = 1$ is proved. If $s > 1$, then, by $g(x_i/s^t) \in F$ for every $t = 1, \ldots, m^K$, there are integers $1 \le r < t \le m^K$ and there is an element $b \in F$ such that $g(x_i/s^r) = g(x_i/s^t) = b$. Then

$$c = g(x_i) = b^{s^t} = b^{s^r \cdot s^{t-r}} = c^{s^{t-r}} = 1,$$

since $c^s = 1$. This proves $g(x_i) = 1$ for every $i = 1, \ldots, n$.

Then, applying (6) with $x = 0$ and $u = 1$, we obtain $\sum_{i=1}^n \alpha_i = 0$ which is impossible. This contradiction completes the proof. □

Proof of Theorem 4.5. Let $E = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are collinear with commensurable distances. Let $\alpha_1, \ldots, \alpha_n$ be complex numbers with $\sum_{i=1}^n \alpha_i \ne 0$, and let $f \colon \mathbb{C} \to \mathbb{C}$ satisfy $\sum_{i=1}^n \alpha_i f(\sigma(x_i)) = 0$ for every rigid motion σ. Applying Lemma 4.6 with $k = n$, $y_i = x_i$ and $\beta_i = \alpha_i$ $(i = 1, \ldots, n)$, we obtain that f is identically zero. □

Remark 4.7. The isometry group of $\mathbb{R}$ consists of translations and reflections. Since no finite subset of $\mathbb{R}$ has the discrete Pompeiu property w.r.t. translations by Proposition 1.1, and every reflected copy of the set $\{1, \ldots, n\}$ is also a translated copy, it follows that the set $\{1, \ldots, n\}$ does not have the discrete Pompeiu property w.r.t. isometries of $\mathbb{R}$. (This is why we had to step out from $\mathbb{R}$ into the plane in the proof of Theorem 4.5.) Note, however, that there are subsets of $\mathbb{Z}$ which have the discrete Pompeiu property w.r.t. isometries of $\mathbb{R}$. The set of integers $0 = z_0 < z_1 < \ldots < z_k$ has this property if and only if the polynomials $p(x) = \sum_{i=0}^k x^{z_i}$ and $q(x) = \sum_{i=0}^k x^{z_k - z_i}$ have no common roots. (This follows immediately from Zeilberger's theorem [13].) This condition is clearly satisfied if the set $\{z_0, z_1, \ldots, z_k\}$ is not symmetric about the point $(z_0 + z_k)/2$, and if p is irreducible in $\mathbb{Z}[x]$.

Since each coefficient of p is 0 or 1, it is easy to decide whether p is irreducible or not. If there is an $n \ge 3$ such that $p(n)$ is prime, then p is irreducible (see [8]). By the Buniakowski-Schinzel conjecture, this condition is also necessary for the irreducibility of p.

5   Quadrangles under isometries

Theorem 5.1. *The set of vertices of any parallelogram has the discrete Pompeiu property w.r.t. rigid motions of the plane.*

Proof. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We put $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $S^1 = \{u \in \mathbb{C} : |u| = 1\}$. Let E be a set of vertices of a parallelogram. Without loss of generality we may assume that $0 \in E$. Then $E = \{0, a, b, a + b\}$, where $0 \neq a, b \in \mathbb{C}$ and $a \neq b$. Clearly, it is enough to prove that if $f \colon \mathbb{C} \to \mathbb{C}$ is such that

$$f(x) + f(x + ay) + f(x + by) + f(x + (a + b)y) = 0 \tag{7}$$

for every $x \in \mathbb{C}$ and $y \in S^1$, then $f \equiv 0$. Suppose that there exists a nonzero f satisfying (7), and let $z_0 \in \mathbb{C}$ be such that $f(z_0) \neq 0$.

Let F be a finite subset of $\mathbb{C}$, and let G denote the additive subgroup of $\mathbb{C}$ generated by $F \cup \{z_0\}$. Let V denote the set of functions $f \colon G \to \mathbb{C}$ satisfying (7) for every $x \in G$ and $y \in S_G^1 = \{y \in S^1 : ay, by \in G\}$. Since $f|_G \in V$ and $z_0 \in G$, it follows that $V \neq 0$.

By Lemma 2.1, there exists an exponential function g in V. Since g satisfies (7) and $g(x + ay) = g(x)g(ay)$ and $g(x + (a + b)y) = g(x)g(ay)g(by)$, we obtain $g(x)(1 + g(ay) + g(by) + g(ay)g(by)) = 0$ whenever $x \in G$ and $y \in S_G^1$. Since $g(x) \neq 0$, we get $(1 + g(ay))(1 + g(by)) = 0$ for every $y \in S_G^1$. That is, we have either $g(ay) = -1$ or $g(by) = -1$ for every $y \in S_G^1$.

