HOW LARGE DIMENSION GUARANTEES A GIVEN ANGLE?

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Abstract. We study the following two problems:

(1) Given $n \geq 2$ and $0 \leq \alpha \leq 180°$, how large Hausdorff dimension can a compact set $A \subset \mathbb{R}^n$ have if A does not contain three points that form an angle α?

(2) Given α and δ, how large Hausdorff dimension can a subset A of a Euclidean space have if A does not contain three points that form an angle in the δ-neighborhood of α?

An interesting phenomenon is that different angles show different behaviour in the above problems. Apart from the clearly special extreme angles 0 and $180°$, the angles $60°, 90°$ and $120°$ also play special role in problem (2): the maximal dimension is smaller for these angles than for the other angles. In problem (1) the angle $90°$ seems to behave differently from other angles.

1. Introduction

The task of guaranteeing given patterns in a sufficiently large set has been a central problem in different areas of mathematics for a long time. Perhaps the most famous example is the celebrated theorem of Szemerédi [14], which states that any sequence of positive integers with positive upper density contains arbitrarily long arithmetic progressions.

More closely related to the present paper are the results of Falconer [3], Keleti [9] and Maga [10], which state that for any three points in $\mathbb{R}$ or in $\mathbb{R}^2$ there exists a set of full Hausdorff dimension that contains no similar copy to the three given points. It is open whether the analogous result holds in higher dimensions. In case of a negative answer it would be natural to ask what Hausdorff dimension guarantees a similar copy of three given points. Since the similar copy of a triangle has the same angles as the original one, the following question arose.

Question 1.1. *For given n and α, what is the smallest d for which any compact set $A \subset \mathbb{R}^n$ with Hausdorff dimension larger than d contains three points that form an angle α?*

We use the following terminology.

Definition 1.2. We say that the set $A \subset \mathbb{R}^n$ *contains the angle* $\alpha \in [0, 180°]$ if there exist distinct points $x, y, z \in A$ such that the angle between the vectors $y - x$ and $z - x$ is α.

Definition 1.3. By dim we denote the Hausdorff dimension.

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If $n \geq 2$ is an integer and $\alpha \in [0, 180°]$, then let

$$C(n,\alpha) = \sup\{s : \exists A \subset \mathbb{R}^n \text{ compact such that } \dim(A) = s \text{ and } A \text{ does not contain the angle } \alpha\}.$$

Clearly, answering Question 1.1 is the same as finding $C(n,\alpha)$. Somewhat surprisingly our results depend very much on the given angle. For 90° we show (Theorem 2.4) that $C(n, 90°) \leq [(n+1)/2]$ (where $[a]$ denotes the integer part of a) while for other angles we prove (Theorem 2.2) only $C(n,\alpha) \leq n-1$, which is sharp for $\alpha = 0$ and $\alpha = 180°$.

In the other direction, Máthé constructed compact sets of Hausdorff dimension $n/8$ not containing α; this construction is published separately in [12]. He obtains a better result ($n/4$) in the special case when $\cos^2\alpha$ is rational, and an even better one ($n/2$) when $\alpha = 90°$. Table 1 shows the best known bounds for $C(n,\alpha)$.

TABLE 1. Best known bounds for $C(n,\alpha)$

α	lower bound	upper bound
$0, 180°$	$n-1$	$n-1$
$90°$	$n/2$ [12, Thm 3.1]	$[(n+1)/2]$
$\cos^2\alpha \in \mathbb{Q}$	$n/4$ [12, Thm 3.2]	$n-1$
other angles	$n/8$ [12, Thm 3.4]	$n-1$

In the present paper for any $\alpha \in (0, 180°) \setminus \{60°, 90°, 120°\}$ we construct (Theorem 3.4) a self-similar compact set with Hausdorff dimension $c(\alpha)\log n$ that does not contain the angle α. This set is of smaller dimension than in Máthé's construction, but it avoids not just the angle α, but an interval of angles around α.

In light of the above construction it is natural to ask what can be said if we only want to guarantee an angle near to a given angle. In Section 4 we show that the previously mentioned special angles $(0, 60°, 90°, 120°, 180°)$ are really very special. If we fix α and a sufficiently small δ (but do not fix n) then for all other angles the above-mentioned self-similar construction (Theorem 3.4) gives a compact set with arbitrarily large Hausdorff dimension that does not contain any angle from the δ-neighborhood of α, while for the special angles this is not the case. More precisely, we show that any set with Hausdorff dimension larger than 1 contains angles arbitrarily close to the right angle (Theorem 4.1), and that any set with Hausdorff dimension larger than $\frac{C}{\delta}\log(\frac{1}{\delta})$ (with an absolute constant C) contains angles from the δ-neighborhoods of 60° and 120° (Corollary 4.7 and Theorem 4.12). For the angles 0 and 180° Erdős and Füredi showed [2] that any infinite set contains angles arbitrarily close to 0 and angles arbitrarily close to 180°.

Note that the previous two upper bounds were independent of n, the dimension of the ambient space. To express the results we introduce the following function $\widetilde{C}$.

Definition 1.4. If $\alpha \in [0, 180°]$ and $\delta > 0$, then let

$$\widetilde{C}(\alpha,\delta) = \sup\{\dim(A) : A \subset \mathbb{R}^n \text{ for } \textit{some } n; \ A \text{ does not contain any angle from } (\alpha-\delta, \alpha+\delta)\}.$$

Theorem 3.4 implies that $\widetilde{C}(\alpha,\delta) = \infty$ if α is different from the special angles 0, 60°, 90°, 120°, 180° and δ is smaller than the distance of α from the special angles. A construction of Harangi [7] shows that $\widetilde{C}(\alpha,\delta) \geq \frac{c}{\delta}/\log(\frac{1}{\delta})$ for the angles $\alpha = 60°, 120°$. We summarize the above results in Table 2.

We emphasize the difference between the tasks of finding an angle precisely and finding it approximately. For example, we can find angles arbitrarily close to 90°

TABLE 2. Smallest dimensions that guarantee an angle in the δ-neighborhood of α

α	$\widetilde{C}(\alpha,\delta)$	
$0, 180°$	$= 0$	
$90°$	$= 1$	
$60°, 120°$	$\approx 1/\delta$	apart from a multiplicative error $C \cdot \log(1/\delta)$
other angles	$= \infty$	provided that δ is sufficiently small

given that the dimension of the set is greater than 1, while if we want to find $90°$ precisely in the set, we need to know that its dimension is greater than $n/2$.

A related question is: How large does the Hausdorff dimension of a compact subset of $\mathbb{R}^n$ need to be to ensure that the set of angles contained in the set has positive Lebesgue measure? In [8] it is proved that larger than $\frac{n+1}{2}$ is enough and in [12] that $n/6$ is not enough.

