

Computations of discrete orthogonal polynomials with *Wolfram Mathematica* (Krisztina Kiss, Ilona Nagy)

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The file consists of the following chapters:

1. Norms of discrete orthogonal polynomials (short version but slow method)
2. Norms of discrete orthogonal polynomials (long version but fast method)
3. The $p_{m,j}$ polynomials
4. The Ξ_2 matrix
5. Lambda matrix
6. Z_m matrices

In chapters 1 and 2 we provide two different methods to calculate the norm squares of the $p_{m,j}$ polynomials for $m = 1, \dots, 6$ and $m = 1, \dots, 5$ and at the end of the chapters we give a list of the results in a product form. Chapters 1 and 2 correspond to section 7 Appendix. **We used M instead of N in formula (2)** because N is a protected symbol in *Mathematica*.

Chapter 3 corresponds to section 7 Appendix. Chapter 4 corresponds to Remark 1.

Chapter 5 corresponds to section 3 and 5.1. Chapter 6 corresponds to section 2.3 and section 7 Appendix.

1. Norms of discrete orthogonal polynomials (short version but slow method)

Definitions

Polynomials

```
n = 100;
```

```
Maincoeff = Flatten[Table[a[i, j, j] → 1, {i, 1, n}, {j, 0, n}]];
```

$$\text{alpha}[m_, n_] := \frac{2^n \text{Product}[2 i - 1, \{i, 1, n\}]}{n!} \frac{1}{\text{Product}[M + i, \{i, 1, n\}]}$$

```
p[m_, n_, x_] := alpha[m, n] Sum[a[m, n, k] x^k, {k, 0, n}] /. Maincoeff
```

```
p[m_, i_, 0] := alpha[m, 0] a[m, i, 0] /. Maincoeff
```

Scalar product for $m = 1, \dots, 6$

```
sum[1, i_, j_] := sum[1, i, j] = Sum[p[1, i, x] p[1, j, x], {x, 0, M - 1}] /. Maincoeff
```

```
sum[2, i_, j_] :=
```

```
sum[2, i, j] = Sum[p[2, i, x] p[2, j, x], {x2, 0, M - 1}, {x, 0, x2}] /. Maincoeff
```

```
sum[3, i_, j_] := sum[3, i, j] =
```

```
Sum[p[3, i, x] p[3, j, x], {x3, 0, M - 1}, {x2, 0, x3}, {x, 0, x2}] /. Maincoeff
```

```
sum[4, i_, j_] := sum[4, i, j] = Sum[p[4, i, x] p[4, j, x],
```

```
{x4, 0, M - 1}, {x3, 0, x4}, {x2, 0, x3}, {x, 0, x2}] /. Maincoeff
```

```
sum[5, i_, j_] := sum[5, i, j] = Sum[p[5, i, x] p[5, j, x], {x5, 0, M - 1},
```

```
{x4, 0, x5}, {x3, 0, x4}, {x2, 0, x3}, {x, 0, x2}] /. Maincoeff
```

```
sum[6, i_, j_] := sum[6, i, j] = Sum[p[6, i, x] p[6, j, x], {x6, 0, M - 1},
```

```
{x5, 0, x6}, {x4, 0, x5}, {x3, 0, x4}, {x2, 0, x3}, {x, 0, x2}] /. Maincoeff
```

List of the coefficients

```
coeff[m_, n_] := Flatten[Table[a[m, j, i], {j, 1, n}, {i, 0, j - 1}], 1]
```

List of scalar products for the orthogonal polynomials

```
eq[m_, n_] := eq[m, n] = Flatten[Table[sum[m, i, j], {j, 1, n}, {i, 0, j - 1}], 1]
```

Substitution rules for the coefficients

```
rules[m_, n_] := rules[m, n] = Solve[eq[m, n] == 0, coeff[m, n]];
```

Norm squares

```
normsquare[m_, n_] :=
  normsquare[m, n] = Table[sum[m, i, i], {i, 0, n}] /. rules[m, n] // Factor
```

Calculations (up to degree 10)

The first output represents the total time to do the computation.

$m = 1$

```
m1 = AbsoluteTiming[normsquare[1, 10]]
```

```
{441.301900, {{M,  $\frac{(-1+M)M}{3(1+M)}$ ,  $\frac{(-2+M)(-1+M)M}{5(1+M)(2+M)}$ ,  $\frac{(-3+M)(-2+M)(-1+M)M}{7(1+M)(2+M)(3+M)}$ ,
 $\frac{(-4+M)(-3+M)(-2+M)(-1+M)M}{9(1+M)(2+M)(3+M)(4+M)}$ ,  $\frac{(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{11(1+M)(2+M)(3+M)(4+M)(5+M)}$ ,
 $\frac{(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{13(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}$ ,
 $\frac{(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{15(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)}$ ,
 $\frac{(-8+M)(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{17(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)(8+M)}$ ,
 $\frac{(-9+M)(-8+M)(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{19(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)(8+M)(9+M)}$ ,
 $\frac{(-10+M)(-9+M)(-8+M)(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{21(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)(8+M)(9+M)(10+M)}$ }}
```

$m = 2$

```
m2 = AbsoluteTiming[normsquare[2, 10]]
```

```
{452.048856, {{ $\frac{1}{2}M(1+M)$ ,  $\frac{(-1+M)M(2+M)}{9(1+M)}$ ,  $\frac{3(-2+M)(-1+M)M(3+M)}{50(1+M)(2+M)}$ ,
 $\frac{2(-3+M)(-2+M)(-1+M)M(4+M)}{49(1+M)(2+M)(3+M)}$ ,  $\frac{5(-4+M)(-3+M)(-2+M)(-1+M)M(5+M)}{162(1+M)(2+M)(3+M)(4+M)}$ ,
 $\frac{3(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M(6+M)}{121(1+M)(2+M)(3+M)(4+M)(5+M)}$ ,
 $\frac{7(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M(7+M)}{338(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}$ ,
 $\frac{4(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M(8+M)}{225(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)}$ ,
 $\frac{9(-8+M)(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M(9+M)}{578(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)(8+M)}$ ,
 $\frac{5(-9+M)(-8+M)(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M(10+M)}{(361(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)(8+M)(9+M))}$ ,
 $\frac{11(-10+M)(-9+M)(-8+M)(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M(11+M)}{(882(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)(8+M)(9+M)(10+M))}}$ 
```

$m = 3$

m3 = AbsoluteTiming[normsquare[3, 10]]

$$\{517.624607, \left\{ \left\{ \frac{1}{6} M (1+M) (2+M), \frac{(-1+M) M (2+M) (3+M)}{40 (1+M)}, \right. \right.$$

$$\frac{2 (-2+M) (-1+M) M (3+M) (4+M)}{175 (1+M) (2+M)}, \frac{25 (-3+M) (-2+M) (-1+M) M (4+M) (5+M)}{3528 (1+M) (2+M) (3+M)},$$

$$\frac{(-4+M) (-3+M) (-2+M) (-1+M) M (5+M) (6+M)}{198 (1+M) (2+M) (3+M) (4+M)},$$

$$(49 (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (6+M) (7+M)) /$$

$$(12584 (1+M) (2+M) (3+M) (4+M) (5+M)),$$

$$(8 (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (7+M) (8+M)) /$$

$$(2535 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M)),$$

$$(9 (-7+M) (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (8+M) (9+M)) /$$

$$(3400 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M)),$$

$$(25 (-8+M) (-7+M) (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (9+M)$$

$$(10+M)) / (10982 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M) (8+M)),$$

$$(121 (-9+M) (-8+M) (-7+M) (-6+M) (-5+M) (-4+M) (-3+M)$$

$$(-2+M) (-1+M) M (10+M) (11+M)) /$$

$$(60648 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M) (8+M) (9+M)),$$

$$(2 (-10+M) (-9+M) (-8+M) (-7+M) (-6+M) (-5+M)$$

$$(-4+M) (-3+M) (-2+M) (-1+M) M (11+M) (12+M)) /$$

$$(1127 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M) (8+M) (9+M) (10+M)) \left. \right\} \left. \right\}$$