Let P be an arbitrary parallelogram obtained from E by a rigid motion and having vertices in G. Then the vertices of P are $c = x$, $d = x + ay$, $e = x + (a + b)y$, $f = x + by$ with a suitable $x \in G$ and $y \in S_G^1$. Then we have either $g(d)/g(c) = g(e)/g(f) = -1$ or $g(f)/g(c) = g(e)/g(d) = -1$. In other words, the values of g at the points c, d, e, f are either $g(c), -g(c), g(e), -g(e)$ or $g(c), g(d), -g(d), -g(c)$. Therefore, the vertex set of the parallelogram can be decomposed into two pairs with g-values of the form $(x, -x)$ in each pair.

Let $\mathbb{C}^* = X_1 \cup X_2$ be a decomposition of $\mathbb{C}^*$ such that $X_1 = -X_2$. Let $h(x) = 1$ if $g(x) \in X_1$, and $h(x) = -1$ if $g(x) \in X_2$. Then $h \colon G \to \{1, -1\}$ has the following property: if σ is a rigid motion and if $\sigma(E) \subset G$, then there are two elements of $\sigma(E)$ where the function h takes the value 1, and at the other two elements of $\sigma(E)$ the function h takes the value -1.

Since this is true for every group generated by any finite subsets of $\mathbb{R}^2$, we may apply Rado's selection principle [2]. We find that there exists a function $h \colon \mathbb{R}^2 \to \{1, -1\}$ such that whenever σ is a rigid motion, then there are two elements of $\sigma(E)$ where the function h takes the value 1, and at the other two elements of $\sigma(E)$ the function h takes the value -1.

The existence of such a function, however, contradicts a known fact of Euclidean Ramsey theory. By a theorem of Shader [11, Theorem 3], for every 2-coloring of the plane, and for every parallelogram E, there is a congruent copy P of E such that at least three vertices of P has the same color. It is clear from the proof that P can be obtained from E by a rigid motion. (See the Remark on p. 563 in [1].) This contradicts the existence of the function h with the properties described, proving that f must be identically zero. $\square$

Our next aim is to prove the following.

Theorem 5.2. *Every set $E \subset \mathbb{R}^2$ of four points having rational coordinates has the weighted discrete Pompeiu property w.r.t. the group of isometries of $\mathbb{R}^2$.*

Proof. If the points of E are collinear, then the statement is a consequence of Theorem 4.5. If there are three collinear points of E, then the statement follows from Proposition 4.3. Therefore, we may assume that the points of E are in general position. Let $E = (x_1, \ldots, x_4)$. By changing the order of the indices we may assume that x_1 and x_2 are vertices of the convex hull of E.

Let $\alpha_1, \ldots, \alpha_4$ be complex numbers such that $\sum_{j=1}^{4} \alpha_j \neq 0$, and let $f \colon \mathbb{C} \to \mathbb{C}$ be such that

$$\sum_{j=1}^{4} \alpha_j f(\sigma(x_j)) = 0 \tag{8}$$

for every isometry σ. We have to show that f is identically zero. If any of the numbers $\alpha_1, \ldots, \alpha_4$ is zero then $f \equiv 0$ follows from Remark 4.2. Therefore, we may assume that $\alpha_4 \neq 0$.

Let σ_1 be the reflection about the line ℓ_1 going through the points x_1, x_2. Let $y_1 = \sigma_1(x_4)$ and $x_5 = \sigma_1(x_3)$, then y_1 and x_5 have rational coordinates. We have, for every σ, $(\sigma \circ \sigma_1)(x_i) = \sigma(x_i)$ for $i = 1, 2$, $(\sigma \circ \sigma_1)(x_3) = \sigma(x_5)$ and $(\sigma \circ \sigma_1)(x_4) = \sigma(y_1)$. Therefore

$$\alpha_1 f(\sigma(x_1)) + \alpha_2 f(\sigma(x_1)) + \alpha_3 f(\sigma(x_5)) + \alpha_4 f(\sigma(y_1)) = 0$$

for every isometry σ. Subtracting (8) we obtain

$$\alpha_3 f(\sigma(x_5)) + \alpha_4 f(\sigma(y_1)) - \alpha_3 f(\sigma(x_3)) - \alpha_4 f(\sigma(x_4)) = 0 \tag{9}$$

for every isometry σ. Suppose that the line going through the points x_3 and x_4 is perpendicular to ℓ_1. Then the points x_5, y_1, x_3, x_4 are collinear. They have rational coordinates, so the distances between them are commensurable. Now f satisfies both (8) and (9) for every isometry σ, and thus, by Lemma 4.6, $f \equiv 0$.