Conway, Croft, Erdős and Guy [1] studied the distribution of values of angles determined by finite sets. The angles $60°$, $90°$ and $120°$ also have a special role in their results.

Notation 1.5. We denote the s-dimensional Hausdorff measure by $\mathcal{H}^s$.

Recall that compact sets having the property $0 < \mathcal{H}^s(K) < \infty$ are called *compact s-sets.*

Using the fact that an analytic set A with positive $\mathcal{H}^s$ measure contains a compact s-set (see e.g. [4, 2.10.47-48]) we get that in all of the above-mentioned results instead of compactness it is enough to assume that the set is analytic (or Borel). Similarly, we can always suppose that the given compact or analytic set is a compact s-set. Thus $C(n, \alpha)$ can be also expressed as

$$C(n,\alpha) = \sup\{s : \exists A \subset \mathbb{R}^n \text{ analytic such that } \dim(A) = s \text{ and } A \text{ does not contain the angle } \alpha\},$$

or

$$C(n,\alpha) = \sup\{s : \exists K \subset \mathbb{R}^n \text{ compact such that } 0 < \mathcal{H}^s(K) < \infty \text{ and } K \text{ does not contain the angle } \alpha\}.$$

However, some assumption on the set is necessary: for any given n and α we construct by transfinite induction a set in $\mathbb{R}^n$ of full Lebesgue outer measure that does not contain the angle α (Theorem 5.1).

Note that in the definition of $\widetilde{C}(\alpha, \delta)$ (Definition 1.4) there is no assumption on the set A. This is simply because the closure of A contains an angle in $(\alpha-\delta, \alpha+\delta)$ if and only if A does, so in these problems we can always assume that A is closed. Combining this with the above-mentioned fact that any analytic set with positive $\mathcal{H}^s$ measure contains a compact s-set we get

$$\widetilde{C}(\alpha,\delta) = \sup\{s : \exists n\ \exists K \subset \mathbb{R}^n \text{ compact such that } 0 < \mathcal{H}^s(K) < \infty \text{ and } K \text{ does not contain any angle from } (\alpha-\delta, \alpha+\delta)\}. \tag{1}$$

In fact, when we want to find an angle near to a given angle, then we get the same results if we replace Hausdorff dimension by upper Minkowski dimension, but this is not as clear as the above observation (see Corollary 5.7).

The following theorem, which is the first statement of [13, Theorem 10.11], is essential in some of our proofs.

Notation 1.6. The set of k-dimensional subspaces of $\mathbb{R}^n$ will be denoted by $G(n,k)$ and the natural probability measure on it by $\gamma_{n,k}$ (see e.g. [13] for more details).

Theorem 1.7. *If $m < s < n$ and A is an $\mathcal{H}^s$ measurable subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$, then*
$$\dim\bigl(A \cap (W + x)\bigr) = s - m$$
for $\mathcal{H}^s \times \gamma_{n,n-m}$ almost all $(x, W) \in A \times G(n, n-m)$.

In two dimensions this says that for $\mathcal{H}^s$ almost all $x \in A$, almost all lines through x intersect A in a set of dimension $s-1$. As one would expect, this theorem also holds for half-lines instead of lines (Marstrand [11, Lemma 17]). Note that Marstrand stated the result only for lines but he actually proved this for half-lines. Therefore the following theorem is also true.

Theorem 1.8. *Let $1 < s < 2$ and let $A \subset \mathbb{R}^2$ be $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$. For any $x \in \mathbb{R}^2$ and $\vartheta \in [0, 360°)$ let $L_{x,\vartheta}$ denote the half-line from x at angle ϑ. Then*
$$\dim\bigl(A \cap L_{x,\vartheta}\bigr) = s - 1$$
for $\mathcal{H}^s \times \lambda$ almost all $(x, \vartheta) \in A \times [0, 360°)$.

2. Finding a given angle

In this section we give estimates on $C(n, \alpha)$. For $n = 2$ we get the following exact result.

Theorem 2.1. *For any $\alpha \in [0, 180°]$ we have $C(2, \alpha) = 1$.*

Proof. A line has dimension 1 and it contains only the angles 0 and 180°. A circle also has dimension 1, but does not contain the angles 0 and 180°. Therefore $C(2, \alpha) \geq 1$ for all $\alpha \in [0, 180°]$.

For the other direction let $\alpha \in [0, 180°]$ and $s > 1$ fixed. We have to prove that any compact s-set contains the angle α. By Theorem 1.8, there exists $x \in K$ such that $\dim(K \cap L_{x,\vartheta}) = s - 1$ for almost all $\vartheta \in [0, 360°)$, where $L_{x,\vartheta}$ denotes the half-line from x at angle ϑ. Hence we can take $\vartheta_1, \vartheta_2 \in [0, 360°)$ such that $|\vartheta_1 - \vartheta_2| = \alpha$, and $\dim(K \cap L_{x,\vartheta_i}) = s - 1$ for $i = 1, 2$. If $x_i \in L_{x,\vartheta_i} \setminus \{x\}$ then the angle between the vectors $x_1 - x$ and $x_2 - x$ is α, so indeed, K contains the angle α. ☐

An analogous theorem holds for higher dimensions.

Theorem 2.2. *If $n \geq 2$ and $\alpha \in [0, 180°]$ then $C(n, \alpha) \leq n - 1$.*

Proof. We have already seen the case $n = 2$, so we may assume that $n \geq 3$. It is enough to show that if $s > n - 1$ and K is a compact s-set, then K contains the angle α. By Theorem 1.7, there exists $x \in K$ such that there exists a $W \in G(n, 2)$ with $\dim(B) = s - n + 2 > 1$ for $B \stackrel{\text{def}}{=} A \cap (W + a)$. The set B lies in a two-dimensional plane, so we can think about B as a subset of $\mathbb{R}^2$. Applying Theorem 2.1 completes the proof. ☐

Now we are able to give the exact value of $C(n, 0)$ and $C(n, 180°)$.

Theorem 2.3. $C(n, 0) = C(n, 180°) = n - 1$ *for all $n \geq 2$.*

Proof. One of the inequalities was proven in the previous theorem, while the other one is shown by the $(n-1)$-dimensional sphere. ☐

We prove a better upper bound for $C(n, 90°)$.