$m = 4$

m4 = AbsoluteTiming[normsquare[4, 10]

```
{510.412194, { {  $\frac{1}{24} M (1 + M) (2 + M) (3 + M)$ ,
 $\frac{(-1 + M) M (2 + M) (3 + M) (4 + M)}{225 (1 + M)}$ ,  $\frac{(-2 + M) (-1 + M) M (3 + M) (4 + M) (5 + M)}{588 (1 + M) (2 + M)}$ ,
 $\frac{5 (-3 + M) (-2 + M) (-1 + M) M (4 + M) (5 + M) (6 + M)}{5292 (1 + M) (2 + M) (3 + M)}$ ,
 $\frac{(49 (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (5 + M) (6 + M) (7 + M))}{(78408 (1 + M) (2 + M) (3 + M) (4 + M))}$ ,
 $\frac{(28 (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (6 + M) (7 + M) (8 + M))}{(61347 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M))}$ ,
 $\frac{(3 (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (7 + M) (8 + M) (9 + M))}{(8450 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M))}$ ,
 $\frac{((-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (8 + M) (9 + M) (10 + M))}{(3468 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M))}$ ,  $\frac{(605 (-8 + M) (-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (9 + M) (10 + M) (11 + M))}{(2503896 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M) (8 + M))}$ ,
 $\frac{(11 (-9 + M) (-8 + M) (-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (10 + M) (11 + M) (12 + M))}{(53067 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M) (8 + M) (9 + M))}$ ,
 $\frac{(169 (-10 + M) (-9 + M) (-8 + M) (-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (11 + M) (12 + M) (13 + M))}{(933156 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M) (8 + M) (9 + M) (10 + M))}$  } } }
```

$m = 5$ **m5 = AbsoluteTiming[normsquare[5, 10]]**

```
{547.640323,
  { { 1
      M (1 + M) (2 + M) (3 + M) (4 + M),
      (-1 + M) M (2 + M) (3 + M) (4 + M) (5 + M),
      (-2 + M) (-1 + M) M (3 + M) (4 + M) (5 + M) (6 + M),
      (-3 + M) (-2 + M) (-1 + M) M (4 + M) (5 + M) (6 + M) (7 + M) } /
    { 120, 1512 (1 + M), 4704 (1 + M) (2 + M), 9504 (1 + M) (2 + M) (3 + M) },
  { 49 (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (5 + M) (6 + M) (7 + M) (8 + M) } /
    { 764 478 (1 + M) (2 + M) (3 + M) (4 + M) },
  { 9 (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (6 + M) (7 + M) (8 + M) (9 + M) } /
    { 204 490 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) },
  { 3 (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (7 + M) (8 + M) (9 + M) (10 + M) } /
    { 91 936 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) },
  { 121 (-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (8 + M) (9 + M)
    (10 + M) (11 + M) } / { 4 744 224 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M) },
  { 121 (-8 + M) (-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M)
    (-1 + M) M (9 + M) (10 + M) (11 + M) (12 + M) } /
    { 5 842 424 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M) (8 + M) },
  { 169 (-9 + M) (-8 + M) (-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M)
    (-2 + M) (-1 + M) M (10 + M) (11 + M) (12 + M) (13 + M) } /
    { 9 764 328 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M) (8 + M) (9 + M) },
  { 169 (-10 + M) (-9 + M) (-8 + M) (-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M)
    (-2 + M) (-1 + M) M (11 + M) (12 + M) (13 + M) (14 + M) } / { 11 426 400 (1 + M)
    (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M) (8 + M) (9 + M) (10 + M) } } }
```

$m = 6$

m6 = AbsoluteTiming[normsquare[6, 10]]

```
{732.271884, { { 1/720 M (1+M) (2+M) (3+M) (4+M) (5+M),
(-1+M) M (2+M) (3+M) (4+M) (5+M) (6+M) /
11760 (1+M),
(-2+M) (-1+M) M (3+M) (4+M) (5+M) (6+M) (7+M) /
43200 (1+M) (2+M),
((-3+M) (-2+M) (-1+M) M (4+M) (5+M) (6+M) (7+M) (8+M)) /
(98010 (1+M) (2+M) (3+M)),
(7 (-4+M) (-3+M) (-2+M) (-1+M) M (5+M) (6+M) (7+M) (8+M) (9+M)) /
(1226940 (1+M) (2+M) (3+M) (4+M)),
(3 (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (6+M) (7+M) (8+M) (9+M) (10+M)) /
(817960 (1+M) (2+M) (3+M) (4+M) (5+M)),
(121 (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (7+M) (8+M) (9+M)
(10+M) (11+M)) / (46887360 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M)),
(121 (-7+M) (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M)
M (8+M) (9+M) (10+M) (11+M) (12+M)) /
(62597400 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M)),
(1859 (-8+M) (-7+M) (-6+M) (-5+M) (-4+M) (-3+M) (-2+M)
(-1+M) M (9+M) (10+M) (11+M) (12+M) (13+M)) /
(1226909040 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M) (8+M)),
(169 (-9+M) (-8+M) (-7+M) (-6+M) (-5+M) (-4+M) (-3+M)
(-2+M) (-1+M) M (10+M) (11+M) (12+M) (13+M) (14+M)) /
(137497680 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M) (8+M) (9+M)),
(13 (-10+M) (-9+M) (-8+M) (-7+M) (-6+M) (-5+M) (-4+M) (-3+M)
(-2+M) (-1+M) M (11+M) (12+M) (13+M) (14+M) (15+M)) / (12696000
(1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M) (8+M) (9+M) (10+M)) } } }
```

2. Norms of discrete orthogonal polynomials (long version but fast method)

Formulas

Quit;

```

ClearAll[S, list, listi, listj, listk, listl, listm,
  listn, listM, s, i, j, k, l, m, n, M, Maincoeff, p, alpha];
S[i_] := Sum[ni, {n, 0, s}]

list = Table[S[i], {i, 0, 16}];
listi = list /. s → i;
listj = list /. s → j;
listk = list /. s → k;
listl = list /. s → l;
listm = list /. s → m;
listn = list /. s → n;
listM = list /. s → M - 1;

Maincoeff = {a[1, 0, 0] → 1, a[2, 0, 0] → 1, a[3, 0, 0] → 1, a[4, 0, 0] → 1,
  a[5, 0, 0] → 1, a[6, 0, 0] → 1, a[7, 0, 0] → 1, a[8, 0, 0] → 1,
  a[1, 1, 1] → 1, a[1, 2, 2] → 1, a[1, 3, 3] → 1, a[1, 4, 4] → 1,
  a[1, 5, 5] → 1, a[1, 6, 6] → 1, a[1, 7, 7] → 1, a[1, 8, 8] → 1,
  a[2, 1, 1] → 1, a[2, 2, 2] → 1, a[2, 3, 3] → 1, a[2, 4, 4] → 1,
  a[2, 5, 5] → 1, a[2, 6, 6] → 1, a[2, 7, 7] → 1, a[2, 8, 8] → 1,
  a[3, 1, 1] → 1, a[3, 2, 2] → 1, a[3, 3, 3] → 1, a[3, 4, 4] → 1,
  a[3, 5, 5] → 1, a[3, 6, 6] → 1, a[3, 7, 7] → 1, a[3, 8, 8] → 1,
  a[4, 1, 1] → 1, a[4, 2, 2] → 1, a[4, 3, 3] → 1, a[4, 4, 4] → 1,
  a[4, 5, 5] → 1, a[4, 6, 6] → 1, a[4, 7, 7] → 1, a[4, 8, 8] → 1,
  a[5, 1, 1] → 1, a[5, 2, 2] → 1, a[5, 3, 3] → 1, a[5, 4, 4] → 1,
  a[5, 5, 5] → 1, a[5, 6, 6] → 1, a[5, 7, 7] → 1, a[5, 8, 8] → 1,
  a[6, 1, 1] → 1, a[6, 2, 2] → 1, a[6, 3, 3] → 1, a[6, 4, 4] → 1,
  a[6, 5, 5] → 1, a[6, 6, 6] → 1, a[6, 7, 7] → 1, a[6, 8, 8] → 1};

p[m_, n_, x_] := alpha[m, n] Sum[a[m, n, k] xk, {k, 0, n}]
p[m_, i_, 0] := alpha[m, 0] a[m, i, 0]
alpha[m_, n_] := 
$$\frac{2^n \text{Product}[2 i - 1, \{i, 1, n\}]}{n!} \frac{1}{\text{Product}[M + i, \{i, 1, n\}]}$$