Therefore, we may assume that the line going through the points x_3 and x_4 is not perpendicular to ℓ_1. Let σ_2 be the reflection about the line ℓ_2 going through the points

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x_3, x_5. Note that the lines ℓ_1 and ℓ_2 are perpendicular. We put $y_2 = \sigma_2(y_1)$, $y_3 = \sigma_2(x_4)$ and $y_4 = x_4$. Then y_1, y_2, y_3, y_4 are the vertices of a rectangle R listed either clockwise or counter-clockwise. It is clear that y_1, y_2, y_3, y_4 have rational coordinates. We claim that

$$f(\sigma(y_1)) - f(\sigma(y_2)) + f(\sigma(y_3)) - f(\sigma(y_4)) = 0 \tag{10}$$

holds for every isometry σ. Indeed, $(\sigma \circ \sigma_2)(x_5) = \sigma(x_5)$, $(\sigma \circ \sigma_2)(x_3) = \sigma(x_3)$, $(\sigma \circ \sigma_2)(y_1) = \sigma(y_2)$ and $(\sigma \circ \sigma_2)(x_4) = \sigma(y_3)$ and thus, by (9) we obtain

$$\alpha_3 f(\sigma(x_5)) + \alpha_4 f(\sigma(y_2)) - \alpha_3 f(\sigma(x_3)) - \alpha_4 f(\sigma(y_3)) = 0.$$

Subtracting (9) and dividing by $-\alpha_4$ we obtain (10) for every isometry σ.

Since the coordinates of $y_1, \ldots, y_4$ are rational, it follows that the side lengths of R are commensurable. (The side lengths themselves can be irrational.) Thus, there exists a square Q with vertices $z_1, \ldots, z_4$ such that Q can be decomposed into finitely many translated copy of R. If we add the equations (10) for those translations σ that map R into these translated copies, then we get

$$f(z_1) - f(z_2) + f(z_3) - f(z_4) = 0, \tag{11}$$

since all other terms cancel out. By rescaling the set E and also the function f if necessary, we may assume that the side length of Q is 1. Clearly, (11) must hold whenever $z_1, \ldots, z_4$ are the vertices of a square of unit side length. That is, we have

$$f(x) - f(x + u) - f(x + u \cdot i) + f(x + u + u \cdot i) = 0$$

for every $x \in \mathbb{C}$ and $u \in S^1$.

Now we turn to the proof of $f \equiv 0$. Suppose this is not true, and fix a $z_0 \in \mathbb{C}$ such that $f(z_0) \neq 0$. Let $a_1, \ldots, a_N$ be vectors of length 12 such that each of the numbers $x_1, \ldots, x_4$ is the sum of some of the a_j's. Let $u_j = a_j/12$ and $v_j = (3u_j + 4u_j \cdot i)/5$ for every $j = 1, \ldots, N$. Then u_j, v_j are unit vectors for every j. Let U denote the set of vectors

$$u_j, \ u_j \cdot i, \ v_j, \ v_j \cdot i \qquad (j = 1, \ldots, N),$$

and let G denote the additive group generated by the set $U \cup \{x_j u \colon j = 1, \ldots, 4, \ u \in U\} \cup \{z_0\}$. Then G is a finitely generated group. Let V be the set of functions $g \colon G \to \mathbb{C}$ satisfying the following condition:

$$\sum_{j=1}^{4} \alpha_j g(x + x_j \cdot u) = 0$$

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and

$$g(x) - g(x + u) - g(x + u \cdot i) + g(x + u + u \cdot i) = 0 \qquad (12)$$

for every $x \in G$ and $u \in U$.

The set V contains a non-identically zero function (namely the restriction of f to G), so by Lemma 2.1, V contains an exponential function g. Then (12) implies $(1 - g(u)) \cdot (1 - g(u \cdot i)) = 0$, and thus we have either $g(u) = 1$ or $g(u \cdot i) = 1$ for every $u \in U$.

Now we show that $g(a_j) = 1$ for every $j = 1, \ldots, N$. If $g(u_j) = 1$, then this follows from $g(a_j) = g(12u_j) = g(u_j)^{12}$. Therefore we may assume that $g(u_j \cdot i) = 1$. Since $5v_j = 3u_j + 4u_j \cdot i$ and $5v_j \cdot i = -4u_j + 3 \cdot u_j \cdot i$, we have $g(v_j)^5 = g(u_j)^3 \cdot g(u_j \cdot i)^4 = g(u_j)^3$ and $g(v_j \cdot i)^5 = g(u_j)^{-4} \cdot g(u_j \cdot i)^3 = g(u_j)^{-4}$. Now we have either $g(v_j) = 1$ or $g(v_j \cdot i) = 1$. Thus at least one of $g(u_j)^3 = 1$ and $g(u_j)^{-4} = 1$ must hold. Thus $g(u_j)^{12} = 1$ in both cases, which gives $g(a_j) = g(12u_j) = g(u_j)^{12} = 1$.