Theorem 2.4. *If n is even then $C(n, 90°) \leq n/2$. If n is odd then $C(n, 90°) \leq (n+1)/2$.*

Proof. Since $\mathbb{R}^n$ embeds into $\mathbb{R}^{n+1}$, we may assume that n is even. Let $s > n/2$ and let K be a compact s-set. From Theorem 1.7 we know that there exists a point $x \in K$ such that

$$\dim\bigl(K \cap (x + W)\bigr) = s - n/2 > 0 \tag{2}$$

for $\gamma_{n,n/2}$ almost all $W \in G(n, n/2)$. There exists a $W \in G(n, n/2)$ such that (2) holds both for W and $W^\perp$. As $(x + W) \cap (x + W^\perp) = \{x\}$, by choosing a $y \in K \cap (x + W)$ and $z \in K \cap (x + W^\perp)$ such that $x \neq y$ and $x \neq z$, we find a right angle at x in the triangle xyz. □

Remark 2.5. In fact, if n is even, $C(n, 90°) = n/2$. This follows from the following result of Máthé [12]: for any n there exists a compact set of Hausdorff dimension $n/2$ in $\mathbb{R}^n$ that does not contain $90°$.

The construction uses number theoretic ideas and the set contains angles arbitrarily close to $90°$. In the next section we will present a different approach where the constructed sets avoid not only a certain angle α but also a whole neighborhood of α.

3. A self-similar construction

In this section we construct a self-similar set in $\mathbb{R}^n$ with large dimension such that it does not contain a certain angle $\alpha \in (0, 180°)$. On the negative side, our method does not work for the angles $60°$, $90°$ and $120°$. On the positive side, the presented sets avoid a whole neighborhood of α, not only α.

We start with two simple lemmas.

Lemma 3.1. *Let $P_0, \dots, P_n$ be the vertices of a regular n-dimensional simplex. For any quadruple of indices (i, j, k, l) with $i \neq j$ and $k \neq l$, the angle between the lines P_iP_j and P_kP_l is either 0, $60°$ or $90°$.*

Proof. The set $\{P_i, P_j, P_k, P_l\}$ is the set of vertices of a one-, two-, or three-dimensional regular simplex. Our assertion is clear in each of these cases. □

Definition 3.2. A nonempty compact set K is *self-similar* if there exist contracting similarities $S_0, \dots, S_k$ such that $K = S_0(K) \cup \dots \cup S_k(K)$. If, in addition, the sets $S_i(K)$ are pairwise disjoint then K is said to satisfy the *strong separation condition.*

We say that the transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *homothety* if there exist a fixed point O and a nonzero real number r such that for any point P we have $f(P) - O = r(P - O)$. The number r is called the *ratio of magnification* and if $r \neq 1$, the single fixed point O is called the *center of the homothety.* We call a self-similar set $K = S_0(K) \cup \dots \cup S_k(K)$ *self-homothetic* if each S_i is a homothety.

Lemma 3.3. *Let $K = S_0(K) \cup \dots \cup S_k(K)$ be a self-homothetic set. Then, for any $x_0, x_1 \in K$, $x_0 \neq x_1$ there exist y_0, y_1 and $i \neq j$ such that $y_0 \in S_i(K)$ and $y_1 \in S_j(K)$ and $y_0 - y_1$ is parallel to $x_0 - x_1$.*

Proof. Since $x_0, x_1 \in K$, there exist sequences $i_1, i_2, \dots$ and $j_1, j_2, \dots$ such that

$$x_0 \in S_{i_1}\Bigl(S_{i_2}\bigl(\cdots S_{i_k}(K)\bigr)\Bigr) \quad \text{and} \quad x_1 \in S_{j_1}\Bigl(S_{j_2}\bigl(\cdots S_{j_k}(K_1)\bigr)\Bigr)$$

for every positive integer k.

Let k be the smallest positive integer such that $i_k \neq j_k$ (such a k exists else x_0 and x_1 would coincide). Set

$$S \stackrel{\text{def}}{=} S_{i_1}\Bigl(S_{i_2}\bigl(\cdots S_{i_{k-1}}(\cdot)\bigr)\Bigr).$$

There exist $y_0 \in S_{i_k}(K)$ and $y_1 \in S_{j_k}(K)$ such that $x_0 = S(y_0)$ and $x_1 = S(y_1)$. Since S is also a homothety, $y_0 - y_1$ is parallel to $x_0 - x_1$. □

Theorem 3.4. *For any $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that for every $n \geq 2$ there exists a compact self-similar (in fact, self-homothetic) set $K \subset \mathbb{R}^n$ with $\dim(K) \geq c_\varepsilon \log n$ and with the property that all angles occurring in the set lie in the ε-neighborhood of the angles $\{0, 60°, 90°, 120°, 180°\}$.*

In particular, for any $\alpha \in (0, 180°) \setminus \{60°, 90°, 120°\}$ we construct a compact set of dimension $c(\alpha) \log n$ that does not contain the angle α; moreover, the set avoids a small neighborhood of α.

Proof. Our set K will be a modified version of the Sierpiński gasket. Take a regular n-dimensional simplex with unit edge length in $\mathbb{R}^n$, denote its vertices by $P_0, \ldots, P_n$ and let $K_1 \overset{\text{def}}{=} \operatorname{conv}(\{P_0, \ldots, P_n\})$. Fix a $0 < \delta < 1/2$ and denote by S_i the homothety of ratio δ centered at P_i ($i = 0, \ldots, n$). The similarities S_i ($i = 0, \ldots, n$) uniquely determine a self-similar set K which can also be written in the following form:

$$K \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} \bigcup_{(i_1,\ldots,i_k)\in\{0,\ldots,n\}^k} S_{i_1}\Big(S_{i_2}\big(\cdots S_{i_k}(K_1)\big)\Big).$$

The set K clearly satisfies the strong separation condition. By [13, Theorem 4.14], the dimension of K is the unique positive number s for which $(n+1)\delta^s = 1$, therefore

$$\dim(K) = \frac{\log(n+1)}{\log \frac{1}{\delta}}.$$

We say that a *direction* $V \in G(n, 1)$ *occurs in a set* $H \subset \mathbb{R}^n$ if there are $x, y \in H$, $x \neq y$ such that $x - y$ is parallel to V. We will show that the directions occurring in K are actually *close* to the directions occurring in $\{P_0, \ldots, P_n\}$.

Let $V \in G(n, 1)$ which occurs in K and let $x_0, x_1 \in K$, $x_0 \neq x_1$ such that $x_0 - x_1$ is parallel to V. By Lemma 3.3 there exist $y_0, y_1 \in K$, $y_0 \neq y_1$ such that $y_0 - y_1$ is also parallel to V and there exist $i \neq j$ with $y_0 \in S_i(K)$ and $y_1 \in S_j(K)$.