```


$m = 1$

Calculations

Program

```
sum1[y0_, z0_] :=
Module[{y = y0, z = z0, poly0, poly1, scoeff, spowers, icoeff, ipowers},
  poly0 = y z;
  scoeff = CoefficientList[poly0, s];
  spowers = Take[listM, Length[scoeff]];

  scoeff.spowers
]
```

Basis polynomials

```
ClearAll[poly10, poly20, poly21, poly30, poly31, poly32, poly40, poly41, poly42,
  poly43, poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62,
  poly63, poly64, poly65, poly70, poly71, poly72, poly73, poly74, poly75,
  poly76, poly80, poly81, poly82, poly83, poly84, poly85, poly86, poly87];
poly10 = sum1[1, p[1, 1, s]]; poly20 = sum1[1, p[1, 2, s]];
poly21 = sum1[p[1, 1, s], p[1, 2, s]]; poly30 = sum1[1, p[1, 3, s]];
poly31 = sum1[p[1, 1, s], p[1, 3, s]]; poly32 = sum1[p[1, 2, s], p[1, 3, s]];
poly40 = sum1[1, p[1, 4, s]]; poly41 = sum1[p[1, 1, s], p[1, 4, s]];
poly42 = sum1[p[1, 2, s], p[1, 4, s]]; poly43 = sum1[p[1, 3, s], p[1, 4, s]];
poly50 = sum1[1, p[1, 5, s]]; poly51 = sum1[p[1, 1, s], p[1, 5, s]];
poly52 = sum1[p[1, 2, s], p[1, 5, s]]; poly53 = sum1[p[1, 3, s], p[1, 5, s]];
poly54 = sum1[p[1, 4, s], p[1, 5, s]]; poly60 = sum1[1, p[1, 6, s]];
poly61 = sum1[p[1, 1, s], p[1, 6, s]]; poly62 = sum1[p[1, 2, s], p[1, 6, s]];
poly63 = sum1[p[1, 3, s], p[1, 6, s]]; poly64 = sum1[p[1, 4, s], p[1, 6, s]];
poly65 = sum1[p[1, 5, s], p[1, 6, s]]; poly70 = sum1[1, p[1, 7, s]];
poly71 = sum1[p[1, 1, s], p[1, 7, s]]; poly72 = sum1[p[1, 2, s], p[1, 7, s]];
poly73 = sum1[p[1, 3, s], p[1, 7, s]]; poly74 = sum1[p[1, 4, s], p[1, 7, s]];
poly75 = sum1[p[1, 5, s], p[1, 7, s]]; poly76 = sum1[p[1, 6, s], p[1, 7, s]];
poly80 = sum1[1, p[1, 8, s]]; poly81 = sum1[p[1, 1, s], p[1, 8, s]];
poly82 = sum1[p[1, 2, s], p[1, 8, s]]; poly83 = sum1[p[1, 3, s], p[1, 8, s]];
poly84 = sum1[p[1, 4, s], p[1, 8, s]]; poly85 = sum1[p[1, 5, s], p[1, 8, s]];
poly86 = sum1[p[1, 6, s], p[1, 8, s]]; poly87 = sum1[p[1, 7, s], p[1, 8, s]];
```

The coefficients as a function of M

rules1 =

```
Solve[{poly10, poly20, poly21, poly30, poly31, poly32, poly40, poly41, poly42,
  poly43, poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62,
  poly63, poly64, poly65, poly70, poly71, poly72, poly73, poly74, poly75,
  poly76, poly80, poly81, poly82, poly83, poly84, poly85, poly86, poly87} == 0,
{a[1, 1, 0], a[1, 2, 1], a[1, 2, 0], a[1, 3, 2], a[1, 3, 1], a[1, 3, 0],
  a[1, 4, 3], a[1, 4, 2], a[1, 4, 1], a[1, 4, 0], a[1, 5, 4], a[1, 5, 3],
  a[1, 5, 2], a[1, 5, 1], a[1, 5, 0], a[1, 6, 5], a[1, 6, 4], a[1, 6, 3],
  a[1, 6, 2], a[1, 6, 1], a[1, 6, 0], a[1, 7, 6], a[1, 7, 5], a[1, 7, 4],
  a[1, 7, 3], a[1, 7, 2], a[1, 7, 1], a[1, 7, 0], a[1, 8, 7], a[1, 8, 6],
  a[1, 8, 5], a[1, 8, 4], a[1, 8, 3], a[1, 8, 2], a[1, 8, 1], a[1, 8, 0]}][[1]]
{a[1, 1, 0] -> 1/2 (a[1, 1, 1] - M a[1, 1, 1]), a[1, 2, 1] -> a[1, 2, 2] - M a[1, 2, 2],
  a[1, 2, 0] -> 1/6 (2 - 3 M + M^2) a[1, 2, 2], a[1, 3, 2] -> -3/2 (-a[1, 3, 3] + M a[1, 3, 3]),
  a[1, 3, 1] -> 1/10 (11 - 15 M + 6 M^2) a[1, 3, 3],
  a[1, 3, 0] -> -1/20 (-6 + 11 M - 6 M^2 + M^3) a[1, 3, 3],
  a[1, 4, 3] -> -2 (-a[1, 4, 4] + M a[1, 4, 4]), a[1, 4, 2] -> 1/7 (17 - 21 M + 9 M^2) a[1, 4, 4],
  a[1, 4, 1] -> -1/7 (-10 + 17 M - 9 M^2 + 2 M^3) a[1, 4, 4],
  a[1, 4, 0] -> 1/70 (24 - 50 M + 35 M^2 - 10 M^3 + M^4) a[1, 4, 4],
  a[1, 5, 4] -> -5/2 (-a[1, 5, 5] + M a[1, 5, 5]), a[1, 5, 3] -> 5/9 (8 - 9 M + 4 M^2) a[1, 5, 5],
  a[1, 5, 2] -> -5/6 (-5 + 8 M - 4 M^2 + M^3) a[1, 5, 5],
  a[1, 5, 1] -> 1/126 (274 - 525 M + 365 M^2 - 105 M^3 + 15 M^4) a[1, 5, 5],
  a[1, 5, 0] -> -1/252 (-120 + 274 M - 225 M^2 + 85 M^3 - 15 M^4 + M^5) a[1, 5, 5],
  a[1, 6, 5] -> -3 (-a[1, 6, 6] + M a[1, 6, 6]),
  a[1, 6, 4] -> 5/22 (32 - 33 M + 15 M^2) a[1, 6, 6],
  a[1, 6, 3] -> -5/11 (-21 + 32 M - 15 M^2 + 4 M^3) a[1, 6, 6],
  a[1, 6, 2] -> 1/22 (178 - 315 M + 215 M^2 - 60 M^3 + 10 M^4) a[1, 6, 6],
  a[1, 6, 1] -> -1/22 (-84 + 178 M - 140 M^2 + 55 M^3 - 10 M^4 + M^5) a[1, 6, 6],
  a[1, 6, 0] -> 1/924 (720 - 1764 M + 1624 M^2 - 735 M^3 + 175 M^4 - 21 M^5 + M^6) a[1, 6, 6],
  a[1, 7, 6] -> -7/2 (-a[1, 7, 7] + M a[1, 7, 7]),
  a[1, 7, 5] -> 7/26 (41 - 39 M + 18 M^2) a[1, 7, 7],
  a[1, 7, 4] -> -35/52 (-28 + 41 M - 18 M^2 + 5 M^3) a[1, 7, 7],
```