Since each x_j is the sum of some of the numbers $a_1, \ldots, a_N$, it follows that $g(x_j)$ is the product of some of the numbers $g(a_1), \ldots, g(a_N)$. Thus $g(x_j) = 1$ for every $j = 1, \ldots, 4$. However, by $\sum_{j=1}^{4} \alpha_j g(x_j) = 0$ this implies $\sum_{j=1}^{4} \alpha_j = 0$, which contradicts the assumption $\sum_{j=1}^{4} \alpha_j \neq 0$. This contradiction proves that $f \equiv 0$. $\square$

Finally, we present a family of quadrangles depending on a continuous parameter such that each member of the family has the discrete Pompeiu property w.r.t. the isometry group.

Let a non-regular triangle $ABC\triangle$ be given in the plane. The steps of the construction are summarized as follows:
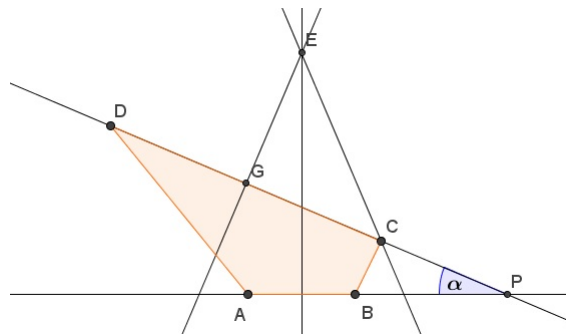

Figure 1: A Pompeiu quadrangle belonging to $\alpha = 23°$.

- since $ABC\triangle$ is non-regular we can suppose that the point C is not on the perpendicular bisector of AB; especially C and B are supposed to be on the same side of the perpendicular bisector of AB.

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- let $0 < \alpha < 45°$ be a given angle and choose a point P on the line AB such that A and P are separated by the point B and the angle enclosed by the lines PB and PC is of measure α (see Figure 1).

- E is the point of the perpendicular bisector of AB such that the line CE intersects the bisector under an angle of measure α (see Figure 1).

- G is the point on the line PC such that the triangle $EGC\triangle$ has a right angle at G. Then, necessarily, the perpendicular bisector of AB is the bisector of the angle of $EGC\triangle$ at the vertex E.

- D_α is the reflection of C about the point G.

Theorem 5.3. *The set $H_\alpha = \{A, B, C, D_\alpha\}$ has the Pompeiu property w.r.t. the isometry group.*

Proof. Suppose that the angle α is given, and let $D := D_\alpha$ for the sake of simplicity. For any point P let P' be the image of P under the reflection about the perpendicular bisector of AB. Then $A' = B$, $B' = A$ and the points C, C', D and D' form a symmetric trapezium such that $D'C = CC' = C'D$; see Figure 2. Using that

$$f(A) + f(B) + f(C) + f(D) = 0 \ \text{ and } \ f(A') + f(B') + f(C') + f(D') = 0$$

it follows that the alternating sum of the values of f at the vertices of the trapezium $CC'DD'$ vanishes, i.e.

$$f(C) - f(C') + f(D) - f(D') = 0. \tag{13}$$

Since equation (13) holds on any congruent copy of the trapezium $CC'DD'$ we have

$$f(C) - f(C') + f(D) - f(D') = 0 \ \text{ and } \ f(C') - f(D) + f(H') - f(C) = 0 \tag{14}$$

as Figure 2 shows: the trapezium $CC'DH'$ comes by a translation $C \mapsto C'$ and a rotation about the point D. Therefore

$$f(D') = f(H') \tag{15}$$

and equation (15) holds on any congruent copy of the segment $D'H'$ of measure r. This means that f takes the same values at any pair of points having distance r. Since any pair of points can be joined by a (finite) chain of circles with radius r it follows that f is a constant function. Especially, the constant must be zero. $\square$
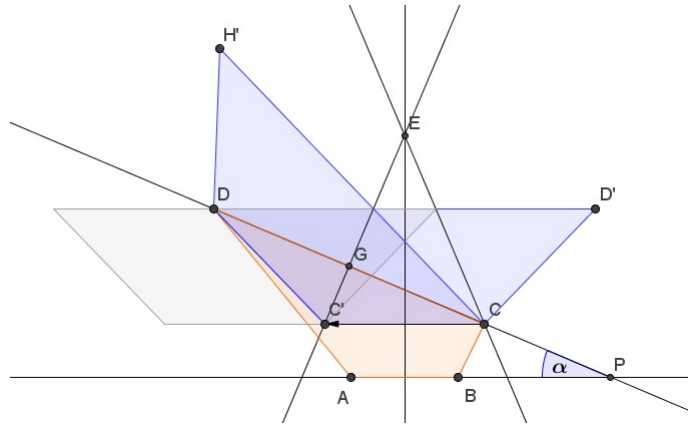
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Figure 2: The proof of Theorem 5.3.

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