We may assume without loss of generality that $y_0 \in S_0(K)$, $y_1 \in S_1(K)$. We will show that the angle φ between $y_0 - y_1$ and $P_0 - P_1$ is small, which is equivalent to $\cos \varphi$ being close to 1. Let $h_i = y_i - P_i$. We have $||h_i|| \leq \delta$ ($i = 0, 1$), hence

$$\cos \varphi = \frac{\langle y_0 - y_1, P_0 - P_1 \rangle}{||y_0 - y_1|| \cdot ||P_0 - P_1||} = \frac{1 + \langle h_0 - h_1, P_0 - P_1 \rangle}{||(P_0 - P_1) + (h_0 - h_1)||} \geq \frac{1 - 2\delta}{1 + 2\delta}.$$

Set $\varepsilon(\delta) = 2 \arccos(\frac{1-2\delta}{1+2\delta})$. Lemma 3.1 implies that the angles occurring in K are in the union of the following intervals: $[0, \varepsilon]$, $[60° - \varepsilon, 60° + \varepsilon]$, $[90° - \varepsilon, 90° + \varepsilon]$, $[120° - \varepsilon, 120° + \varepsilon]$, $[180° - \varepsilon, 180°]$. If δ, and therefore ε is sufficiently small, then none of these intervals contain α. □

Harangi [7] improved this result: he used the same methods to show that there exists a set with the same properties and with dimension $c_\varepsilon n$. Moreover, even for the angles 60° and 120° it is possible to construct large dimensional self-homothetic sets avoiding these angles.

However, as the next theorem shows, one cannot avoid the right angle with similar constructions.

Theorem 3.5. *Suppose $K = S_0(K) \cup S_1(K) \cup \cdots \cup S_k(K)$ is a self-homothetic set satisfying the strong separation condition (that is, $S_0, \ldots, S_k$ are homotheties with ratios less than 1 and the sets $S_i(K)$ are pairwise disjoint) and $\dim(K) > 1$. Then K contains four points that form a non-degenerate rectangle.*

Proof. We begin the proof by defining the following map:

$$D: \quad K \times K \setminus \{(x, x) : x \in K\} \to S^{n-1}; \quad (x, y) \mapsto \frac{x - y}{||x - y||}.$$

We denote the range of D by $\mathrm{Range}(D)$. The set $\mathrm{Range}(D)$ can be considered as the set of directions in K.

First, we prove that if K is such a self-similar set then $\mathrm{Range}(D)$ is closed. By Lemma 3.3, for any $x, y \in K$, $x \neq y$ there exist $x' \in S_i(K)$ and $y' \in S_j(K)$ for some $i \neq j$ such that $x' - y'$ is parallel to $x - y$. If $d(\cdot,\cdot)$ denotes the Euclidean distance then

$$\min_{0 \le i < j \le k} d(S_i(K), S_j(K)) = c > 0,$$

so $\mathrm{Range}(D)$ actually equals to the image of D restricted to the set $K_0 = K \times K \setminus \{(x,y) : d(x,y) < c\}$. As K_0 is compact and D is continuous, $\mathrm{Range}(D) = D(K_0)$ is indeed compact.

Next we show that for any $v \in S^{n-1}$ there exist $x, y \in K$, $x \neq y$ such that the vectors v and $D(x,y)$ are perpendicular. If this was not true, the compactness of $\mathrm{Range}(D)$ would imply that the orthogonal projection p to a line parallel to v would be a one-to-one map on K with p^{-1} being a Lipschitz map on $p(K)$. This would imply $\dim(K) \le 1$, which is a contradiction.

The homotheties $S_0 \circ S_1$ and $S_1 \circ S_0$ have the same ratio. Denote their fixed points by P and Q, respectively. Since $P \neq Q$, there are $x, y \in K$, $x \neq y$ such that $x - y$ is perpendicular to $P - Q$. It is easy to check that the points $S_0(S_1(x))$, $S_0(S_1(y))$, $S_1(S_0(y))$ and $S_1(S_0(x))$ form a non-degenerate rectangle. □

4. Finding angles close to a given angle

In this section we show the remaining claims made in Table 2.

We start by proving that a set that does not contain angles near to 90° must be very small, it cannot have Hausdorff dimension bigger than 1. This makes 90° very special since the analogous statement would be false for any other angle $\alpha \in (0, 180°)$ (see Theorem 3.4 and Remark 4.8). This result is clearly sharp since a line segment contains only 0 and 180°.

Theorem 4.1. *Any set $A \subset \mathbb{R}^n$ with Hausdorff dimension greater than 1 contains angles arbitrarily close to the right angle. Thus $\widetilde{C}(90°, \delta) = 1$ for any $\delta > 0$.*

Proof. By the equivalent definition (1) of $\widetilde{C}$ we can assume that A is compact and $0 < \mathcal{H}^s(A) < \infty$ for some $s > 1$. Applying Theorem 1.7 for $m = 1$ we obtain that for $\mathcal{H}^s$ almost all $x \in A$ the set $A \cap (W + x)$ has positive dimension for $\gamma_{n,n-1}$ almost all $W \in G(n, n-1)$. Let us fix a point x with this property and let $y \neq x$ be an arbitrary point in A.

Since for any fixed $\delta > 0$ the subspaces forming an angle at least $90° - \delta$ with $x - y$ have positive measure, and the exceptional set in Theorem 1.7 is of measure zero, the theorem follows. □

Now we prove the same result for upper Minkowski dimension instead of Hausdorff dimension. Note that the upper Minkowski dimension is always greater or equal than the Hausdorff dimension (see e.g. in [13]). Hence the following theorem is stronger than the previous one.

Theorem 4.2. *Any bounded set A in $\mathbb{R}^n$ with upper Minkowski dimension greater than 1 contains angles arbitrarily close to the right angle.*

The upper Minkowski dimension can be defined in many different ways, we will use the following definition (see [13, Section 5.3] for details).

Definition 4.3. By $B(x, r)$ we denote the closed ball with center $x \in \mathbb{R}^n$ and radius r. For a non-empty bounded set $A \subset \mathbb{R}^n$ let $P(A, \varepsilon)$ denote the greatest integer k for which there exist k disjoint balls $B(x_i, \varepsilon)$ with $x_i \in A$, $i = 1, \ldots, k$. The *upper Minkowski dimension* of A is defined as

$$\overline{\dim}_{\mathrm{M}}(A) \stackrel{\text{def}}{=} \sup\{s : \limsup_{\varepsilon \to 0+} P(A, \varepsilon)\varepsilon^s = \infty\}.$$

Note that we get an equivalent definition if we consider the lim sup through $\varepsilon = 2^{-k}$, $k \in \mathbb{N}$.

The following technical lemma is needed not only for the proof of Theorem 4.2 but also for the result about finding angles near to 60°. It roughly says that in a set of large upper Minkowski dimension one can find many points such that the distance between each pair is more or less the same.

Lemma 4.4. *Suppose that $\overline{\dim}_{\mathrm{M}}(A) > t$ for a bounded set $A \subset \mathbb{R}^n$ and a positive real t. Then for infinitely many positive integers k it is the case that for any integer $0 < l < k$ there are more than $2^{(k-l)t}$ points in A with the property that the distance of any two of them is between 2^{-k+1} and 2^{-l+2}.*

Proof. Let

$$r_k = P(A, 2^{-k})2^{-kt}.$$

Due to the previous definition $\limsup_{k\to\infty} r_k = \infty$. It follows that there are infinitely many values of k such that $r_k > r_l$ for all $l < k$. Let us fix such a k and let $0 < l < k$ be arbitrary.