$$\begin{aligned}
 a[1, 7, 3] &\rightarrow \frac{7}{286} (937 - 1540 M + 1030 M^2 - 275 M^3 + 50 M^4) a[1, 7, 7], a[1, 7, 2] \rightarrow \\
 &-\frac{7}{572} (-1414 + 2811 M - 2100 M^2 + 835 M^3 - 150 M^4 + 18 M^5) a[1, 7, 7], a[1, 7, 1] \rightarrow \frac{1}{858} \\
 &(6534 - 14\,847 M + 13\,223 M^2 - 5880 M^3 + 1505 M^4 - 189 M^5 + 14 M^6) a[1, 7, 7], a[1, 7, 0] \rightarrow \\
 &-\frac{1}{3432} (-5040 + 13\,068 M - 13\,132 M^2 + 6769 M^3 - 1960 M^4 + 322 M^5 - 28 M^6 + M^7) a[1, 7, 7], \\
 a[1, 8, 7] &\rightarrow -4 (-a[1, 8, 8] + M a[1, 8, 8]), \\
 a[1, 8, 6] &\rightarrow \frac{14}{15} (17 - 15 M + 7 M^2) a[1, 8, 8], \\
 a[1, 8, 5] &\rightarrow -\frac{14}{5} (-12 + 17 M - 7 M^2 + 2 M^3) a[1, 8, 8], \\
 a[1, 8, 4] &\rightarrow \frac{7}{39} (305 - 468 M + 307 M^2 - 78 M^3 + 15 M^4) a[1, 8, 8], \\
 a[1, 8, 3] &\rightarrow -\frac{14}{39} (-162 + 305 M - 216 M^2 + 86 M^3 - 15 M^4 + 2 M^5) a[1, 8, 8], \\
 a[1, 8, 2] &\rightarrow \frac{1}{2145} (88\,316 - 187\,110 M + 160\,951 M^2 - 69\,300 M^3 + 18\,375 M^4 - 2310 M^5 + 210 M^6) \\
 &a[1, 8, 8], a[1, 8, 1] \rightarrow -\frac{1}{2145} \\
 &2 (-18\,264 + 44\,158 M - 42\,399 M^2 + 21\,763 M^3 - 6300 M^4 + 1141 M^5 - 105 M^6 + 6 M^7) a[1, 8, 8], \\
 a[1, 8, 0] &\rightarrow \frac{1}{12\,870} (40\,320 - 109\,584 M + 118\,124 M^2 - 67\,284 M^3 + \\
 &22\,449 M^4 - 4536 M^5 + 546 M^6 - 36 M^7 + M^8) a[1, 8, 8] \}
 \end{aligned}$$

Norm squares (for $m = 1$ up to degree 8)

```

m1normsquare = {sum1[1, 1],
  sum1[p[1, 1, s], p[1, 1, s]],
  sum1[p[1, 2, s], p[1, 2, s]],
  sum1[p[1, 3, s], p[1, 3, s]],
  sum1[p[1, 4, s], p[1, 4, s]],
  sum1[p[1, 5, s], p[1, 5, s]],
  sum1[p[1, 6, s], p[1, 6, s]],
  sum1[p[1, 7, s], p[1, 7, s]],
  sum1[p[1, 8, s], p[1, 8, s]]} /. rules1 /. Maincoeff // Factor
{M,  $\frac{(-1+M)M}{3(1+M)}$ ,  $\frac{(-2+M)(-1+M)M}{5(1+M)(2+M)}$ ,  $\frac{(-3+M)(-2+M)(-1+M)M}{7(1+M)(2+M)(3+M)}$ ,
 $\frac{(-4+M)(-3+M)(-2+M)(-1+M)M}{9(1+M)(2+M)(3+M)(4+M)}$ ,  $\frac{(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{11(1+M)(2+M)(3+M)(4+M)(5+M)}$ ,
 $\frac{(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{13(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}$ ,
 $((-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M) /$ 
 $(15(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M))$ ,
 $((-8+M)(-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M) /$ 
 $(17(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)(8+M))$  }

```

$m = 2$

Calculations

Program

```

sum2[y0_, z0_] :=
Module[{y = y0, z = z0, poly0, poly1, scoeff, spowers, icoeff, ipowers},
  poly0 = y z;
  scoeff = CoefficientList[poly0, s];
  spowers = Take[listi, Length[scoeff]];

  poly1 = scoeff.spowers; (*sum 1*)
  icoeff = CoefficientList[poly1, i];
  ipowers = Take[listM, Length[icoeff]] // Expand;

  icoeff.ipowers // Expand(*sum 2*)
]

```

Basis polynomials

```

ClearAll[poly10, poly20, poly21, poly30, poly31, poly32, poly40, poly41, poly42,
  poly43, poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62,
  poly63, poly64, poly65, poly70, poly71, poly72, poly73, poly74, poly75,
  poly76, poly80, poly81, poly82, poly83, poly84, poly85, poly86, poly87];
poly10 = sum2[1, p[2, 1, s]]; poly20 = sum2[1, p[2, 2, s]];
poly21 = sum2[p[2, 1, s], p[2, 2, s]]; poly30 = sum2[1, p[2, 3, s]];
poly31 = sum2[p[2, 1, s], p[2, 3, s]]; poly32 = sum2[p[2, 2, s], p[2, 3, s]];
poly40 = sum2[1, p[2, 4, s]]; poly41 = sum2[p[2, 1, s], p[2, 4, s]];
poly42 = sum2[p[2, 2, s], p[2, 4, s]]; poly43 = sum2[p[2, 3, s], p[2, 4, s]];
poly50 = sum2[1, p[2, 5, s]]; poly51 = sum2[p[2, 1, s], p[2, 5, s]];
poly52 = sum2[p[2, 2, s], p[2, 5, s]]; poly53 = sum2[p[2, 3, s], p[2, 5, s]];
poly54 = sum2[p[2, 4, s], p[2, 5, s]]; poly60 = sum2[1, p[2, 6, s]];
poly61 = sum2[p[2, 1, s], p[2, 6, s]]; poly62 = sum2[p[2, 2, s], p[2, 6, s]];
poly63 = sum2[p[2, 3, s], p[2, 6, s]]; poly64 = sum2[p[2, 4, s], p[2, 6, s]];
poly65 = sum2[p[2, 5, s], p[2, 6, s]]; poly70 = sum2[1, p[2, 7, s]];
poly71 = sum2[p[2, 1, s], p[2, 7, s]]; poly72 = sum2[p[2, 2, s], p[2, 7, s]];
poly73 = sum2[p[2, 3, s], p[2, 7, s]]; poly74 = sum2[p[2, 4, s], p[2, 7, s]];
poly75 = sum2[p[2, 5, s], p[2, 7, s]]; poly76 = sum2[p[2, 6, s], p[2, 7, s]];
poly80 = sum2[1, p[2, 8, s]]; poly81 = sum2[p[2, 1, s], p[2, 8, s]];
poly82 = sum2[p[2, 2, s], p[2, 8, s]]; poly83 = sum2[p[2, 3, s], p[2, 8, s]];
poly84 = sum2[p[2, 4, s], p[2, 8, s]]; poly85 = sum2[p[2, 5, s], p[2, 8, s]];
poly86 = sum2[p[2, 6, s], p[2, 8, s]]; poly87 = sum2[p[2, 7, s], p[2, 8, s]];

```

The coefficients as a function of M

rules2 =

```
Solve[{poly10, poly20, poly21, poly30, poly31, poly32, poly40, poly41, poly42,
poly43, poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62,
poly63, poly64, poly65, poly70, poly71, poly72, poly73, poly74, poly75,
poly76, poly80, poly81, poly82, poly83, poly84, poly85, poly86, poly87} == 0,
{a[2, 1, 0], a[2, 2, 1], a[2, 2, 0], a[2, 3, 2], a[2, 3, 1], a[2, 3, 0],
a[2, 4, 3], a[2, 4, 2], a[2, 4, 1], a[2, 4, 0], a[2, 5, 4], a[2, 5, 3],
a[2, 5, 2], a[2, 5, 1], a[2, 5, 0], a[2, 6, 5], a[2, 6, 4], a[2, 6, 3],
a[2, 6, 2], a[2, 6, 1], a[2, 6, 0], a[2, 7, 6], a[2, 7, 5], a[2, 7, 4],
a[2, 7, 3], a[2, 7, 2], a[2, 7, 1], a[2, 7, 0], a[2, 8, 7], a[2, 8, 6],
a[2, 8, 5], a[2, 8, 4], a[2, 8, 3], a[2, 8, 2], a[2, 8, 1], a[2, 8, 0]}][[1]]
```

$$\{a[2, 1, 0] \rightarrow \frac{1}{3} (a[2, 1, 1] - M a[2, 1, 1]),$$

$$a[2, 2, 1] \rightarrow \frac{1}{5} (3 a[2, 2, 2] - 4 M a[2, 2, 2]), a[2, 2, 0] \rightarrow \frac{1}{10} (2 - 3 M + M^2) a[2, 2, 2],$$

$$a[2, 3, 2] \rightarrow -\frac{3}{7} (-2 a[2, 3, 3] + 3 M a[2, 3, 3]),$$

$$a[2, 3, 1] \rightarrow \frac{1}{7} (5 - 6 M + 3 M^2) a[2, 3, 3],$$

$$a[2, 3, 0] \rightarrow -\frac{1}{35} (-6 + 11 M - 6 M^2 + M^3) a[2, 3, 3],$$

$$a[2, 4, 3] \rightarrow -\frac{2}{9} (-5 a[2, 4, 4] + 8 M a[2, 4, 4]),$$

$$a[2, 4, 2] \rightarrow \frac{1}{3} (5 - 5 M + 3 M^2) a[2, 4, 4],$$

$$a[2, 4, 1] \rightarrow -\frac{1}{63} (-50 + 95 M - 45 M^2 + 12 M^3) a[2, 4, 4],$$

$$a[2, 4, 0] \rightarrow \frac{1}{126} (24 - 50 M + 35 M^2 - 10 M^3 + M^4) a[2, 4, 4],$$

$$a[2, 5, 4] \rightarrow -\frac{5}{11} (-3 a[2, 5, 5] + 5 M a[2, 5, 5]),$$

$$a[2, 5, 3] \rightarrow \frac{5}{11} (7 - 6 M + 4 M^2) a[2, 5, 5],$$

$$a[2, 5, 2] \rightarrow -\frac{5}{33} (-15 + 29 M - 12 M^2 + 4 M^3) a[2, 5, 5],$$

$$a[2, 5, 1] \rightarrow \frac{1}{66} (84 - 150 M + 115 M^2 - 30 M^3 + 5 M^4) a[2, 5, 5],$$

$$a[2, 5, 0] \rightarrow -\frac{1}{462} (-120 + 274 M - 225 M^2 + 85 M^3 - 15 M^4 + M^5) a[2, 5, 5],$$

$$a[2, 6, 5] \rightarrow -\frac{3}{13} (-7 a[2, 6, 6] + 12 M a[2, 6, 6]),$$

$$a[2, 6, 4] \rightarrow \frac{5}{26} (28 - 21 M + 15 M^2) a[2, 6, 6],$$

$$a[2, 6, 3] \rightarrow -\frac{5}{143} (-147 + 287 M - 105 M^2 + 40 M^3) a[2, 6, 6],$$

$$a[2, 6, 2] \rightarrow \frac{1}{286} (1414 - 2205 M + 1785 M^2 - 420 M^3 + 90 M^4) a[2, 6, 6],$$

$$a[2, 6, 1] \rightarrow -\frac{1}{143} (-294 + 665 M - 490 M^2 + 210 M^3 - 35 M^4 + 4 M^5) a[2, 6, 6],$$

$$\begin{aligned}
a[2, 6, 0] &\rightarrow \frac{1}{1716} (720 - 1764 M + 1624 M^2 - 735 M^3 + 175 M^4 - 21 M^5 + M^6) a[2, 6, 6], \\
a[2, 7, 6] &\rightarrow -\frac{7}{15} (-4 a[2, 7, 7] + 7 M a[2, 7, 7]), \\
a[2, 7, 5] &\rightarrow \frac{7}{5} (6 - 4 M + 3 M^2) a[2, 7, 7], \\
a[2, 7, 4] &\rightarrow -\frac{7}{39} (-56 + 110 M - 36 M^2 + 15 M^3) a[2, 7, 7], \\
a[2, 7, 3] &\rightarrow \frac{7}{39} (81 - 112 M + 94 M^2 - 20 M^3 + 5 M^4) a[2, 7, 7], \\
a[2, 7, 2] &\rightarrow -\frac{1}{2145} 7 (-2828 + 6329 M - 4200 M^2 + 1950 M^3 - 300 M^4 + 45 M^5) a[2, 7, 7], \\
a[2, 7, 1] &\rightarrow \frac{1}{2145} \\
&\quad (9132 - 19796 M + 18725 M^2 - 7840 M^3 + 2170 M^4 - 252 M^5 + 21 M^6) a[2, 7, 7], a[2, 7, 0] \rightarrow \\
&\quad -\frac{1}{6435} (-5040 + 13068 M - 13132 M^2 + 6769 M^3 - 1960 M^4 + 322 M^5 - 28 M^6 + M^7) a[2, 7, 7], \\
a[2, 8, 7] &\rightarrow -\frac{4}{17} (-9 a[2, 8, 8] + 16 M a[2, 8, 8]), \\
a[2, 8, 6] &\rightarrow \frac{14}{17} (15 - 9 M + 7 M^2) a[2, 8, 8], \\
a[2, 8, 5] &\rightarrow -\frac{14}{85} (-108 + 213 M - 63 M^2 + 28 M^3) a[2, 8, 8], \\
a[2, 8, 4] &\rightarrow \frac{7}{17} (87 - 108 M + 93 M^2 - 18 M^3 + 5 M^4) a[2, 8, 8], \\
a[2, 8, 3] &\rightarrow -\frac{14}{221} (-486 + 1077 M - 648 M^2 + 318 M^3 - 45 M^4 + 8 M^5) a[2, 8, 8], a[2, 8, 2] \rightarrow \\
&\quad \frac{1}{221} (5260 - 10206 M + 9807 M^2 - 3780 M^3 + 1155 M^4 - 126 M^5 + 14 M^6) a[2, 8, 8], a[2, 8, 1] \rightarrow \\
&\quad -\frac{1}{12155} 2 (-54792 + 138562 M - 127197 M^2 + 69027 M^3 - 18900 M^4 + 3675 M^5 - 315 M^6 + 20 M^7) \\
&\quad a[2, 8, 8], a[2, 8, 0] \rightarrow \frac{1}{24310} (40320 - 109584 M + 118124 M^2 - \\
&\quad 67284 M^3 + 22449 M^4 - 4536 M^5 + 546 M^6 - 36 M^7 + M^8) a[2, 8, 8] \}
\end{aligned}$$