By the definition of r_k, there are $r_k 2^{kt}$ disjoint balls with radii 2^{-k} and centers in A. Let $\mathcal{S}$ denote the set of the centers of these balls. Clearly the distance of any two of them is at least 2^{-k+1}.

Similarly, we can find a maximal system of disjoint balls $B(x_i, 2^{-l})$ with $x_i \in A$, $i = 1, \ldots, r_l 2^{lt}$. Consider the balls $B(x_i, 2^{-l+1})$ of doubled radii. These doubled balls cover the whole A (otherwise the original system would not be maximal). By the pigeonhole principle, one of these doubled balls contains at least

$$\frac{r_k 2^{kt}}{r_l 2^{lt}} = \frac{r_k}{r_l} 2^{(k-l)t} > 2^{(k-l)t}$$

points of $\mathcal{S}$. These points clearly have the desired property. □

Now we are in a position to prove the theorem.

Proof of Theorem 4.2. We can assume that $\operatorname{diam}(A) > 2$. Fix a t such that $\overline{\dim}_{\mathrm{M}}(A) > t > 1$.

Lemma 4.4 tells us that there are arbitrarily large integers k such that for any $l < k$ one can have more than $2^{(k-l)t}$ points in A such that each distance is between 2^{-k+1} and 2^{-l+2}. Let $\mathcal{S}$ be a set of such points and pick an arbitrary point $O \in \mathcal{S}$. Since $\operatorname{diam}(A) > 2$, there exists a point $P \in A$ with $OP \geq 1$. Now we project the points of $\mathcal{S}$ to the line OP. There must be two distinct points $Q_1, Q_2 \in \mathcal{S}$ such that the distance apart of their projections is at most

$$\frac{2^{-l+2}}{2^{(k-l)t}} = 2^{-l+2-(k-l)t},$$

It follows that

$$\cos\angle(\overrightarrow{Q_1Q_2}, \overrightarrow{OP}) \leq \frac{2^{-l+2-(k-l)t}}{2^{-k+1}} = 2^{-(k-l)(t-1)+1}.$$

Since $Q_1O \leq 2^{-l+2}$ and $OP \geq 1$, the angle of the lines OP and Q_1P is at most $C_1 2^{-l}$ with some constant C_1. Combining the previous results we get that

$$|\angle PQ_1Q_2 - 90°| \leq C_1 2^{-l} + C_2 2^{-(k-l)(t-1)}$$

with some constants C_1, C_2. The right hand side can be arbitrarily small since $t-1$ is positive and both l and $k-l$ can be chosen to be large. $\square$

Now we try to find angles close to $60°$. We will do that by finding three points forming an *almost regular* triangle provided that the dimension of the set is sufficiently large.

We will need a simple result from Ramsey theory. Let $R_r(3)$ denote the least positive integer k for which it holds that no matter how we color the edges of a complete graph on k vertices with r colors it contains a monochromatic triangle. The next inequality can be obtained easily:

$$R_r(3) \leq r \cdot R_{r-1}(3) - (r-2).$$

(A more general form of the above inequality can be found in e.g. [6, p. 90, Eq. 2].) It readily implies the following upper bound for $R_r(3)$.

Lemma 4.5. *For any positive integer $r \geq 2$*

$$R_r(3) \leq 3r!,$$

that is, any complete graph on at least $3r!$ vertices edge-colored by r colors contains a monochromatic triangle.

Using this lemma we can prove the following theorem.

Theorem 4.6. *There exists an absolute constant C such that whenever $\overline{\dim}_{\mathrm{M}}(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ for some bounded set $A \subset \mathbb{R}^n$ and $\delta > 0$ the following holds: A contains three points that form a δ-almost regular triangle, that is, the ratio of the length of the longest and shortest sides is at most $1+\delta$.*

As an immediate consequence, we can find angles close to $60°$.

Corollary 4.7. *Suppose that $\overline{\dim}_{\mathrm{M}}(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ for some bounded set $A \subset \mathbb{R}^n$ and $\delta > 0$. Then A contains angles from the interval $(60° - \delta, 60°]$ and also from $[60°, 60° + \delta)$. Therefore $\widetilde{C}(60°, \delta) \leq \frac{C}{\delta}\log(\frac{1}{\delta})$.*

Remark 4.8. The above theorem and even the corollary is essentially sharp: Harangi [7] constructed a set with Hausdorff dimension $\frac{c}{\delta}/\log(\frac{1}{\delta})$ and without any angles from the interval $(60° - \delta, 60° + \delta)$, so we have $\widetilde{C}(60°, \delta) \geq \frac{c}{\delta}/\log(\frac{1}{\delta})$.

Proof of Theorem 4.6. Let $t = \frac{C}{\delta}\log(\frac{1}{\delta})$ and apply Lemma 4.4 for $l = k-1$. We obtain at least 2^t points in A such that each distance is in the interval $[2^{-k+1}, 2^{-k+3}]$. Let $a = 2^{-k+1}$ and divide $[a, 4a]$ into $N = \lceil \frac{3}{\delta} \rceil$ disjoint intervals of length at most δa. Regard the points of A as the vertices of a graph. Color the edges of this graph with N colors according to which interval contains the distance of the corresponding points.

Easy computation shows that $2^t > 3N!$ (with a suitable choice of C). Therefore the above graph contains a monochromatic triangle by Lemma 4.5. It easily follows that the three corresponding points form a δ-almost regular triangle in $\mathbb{R}^n$. $\square$

Remark 4.9. The same proof yields the following: for any positive integer d and positive real δ there is a number $K(d, \delta)$ such that whenever $\overline{\dim}_{\mathrm{M}}(A) > K(d, \delta)$ for some bounded set A, one can find d points in A with the property that the ratio of the largest and the smallest distance among these points is at most $1+\delta$. (One needs to use the fact that the Ramsey number $R_r(d)$ is finite.)

In order to derive similar results for $120°$ instead of $60°$ we show that if large Hausdorff dimension implies the existence of an angle near α, then it also implies the existence of an angle near $180° - \alpha$.

Proposition 4.10. *Suppose that $s = s(\alpha, \delta, n)$ is a positive real number such that any analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $(\alpha - \delta, \alpha + \delta)$. Then any analytic set $B \subset \mathbb{R}^n$ with $\mathcal{H}^s(B) > 0$ contains an angle from the interval $(180° - \alpha - \delta', 180° - \alpha + \delta')$ for any $\delta' > \delta$.*

Proof. Again, we can assume that $0 < \mathcal{H}^s(B) < \infty$. It is well-known that for $\mathcal{H}^s$ almost all $x \in B$ the set $B \cap B(x, r)$ has positive $\mathcal{H}^s$ measure for any $r > 0$ [13, Theorem 6.2]. If we omit the exceptional points from B, this will be true for every point of the obtained set. Assume that B had this property in the first place. Then, by the assumptions of the proposition, any ball around any point of B contains an angle from the δ-neighborhood of α.

We define the points $P_m, Q_m, R_m \in B$ recursively in the following way. Fix a small ε. First take P_0, Q_0, R_0 such that the angle $\angle P_0Q_0R_0$ falls into the interval $(\alpha - \delta, \alpha + \delta)$. If the points P_m, Q_m, R_m are given, then choose points $P_{m+1}, Q_{m+1}, R_{m+1}$ from the $\varepsilon \cdot \min(Q_mP_m, Q_mR_m)$-neighborhood of P_m such that $\angle P_{m+1}Q_{m+1}R_{m+1} \in (\alpha - \delta, \alpha + \delta)$.

We can find two indices $k > l$ such that the angle enclosed by the vectors $\overrightarrow{Q_lP_l}$ and $\overrightarrow{Q_kP_k}$ is less than ε. It is clear that if we choose ε sufficiently small, then $\angle(Q_l, Q_k, R_k) \in (180° - \alpha - \delta', 180° - \alpha + \delta')$. □

Remark 4.11. Proposition 4.10 holds for $\delta' = \delta$ as well. Surprisingly, it even holds for some $\delta' < \delta$. The reason behind is the following. If every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $(\alpha - \delta, \alpha + \delta)$, then there necessarily exists a closed subinterval $[\alpha - \gamma, \alpha + \gamma]$ ($\gamma < \delta$) such that every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval $[\alpha - \gamma, \alpha + \gamma]$. We prove this statement in the Appendix (Theorem 5.3).

This implies that $\widetilde{C}$ satisfies the symmetry property

$$\widetilde{C}(\alpha, \delta) = \widetilde{C}(180° - \alpha, \delta).$$

Theorem 4.12. *There exists an absolute constant C such that any analytic set $A \subset \mathbb{R}^n$ with $\dim(A) > \frac{C}{\delta}\log(\frac{1}{\delta})$ contains an angle from the δ-neighborhood of $120°$. Therefore $\widetilde{C}(120°, \delta) \le \frac{C}{\delta}\log(\frac{1}{\delta})$.*

Proof. The claim readily follows from Corollary 4.7, Proposition 4.10 and the fact that the upper Minkowski dimension is greater or equal than the Hausdorff dimension. □

Remark 4.13. In fact, in Theorem 4.12 it is enough to assume that the upper Minkowski dimension is bigger than $\frac{C}{\delta}\log(\frac{1}{\delta})$. This follows from a more general result that we prove in the Appendix.

To find angles arbitrarily close to 0 and 180°, it suffices to have infinitely many points.

Proposition 4.14. *Any $A \subset \mathbb{R}^n$ of infinite cardinality contains angles arbitrarily close to 0 and angles arbitrarily close to 180°. Therefore $\widetilde{C}(0, \delta) = \widetilde{C}(180°, \delta) = 0$.*

Sketch of the proof. We claim that given N points in $\mathbb{R}^n$ they must contain an angle less than $\delta_1 = \frac{C}{\sqrt[n-1]{N}}$ and an angle greater than $180° - \delta_2$ with $\delta_2 = \frac{C}{\sqrt[n-1]{\log N}}$. The former follows easily from the pigeonhole principle. The latter is a result of Erdős and Füredi [2, Theorem 4.3]. □

5. Appendix

5.1. **A transfinite construction.** We prove the following theorem, which shows that if we allowed arbitrary sets in Definition 1.3 then $C(n, \alpha)$ would be n.

Theorem 5.1. *Let $n \geq 2$. For any $\alpha \in [0, 180°]$ there exists $H \subset \mathbb{R}^n$ such that H does not contain the angle α, and H has full Lebesgue outer measure; that is, its complement does not contain any measurable set with positive measure. In particular,* $\dim(H) = n$.

The proof we present here is shorter than our original proof, this one was suggested by Marianna Csörnyei.

We need the following simple lemma, which might be well-known even for more general sets but for completeness we present a proof. Recall that an algebraic set is the set of solutions of a system of polynomial equations.

Lemma 5.2. *Fewer than continuum many proper algebraic subsets of $\mathbb{R}^n$ cannot cover a Borel set of positive n-dimensional Lebesgue measure.*

Proof. We prove by induction. For $n = 1$ this is clear since proper algebraic subsets of $\mathbb{R}$ are finite and every Borel set of positive Lebesgue measure has cardinality continuum.

Suppose that the lemma holds for $n - 1$ but it is false for n, so there exists a collection $\mathcal{A}$ of less than continuum many proper algebraic subsets of $\mathbb{R}^n$ such that they cover a Borel set $B \subset \mathbb{R}^n$ with positive Lebesgue measure.

Let H^t denote the "horizontal" section $H^t = \{(x_1, \dots, x_{n-1}) : (x_1, \dots, x_{n-1}, t) \in H\}$ of a set $H \subset \mathbb{R}^n$ at "height" $t \in \mathbb{R}$. If A is a proper algebraic subset of $\mathbb{R}^n$ then with finitely many exceptions every A^t is a proper algebraic subset of $\mathbb{R}^{n-1}$. Therefore, by using the assumption that the lemma holds for $n - 1$, we get that $(\cup\mathcal{A})^t$ can contain Borel sets of positive $n - 1$-dimensional Lebesgue measure only for less than continuum many t. Let $f(t)$ denote the $(n-1)$-dimensional Lebesgue measure of the Borel set B^t. Since $B \subset \cup\mathcal{A}$, we obtain that $\{t : f(t) > 0\}$ has cardinality less than continuum.

On the other hand, by Fubini theorem f is measurable and its integral is the Lebesgue measure of B, so it is positive. This implies that $\{t : f(t) > 0\}$ is a measurable set of positive measure, hence it contains an uncountable compact set, so it must have the cardinality of the continuum, contradiction. □

Proof of Theorem 5.1. Take a well-ordering $\{B_\beta : \beta < \mathfrak{c}\}$ of the Borel subsets of $\mathbb{R}^n$ with positive n-dimensional Lebesgue measure. We will construct a sequence of points $\{x_\beta : \beta < \mathfrak{c}\}$ of $\mathbb{R}^n$ using transfinite induction so that

$$x_\beta \in B_\beta \qquad \text{and} \qquad H_\beta = \{x_\delta : \delta \leq \beta\} \text{ does not contain the angle } \alpha \tag{3}$$

for any $\beta < \mathfrak{c}$. This will complete the proof since then $H = \{x_\beta : \beta < \mathfrak{c}\}$ will have all the required properties.