Norm squares (for $m = 2$ up to degree 7)

```

m2normsquare = {sum2[1, 1],
  sum2[p[2, 1, s], p[2, 1, s]],
  sum2[p[2, 2, s], p[2, 2, s]],
  sum2[p[2, 3, s], p[2, 3, s]],
  sum2[p[2, 4, s], p[2, 4, s]],
  sum2[p[2, 5, s], p[2, 5, s]],
  sum2[p[2, 6, s], p[2, 6, s]],
  sum2[p[2, 7, s], p[2, 7, s]]} /. rules2 /. Maincoeff // Factor

```

$$\left\{ \frac{1}{2} M (1+M), \frac{(-1+M) M (2+M)}{9 (1+M)}, \frac{3 (-2+M) (-1+M) M (3+M)}{50 (1+M) (2+M)}, \right.$$

$$\frac{2 (-3+M) (-2+M) (-1+M) M (4+M)}{49 (1+M) (2+M) (3+M)}, \frac{5 (-4+M) (-3+M) (-2+M) (-1+M) M (5+M)}{162 (1+M) (2+M) (3+M) (4+M)},$$

$$\frac{3 (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (6+M)}{121 (1+M) (2+M) (3+M) (4+M) (5+M)},$$

$$(7 (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (7+M)) /$$

$$(338 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M)),$$

$$(4 (-7+M) (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (8+M)) /$$

$$(225 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M)) \left. \right\}$$

$m = 3$

Calculations

Program

```
sum3[y0_, z0_] := Module[{y = y0, z = z0, poly0, poly1,
  poly2, scoeff, spowers, icoeff, ipowers, jcoeff, jpowers},
  poly0 = y z;
  scoeff = CoefficientList[poly0, s];
  spowers = Take[listi, Length[scoeff]];

  poly1 = scoeff.spowers; (*sum 1*)
  icoeff = CoefficientList[poly1, i];
  ipowers = Take[listj, Length[icoeff]] // Expand;

  poly2 = icoeff.ipowers; (*sum 2*)
  jcoeff = CoefficientList[poly2, j];
  jpowers = Take[listM, Length[jcoeff]] // Expand;

  jcoeff.jpowers // Expand(*sum 3*)
]
```

Basis polynomials

```

ClearAll[poly10, poly20, poly21, poly30, poly31, poly32, poly40,
  poly41, poly42, poly43, poly50, poly51, poly52, poly53, poly54,
  poly60, poly61, poly62, poly63, poly64, poly65, poly70,
  poly71, poly72, poly73, poly74, poly75, poly76, poly80,
  poly81, poly82, poly83, poly84, poly85, poly86, poly87];
poly10 = sum3[1, p[3, 1, s]]; poly20 = sum3[1, p[3, 2, s]];
poly21 = sum3[p[3, 1, s], p[3, 2, s]]; poly30 = sum3[1, p[3, 3, s]];
poly31 = sum3[p[3, 1, s], p[3, 3, s]]; poly32 = sum3[p[3, 2, s], p[3, 3, s]];
poly40 = sum3[1, p[3, 4, s]]; poly41 = sum3[p[3, 1, s], p[3, 4, s]];
poly42 = sum3[p[3, 2, s], p[3, 4, s]]; poly43 = sum3[p[3, 3, s], p[3, 4, s]];
poly50 = sum3[1, p[3, 5, s]]; poly51 = sum3[p[3, 1, s], p[3, 5, s]];
poly52 = sum3[p[3, 2, s], p[3, 5, s]]; poly53 = sum3[p[3, 3, s], p[3, 5, s]];
poly54 = sum3[p[3, 4, s], p[3, 5, s]]; poly60 = sum3[1, p[3, 6, s]];
poly61 = sum3[p[3, 1, s], p[3, 6, s]]; poly62 = sum3[p[3, 2, s], p[3, 6, s]];
poly63 = sum3[p[3, 3, s], p[3, 6, s]]; poly64 = sum3[p[3, 4, s], p[3, 6, s]];
poly65 = sum3[p[3, 5, s], p[3, 6, s]]; poly70 = sum3[1, p[3, 7, s]];
poly71 = sum3[p[3, 1, s], p[3, 7, s]]; poly72 = sum3[p[3, 2, s], p[3, 7, s]];
poly73 = sum3[p[3, 3, s], p[3, 7, s]]; poly74 = sum3[p[3, 4, s], p[3, 7, s]];
poly75 = sum3[p[3, 5, s], p[3, 7, s]]; poly76 = sum3[p[3, 6, s], p[3, 7, s]];

```

The coefficients as a function of M

```

rules3 = Solve[
  {poly10, poly20, poly21, poly30, poly31, poly32, poly40, poly41, poly42, poly43,
    poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62, poly63,
    poly64, poly65, poly70, poly71, poly72, poly73, poly74, poly75, poly76} == 0,
  {a[3, 1, 0], a[3, 2, 1], a[3, 2, 0], a[3, 3, 2], a[3, 3, 1], a[3, 3, 0],
    a[3, 4, 3], a[3, 4, 2], a[3, 4, 1], a[3, 4, 0], a[3, 5, 4],
    a[3, 5, 3], a[3, 5, 2], a[3, 5, 1], a[3, 5, 0], a[3, 6, 5], a[3, 6, 4],
    a[3, 6, 3], a[3, 6, 2], a[3, 6, 1], a[3, 6, 0], a[3, 7, 6], a[3, 7, 5],
    a[3, 7, 4], a[3, 7, 3], a[3, 7, 2], a[3, 7, 1], a[3, 7, 0]}][[1]]