Suppose that $\gamma < \mathfrak{c}$ and we have already properly defined x_β for all $\beta < \gamma$ so that (3) holds for all $\beta < \gamma$. For any $p, q \in \mathbb{R}^n$, $p \neq q$, let $A_{p,q}$ denote the set of those $x \in \mathbb{R}^n$ for which one of the angles of the triangle pqx is α. Note that $A_{p,q}$ can be covered by three proper algebraic subsets of $\mathbb{R}^n$. Then, by Lemma 5.2, the sets $A_{x_\delta, x_{\delta'}}$ $(\delta, \delta' < \gamma, x_\delta \neq x_{\delta'})$ cannot cover B_γ, so we can choose a point

$$x_\gamma \in B_\gamma \setminus \cup\{A_{x_\delta, x_{\delta'}} \ : \ \delta, \delta' < \gamma, \ x_\delta \neq x_{\delta'}\}.$$

Then (3) also holds for $\beta = \gamma$.

This way we obtain a sequence $(x_\beta)_{\beta < \mathfrak{c}}$, so that (3) holds for all $\beta < \mathfrak{c}$, which completes the proof. □

5.2. **The size of the neighborhood in the approximative problems.** Now, our goal is to prove the following theorem, which was claimed in Remark 4.11.

Theorem 5.3. *Suppose that $s = s(\alpha, \delta, n)$ is a positive real number such that every analytic set $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ contains an angle from the interval*

$(\alpha-\delta,\alpha+\delta)$. Then there exists a closed subinterval $[\alpha-\gamma,\alpha+\gamma]$ ($\gamma<\delta$) such that every analytic set $A\subset\mathbb{R}^n$ with $\mathcal{H}^s(A)>0$ contains an angle from the interval $[\alpha-\gamma,\alpha+\gamma]$.

To prove this theorem, we need two lemmas. For $r\in(0,\infty]$ let

$$\mathcal{H}^s_r(A)=\inf\left\{\sum_{i=1}^{\infty}\operatorname{diam}(U_i)^s\ :\ \operatorname{diam}(U_i)\le r,\ A\subset\cup_{i=1}^{\infty}U_i\right\},$$

thus $\mathcal{H}^s(A)=\lim_{r\to 0+}\mathcal{H}^s_r(A)$.

Lemma 5.4. *Let A_i be a sequence of compact sets converging in the Hausdorff metric to a set A. Then the following two statements hold.*

(i) $\mathcal{H}^s_\infty(A)\ge\limsup_{i\to\infty}\mathcal{H}^s_\infty(A_i)$.

(ii) *Suppose that for every $i=1,2,\dots$ the set A_i does not contain any angle from $[\alpha-\delta+1/i,\ \alpha+\delta-1/i]$. Then A does not contain any angle from $(\alpha-\delta,\ \alpha+\delta)$.*

Proof. The first statement is well-known and easy. To prove the second, notice that for any three points x,y,z of A there exist three points in A_i arbitrarily close to x,y,z, for sufficiently large i. □

The next lemma follows easily from [4, Theorem 2.10.17 (3)]. For the sake of completeness, we give a short direct proof.

Lemma 5.5. *Let $A\subset\mathbb{R}^n$ be a compact set satisfying $\mathcal{H}^s(A)>0$. Then there exists a ball B such that $\mathcal{H}^s_\infty(A\cap B)\ge c\,diam(B)^s$, where $c>0$ depends only on s.*

Proof. We may suppose without loss of generality that $\mathcal{H}^s(A)<\infty$. (Otherwise we choose a compact subset of A with positive and finite $\mathcal{H}^s$ measure. If the theorem holds for a subset of A then it clearly holds for A as well.)

Choose $r>0$ so that $\mathcal{H}^s_r(A)>\mathcal{H}^s(A)/2$. Cover A by sets U_i of diameter at most $r/2$ such that $\sum_i\operatorname{diam}(U_i)^s\le 2\mathcal{H}^s(A)$. Cover each U_i by a ball B_i of radius at most the diameter of U_i. Then the balls B_i cover A, have diameter at most r, and $\sum_i\operatorname{diam}(B_i)^s\le 2^{1+s}\mathcal{H}^s(A)$.

We claim that one of these balls B_i satisfies the conditions of the Lemma for $c=2^{-2-s}$. Otherwise we have

$$\mathcal{H}^s_\infty(A\cap B_i)<2^{-2-s}\operatorname{diam}(B_i)^s$$

for every i. Since the sets $A\cap B_i$ have diameter at most r, clearly $\mathcal{H}^s_r(A\cap B_i)=\mathcal{H}^s_\infty(A\cap B_i)$. Therefore

$$\begin{aligned}\mathcal{H}^s_r(A)\le\sum_i\mathcal{H}^s_r(A\cap B_i)&<\sum_i 2^{-2-s}\operatorname{diam}(B_i)^s\\&\le 2^{-2-s}2^{1+s}\mathcal{H}^s(A)=\mathcal{H}^s(A)/2,\end{aligned}$$

which contradicts the choice of r. □

Proof of Theorem 5.3. Suppose on the contrary that there exist compact sets $K_i\subset\mathbb{R}^n$ with $\mathcal{H}^s(K_i)>0$ such that K_i does not contain any angle from $[\alpha-\delta+1/i,\ \alpha+\delta-1/i]$. Choose a ball B_i for each compact set K_i according to Lemma 5.5. Let B be a ball of diameter 1. Let K_i' be the image of $K_i\cap B_i$ under a similarity transformation which maps B_i to the ball B. Thus $\mathcal{H}^s_\infty(K_i')\ge c$. Let K denote the limit of a convergent subsequence of the sets K_i. We can apply Lemma 5.4 to this subsequence and obtain $\mathcal{H}^s_\infty(K)\ge c$, implying $\mathcal{H}^s(K)>0$. Also, K does not contain any angle from the interval $(\alpha-\delta,\ \alpha+\delta)$, which is a contradiction. □

5.3. **Replacing Hausdorff dimension by upper Minkowski dimension.** Our final goal is showing that in the problems when we want angles only in a neighborhood of a given angle, Hausdorff dimension can be replaced by Minkowski dimension. This will follow from the following theorem. As pointed out by Pablo Shmerkin, this theorem also follows from a result of Furstenberg [5]. His result is much more general and it is not immediately trivial to see that it implies what we need. Therefore we give a direct self-contained proof.

Theorem 5.6. *Let $A \subset \mathbb{R}^d$ be a bounded set with upper Minkowski dimension $s > 0$. Then there exists a compact set K of Hausdorff dimension s such that all finite subsets of K are limits of homothetic copies of finite subsets of A. (That is, for every finite set $S \subset K$ and $\varepsilon > 0$ there exists a set $S' \subset A$ and $r > 0$, $t \in \mathbb{R}^d$ such that the Hausdorff distance of $t + rS'$ and S is at most ε.)*

Applying Theorem 5.6 to a bounded set A that does not contain any angle from an open interval we get a compact set K with the same property and with $\dim(K) = \overline{\dim}_{\mathrm{M}}(A)$. Thus we get the following.

Corollary 5.7. *For any $\alpha \in [0, 180°]$ and $\delta > 0$, we have*

$$\widetilde{C}(\alpha, \delta) = \sup\{\overline{\dim}_{\mathrm{M}}(A) : A \subset \mathbb{R}^n \text{ for some } n; A \text{ is bounded};\\ A \text{ does not contain any angle from } (\alpha - \delta, \alpha + \delta)\}.$$

Proof of Theorem 5.6. We will need to use a slightly different version of the Hausdorff content $\mathcal{H}^s_\infty(B)$ in this proof. Instead of covering $B \subset \mathbb{R}^d$ with arbitrary sets, we will only consider coverings with homothetic copies of the unit cube $[0,1]^d$. (From now on, a cube is always assumed to be a homothetic copy of the unit cube.) For a cube C, $\operatorname{diam}(C)$ is just a constant multiple of the edge length of C (denoted here by $|C|$). For the sake of simplicity, we will use $|C|$ in our definition: for any $B \subset \mathbb{R}^d$ and $s > 0$ let

$$\widehat{\mathcal{H}}^s_\infty(B) \overset{\text{def}}{=} \inf\left\{\sum_{i=1}^{\infty} |C_i|^s : C_i \text{ is a cube for each } i;\ B \subset \bigcup_{i=1}^{\infty} C_i\right\}.$$

It is easy to see that $d^{-s/2}\mathcal{H}^s_\infty \le \widehat{\mathcal{H}}^s_\infty \le \mathcal{H}^s_\infty$. Also note that $\widehat{\mathcal{H}}^s_\infty([0,1]^d) = 1$ for any $0 < s \le d$.

We may assume that $A \subset [0,1]^d$. For a positive integer n we divide the unit cube into n^d subcubes of edge length $1/n$. Let A_n be the union of the subcubes that intersect A.

We claim that for any fixed $0 < \delta < s/2$, for infinitely many n (depending on δ) there exists a cube C such that

$$|C| \ge n^{\frac{\delta}{2d}}/n \text{ and } \widehat{\mathcal{H}}^{s-2\delta}_\infty(C \cap A_n) \ge 2^{-s-2}|C|^{s-2\delta}. \tag{4}$$

First we show how the theorem follows from this claim. If (4) holds for n and C, then let K_n be the image of $C \cap A_n$ under the homothety that maps C to $[0,1]^d$. Hence $\widehat{\mathcal{H}}^{s-2\delta}_\infty(K_n) \ge 2^{-s-2}$. If $S \subset K_n$ is finite, then there exists S' such that the Hausdorff distance of S and S' is at most $\sqrt{d}n^{-\delta/(2d)}$ and a homothetic image of S' is in A.

For each $\delta = 1/l$ choose $n = n_l \ge l^l$ such that the claim holds. Let $\tilde{K}$ be the limit of a convergent subsequence of K_{n_l}. By Lemma 5.4 the Hausdorff dimension of $\tilde{K}$ is at least s. Let K be a compact subset of $\tilde{K}$ of Hausdorff dimension s. It is easy to check that K satisfies all the required properties.

It remains to prove the claim. Since $\overline{\dim}_{\mathrm{M}}(A) = s$, A_n contains at least $n^{s-\delta}$ subcubes for infinitely many n. Fix such an n with $n \ge 2^{4/\delta}$. Let

$$c = \min\left\{\widehat{\mathcal{H}}^{s-2\delta}_\infty(B)/m : B \text{ is the union of } m \text{ subcubes of } A_n,\ m \ge 1\right\}.$$

Since the unit cube covers A_n, by choosing B as the union of $m \geq n^{s-\delta}$ subcubes of A_n we get $c \leq \widehat{\mathcal{H}}^{s-2\delta}_\infty(B)/m \leq 1/n^{s-\delta}$. (On the other hand, one subcube has content $1/n^{s-2\delta}$, hence the minimum is taken for a set B for which m is at least n^δ.)

Suppose now that B is a set for which the minimum is taken; that is,

$$\widehat{\mathcal{H}}^{s-2\delta}_\infty(B) = cm,$$

where B consists of m subcubes of A_n. It follows that there exists a covering of B with cubes C_i $(i = 1, 2, \ldots)$ such that

$$\sum_{i=1}^{\infty} |C_i|^{s-2\delta} \leq 2cm.$$

Let $k = n^{\delta/(2d)}$. We say that a cube C_i is "bad" if $|C_i| < k/n$, and "good" otherwise. The total volume of the bad cubes is at most

$$\sum_{C_i \text{ is bad}} |C_i|^d = \sum_{C_i \text{ is bad}} |C_i|^{d-s+2\delta}|C_i|^{s-2\delta} \leq (k/n)^{d-s+2\delta} \sum_{i=1}^{\infty} |C_i|^{s-2\delta}$$
$$\leq 2cm(k/n)^{d-s+2\delta} \leq 2mk^{d-s+2\delta}n^{-\delta-d} \leq 2mk^d n^{-\delta-d} = 2mn^{-\frac{\delta}{2}-d} \leq \frac{m}{2}n^{-d},$$

where the last four estimates follow from $c \leq 1/n^{s-\delta}$, $\delta < s/2$, $k = n^{\delta/(2d)}$ and $n \geq 2^{4/\delta}$. So there are at most $m/2$ subcubes that are fully covered by bad cubes. Let B' be the union of the remaining (at least $m/2$) subcubes in B. Since each subcube in B' must intersect a good cube C_i, it follows that the cubes $2C_i$ cover B', where $2C_i$ is the cube with the same center as C_i and double edge length.

Then the definition of c implies that

$$\sum_{C_i \text{ is good}} \widehat{\mathcal{H}}^{s-2\delta}_\infty(2C_i \cap A_n) \geq \widehat{\mathcal{H}}^{s-2\delta}_\infty(B') \geq c\frac{m}{2}.$$

On the other hand, we have

$$\sum_{C_i \text{ is good}} |2C_i|^{s-2\delta} \leq 2^{s-2\delta} \sum_{i=1}^{\infty} |C_i|^{s-2\delta} \leq 2^{s-2\delta}2cm \leq 2^{s+1}cm.$$

Therefore there exists a good cube C_i such that

$$\widehat{\mathcal{H}}^{s-2\delta}_\infty(2C_i \cap A_n) \geq 2^{-s-2}|2C_i|^{s-2\delta}.$$

Thus (4) holds for the cube $C = 2C_i$, which completes the proof. □

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