```


$$\begin{aligned}
& \{a[3, 1, 0] \rightarrow \frac{1}{4} (a[3, 1, 1] - M a[3, 1, 1]), a[3, 2, 1] \rightarrow \frac{1}{3} (a[3, 2, 2] - 2 M a[3, 2, 2]), \\
& a[3, 2, 0] \rightarrow \frac{1}{15} (2 - 3 M + M^2) a[3, 2, 2], a[3, 3, 2] \rightarrow -\frac{3}{8} (-a[3, 3, 3] + 3 M a[3, 3, 3]), \\
& a[3, 3, 1] \rightarrow \frac{1}{56} (31 - 27 M + 18 M^2) a[3, 3, 3], \\
& a[3, 3, 0] \rightarrow -\frac{1}{56} (-6 + 11 M - 6 M^2 + M^3) a[3, 3, 3], \\
& a[3, 4, 3] \rightarrow -\frac{2}{5} (-a[3, 4, 4] + 4 M a[3, 4, 4]), a[3, 4, 2] \rightarrow \frac{1}{5} (7 - 4 M + 4 M^2) a[3, 4, 4], \\
& a[3, 4, 1] \rightarrow -\frac{2}{15} (-3 + 8 M - 3 M^2 + M^3) a[3, 4, 4], \\
& a[3, 4, 0] \rightarrow \frac{1}{210} (24 - 50 M + 35 M^2 - 10 M^3 + M^4) a[3, 4, 4], \\
& a[3, 5, 4] \rightarrow -\frac{5}{12} (-a[3, 5, 5] + 5 M a[3, 5, 5]), \\
& a[3, 5, 3] \rightarrow \frac{5}{66} (37 - 15 M + 20 M^2) a[3, 5, 5], \\
& a[3, 5, 2] \rightarrow -\frac{5}{132} (-25 + 89 M - 24 M^2 + 12 M^3) a[3, 5, 5], \\
& a[3, 5, 1] \rightarrow \frac{1}{198} (177 - 235 M + 230 M^2 - 50 M^3 + 10 M^4) a[3, 5, 5], \\
& a[3, 5, 0] \rightarrow -\frac{1}{792} (-120 + 274 M - 225 M^2 + 85 M^3 - 15 M^4 + M^5) a[3, 5, 5], \\
& a[3, 6, 5] \rightarrow -\frac{3}{7} (-a[3, 6, 6] + 6 M a[3, 6, 6]), \\
& a[3, 6, 4] \rightarrow \frac{5}{91} (89 - 27 M + 45 M^2) a[3, 6, 6], \\
& a[3, 6, 3] \rightarrow -\frac{5}{91} (-33 + 148 M - 30 M^2 + 20 M^3) a[3, 6, 6], \\
& a[3, 6, 2] \rightarrow \frac{(3824 - 3525 M + 4500 M^2 - 750 M^3 + 225 M^4) a[3, 6, 6]}{1001}, \\
& a[3, 6, 1] \rightarrow -\frac{1}{1001} (-996 + 3068 M - 1815 M^2 + 940 M^3 - 135 M^4 + 18 M^5) a[3, 6, 6], \\
& a[3, 6, 0] \rightarrow \frac{1}{3003} (720 - 1764 M + 1624 M^2 - 735 M^3 + 175 M^4 - 21 M^5 + M^6) a[3, 6, 6], \\
& a[3, 7, 6] \rightarrow -\frac{7}{16} (-a[3, 7, 7] + 7 M a[3, 7, 7]), \\
& a[3, 7, 5] \rightarrow \frac{7}{80} (89 - 21 M + 42 M^2) a[3, 7, 7], \\
& a[3, 7, 4] \rightarrow -\frac{7}{16} (-7 + 38 M - 6 M^2 + 5 M^3) a[3, 7, 7], \\
& a[3, 7, 3] \rightarrow \frac{7}{208} (355 - 241 M + 382 M^2 - 50 M^3 + 20 M^4) a[3, 7, 7], a[3, 7, 2] \rightarrow \\
& -\frac{7}{208} (-112 + 441 M - 195 M^2 + 130 M^3 - 15 M^4 + 3 M^5) a[3, 7, 7], a[3, 7, 1] \rightarrow \frac{1}{11440} \\
& (32436 - 53144 M + 63063 M^2 - 22225 M^3 + 7175 M^4 - 735 M^5 + 70 M^6) a[3, 7, 7], a[3, 7, 0] \rightarrow \\
& -\frac{1}{11440} (-5040 + 13068 M - 13132 M^2 + 6769 M^3 - 1960 M^4 + 322 M^5 - 28 M^6 + M^7) a[3, 7, 7] \}
\end{aligned}$$

Norm squares (for $m = 3$ up to degree 7)

```

m3normsquare = {sum3[1, 1],
  sum3[p[3, 1, s], p[3, 1, s]],
  sum3[p[3, 2, s], p[3, 2, s]],
  sum3[p[3, 3, s], p[3, 3, s]],
  sum3[p[3, 4, s], p[3, 4, s]],
  sum3[p[3, 5, s], p[3, 5, s]],
  sum3[p[3, 6, s], p[3, 6, s]],
  sum3[p[3, 7, s], p[3, 7, s]]} /. rules3 /. Maincoeff // Factor
{
 $\frac{1}{6} M (1+M) (2+M)$ ,  $\frac{(-1+M) M (2+M) (3+M)}{40 (1+M)}$ ,
 $\frac{2 (-2+M) (-1+M) M (3+M) (4+M)}{175 (1+M) (2+M)}$ ,  $\frac{25 (-3+M) (-2+M) (-1+M) M (4+M) (5+M)}{3528 (1+M) (2+M) (3+M)}$ ,
 $\frac{(-4+M) (-3+M) (-2+M) (-1+M) M (5+M) (6+M)}{198 (1+M) (2+M) (3+M) (4+M)}$ ,
 $(49 (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (6+M) (7+M)) /$ 
 $(12584 (1+M) (2+M) (3+M) (4+M) (5+M))$ ,
 $(8 (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (7+M) (8+M)) /$ 
 $(2535 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M))$ ,
 $(9 (-7+M) (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (8+M) (9+M)) /$ 
 $(3400 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M))$ 
}

```

$m = 4$

Calculations

Program

```
sum4[y0_, z0_] := Module[{y = y0, z = z0, poly0, poly1, poly2, poly3,
  scoeff, spowers, icoeff, ipowers, jcoeff, jpowers, kcoeff, kpowers},
  poly0 = y z;
  scoeff = CoefficientList[poly0, s];
  spowers = Take[listi, Length[scoeff]];

  poly1 = scoeff.spowers; (*sum 1*)
  icoeff = CoefficientList[poly1, i];
  ipowers = Take[listj, Length[icoeff]] // Expand;

  poly2 = icoeff.ipowers; (*sum 2*)
  jcoeff = CoefficientList[poly2, j];
  jpowers = Take[listk, Length[jcoeff]] // Expand;

  poly3 = jcoeff.jpowers; (*sum 3*)
  kcoeff = CoefficientList[poly3, k];
  kpowers = Take[listM, Length[kcoeff]] // Expand;

  kcoeff.kpowers // Expand(*sum 4*)
]
```

Basis polynomials

```

ClearAll[poly10, poly20, poly21, poly30, poly31, poly32, poly40, poly41, poly42,
  poly43, poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62,
  poly63, poly64, poly65, poly70, poly71, poly72, poly73, poly74, poly75,
  poly76, poly80, poly81, poly82, poly83, poly84, poly85, poly86, poly87];
poly10 = sum4[1, p[4, 1, s]]; poly20 = sum4[1, p[4, 2, s]];
poly21 = sum4[p[4, 1, s], p[4, 2, s]];
poly30 = sum4[1, p[4, 3, s]]; poly31 = sum4[p[4, 1, s], p[4, 3, s]];
poly32 = sum4[p[4, 2, s], p[4, 3, s]]; poly40 = sum4[1, p[4, 4, s]];
poly41 = sum4[p[4, 1, s], p[4, 4, s]]; poly42 = sum4[p[4, 2, s], p[4, 4, s]];
poly43 = sum4[p[4, 3, s], p[4, 4, s]]; poly50 = sum4[1, p[4, 5, s]];
poly51 = sum4[p[4, 1, s], p[4, 5, s]]; poly52 = sum4[p[4, 2, s], p[4, 5, s]];
poly53 = sum4[p[4, 3, s], p[4, 5, s]]; poly54 = sum4[p[4, 4, s], p[4, 5, s]];
poly60 = sum4[1, p[4, 6, s]]; poly61 = sum4[p[4, 1, s], p[4, 6, s]];
poly62 = sum4[p[4, 2, s], p[4, 6, s]]; poly63 = sum4[p[4, 3, s], p[4, 6, s]];
poly64 = sum4[p[4, 4, s], p[4, 6, s]]; poly65 = sum4[p[4, 5, s], p[4, 6, s]];
poly70 = sum4[1, p[4, 7, s]]; poly71 = sum4[p[4, 1, s], p[4, 7, s]];
poly72 = sum4[p[4, 2, s], p[4, 7, s]]; poly73 = sum4[p[4, 3, s], p[4, 7, s]];
poly74 = sum4[p[4, 4, s], p[4, 7, s]]; poly75 = sum4[p[4, 5, s], p[4, 7, s]];
poly76 = sum4[p[4, 6, s], p[4, 7, s]];

```

The coefficients as a function of M

```

rules4 = Solve[
  {poly10, poly20, poly21, poly30, poly31, poly32, poly40, poly41, poly42, poly43,
    poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62, poly63,
    poly64, poly65, poly70, poly71, poly72, poly73, poly74, poly75, poly76} == 0,
  {a[4, 1, 0], a[4, 2, 1], a[4, 2, 0], a[4, 3, 2], a[4, 3, 1], a[4, 3, 0],
    a[4, 4, 3], a[4, 4, 2], a[4, 4, 1], a[4, 4, 0], a[4, 5, 4],
    a[4, 5, 3], a[4, 5, 2], a[4, 5, 1], a[4, 5, 0], a[4, 6, 5], a[4, 6, 4],
    a[4, 6, 3], a[4, 6, 2], a[4, 6, 1], a[4, 6, 0], a[4, 7, 6], a[4, 7, 5],
    a[4, 7, 4], a[4, 7, 3], a[4, 7, 2], a[4, 7, 1], a[4, 7, 0]}][[1]]

```

$$\begin{aligned}
& \{a[4, 1, 0] \rightarrow \frac{1}{5} (a[4, 1, 1] - M a[4, 1, 1]), a[4, 2, 1] \rightarrow \frac{1}{7} (a[4, 2, 2] - 4 M a[4, 2, 2]), \\
& a[4, 2, 0] \rightarrow \frac{1}{21} (2 - 3 M + M^2) a[4, 2, 2], a[4, 3, 2] \rightarrow -M a[4, 3, 3], \\
& a[4, 3, 1] \rightarrow \frac{1}{4} (2 - M + M^2) a[4, 3, 3], a[4, 3, 0] \rightarrow -\frac{1}{84} (-6 + 11 M - 6 M^2 + M^3) a[4, 3, 3], \\
& a[4, 4, 3] \rightarrow -\frac{2}{11} (a[4, 4, 4] + 8 M a[4, 4, 4]), \\
& a[4, 4, 2] \rightarrow \frac{1}{55} (77 - 12 M + 36 M^2) a[4, 4, 4], \\
& a[4, 4, 1] \rightarrow -\frac{2}{165} (-9 + 70 M - 18 M^2 + 8 M^3) a[4, 4, 4], \\
& a[4, 4, 0] \rightarrow \frac{1}{330} (24 - 50 M + 35 M^2 - 10 M^3 + M^4) a[4, 4, 4], \\
& a[4, 5, 4] \rightarrow -\frac{5}{13} (a[4, 5, 5] + 5 M a[4, 5, 5]), \\
& a[4, 5, 3] \rightarrow \frac{5}{39} (23 a[4, 5, 5] + 10 M^2 a[4, 5, 5]), \\
& a[4, 5, 2] \rightarrow -\frac{25}{143} (1 + 17 M - 2 M^2 + 2 M^3) a[4, 5, 5], \\
& a[4, 5, 1] \rightarrow \frac{1}{429} (346 - 210 M + 365 M^2 - 60 M^3 + 15 M^4) a[4, 5, 5], \\
& a[4, 5, 0] \rightarrow -\frac{(-120 + 274 M - 225 M^2 + 85 M^3 - 15 M^4 + M^5) a[4, 5, 5]}{1287}, \\
& a[4, 6, 5] \rightarrow -\frac{3}{5} (a[4, 6, 6] + 4 M a[4, 6, 6]), \\
& a[4, 6, 4] \rightarrow \frac{1}{7} (37 + 3 M + 15 M^2) a[4, 6, 6], \\
& a[4, 6, 3] \rightarrow -\frac{1}{91} (111 + 694 M - 30 M^2 + 80 M^3) a[4, 6, 6], \\
& a[4, 6, 2] \rightarrow \frac{1}{91} (364 - 45 M + 330 M^2 - 30 M^3 + 15 M^4) a[4, 6, 6], \\
& a[4, 6, 1] \rightarrow -\frac{1}{5005} (-828 + 12\,128 M - 4275 M^2 + 3250 M^3 - 375 M^4 + 60 M^5) a[4, 6, 6], \\
& a[4, 6, 0] \rightarrow \frac{1}{5005} (720 - 1764 M + 1624 M^2 - 735 M^3 + 175 M^4 - 21 M^5 + M^6) a[4, 6, 6], \\
& a[4, 7, 6] \rightarrow -\frac{7}{17} (2 a[4, 7, 7] + 7 M a[4, 7, 7]), \\
& a[4, 7, 5] \rightarrow \frac{7}{136} (166 + 21 M + 63 M^2) a[4, 7, 7], \\
& a[4, 7, 4] \rightarrow -\frac{245}{136} (2 + 9 M + M^3) a[4, 7, 7], \\
& a[4, 7, 3] \rightarrow \frac{7}{136} (262 + 25 M + 215 M^2 - 10 M^3 + 10 M^4) a[4, 7, 7], \\
& a[4, 7, 2] \rightarrow -\frac{7 (322 + 3409 M - 480 M^2 + 835 M^3 - 60 M^4 + 18 M^5) a[4, 7, 7]}{1768}, a[4, 7, 1] \rightarrow \\
& \frac{1}{1768} (4644 - 2758 M + 7070 M^2 - 1715 M^3 + 735 M^4 - 63 M^5 + 7 M^6) a[4, 7, 7], a[4, 7, 0] \rightarrow \\
& -\frac{1}{19\,448} (-5040 + 13\,068 M - 13\,132 M^2 + 6769 M^3 - 1960 M^4 + 322 M^5 - 28 M^6 + M^7) a[4, 7, 7] \}
\end{aligned}$$

Norm squares (for $m = 4$ up to degree 7)

```

m4normsquare = {sum4[1, 1],
  sum4[p[4, 1, s], p[4, 1, s]],
  sum4[p[4, 2, s], p[4, 2, s]],
  sum4[p[4, 3, s], p[4, 3, s]],
  sum4[p[4, 4, s], p[4, 4, s]],
  sum4[p[4, 5, s], p[4, 5, s]],
  sum4[p[4, 6, s], p[4, 6, s]]} /. rules4 /. Maincoeff // Factor
{
  1
  24 M (1 + M) (2 + M) (3 + M),
  (-1 + M) M (2 + M) (3 + M) (4 + M)
  225 (1 + M)
  (-2 + M) (-1 + M) M (3 + M) (4 + M) (5 + M)
  588 (1 + M) (2 + M)
  5 (-3 + M) (-2 + M) (-1 + M) M (4 + M) (5 + M) (6 + M)
  5292 (1 + M) (2 + M) (3 + M)
  49 (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (5 + M) (6 + M) (7 + M)
  78408 (1 + M) (2 + M) (3 + M) (4 + M)
  (28 (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (6 + M) (7 + M) (8 + M)) /
  (61347 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M)),
  (3 (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (7 + M) (8 + M) (9 + M)) /
  (8450 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M))
}

```

$m = 5$

Calculations

Program

```

sum5[y0_, z0_] :=
Module[{y = y0, z = z0, poly0, poly1, poly2, poly3, poly4, scoeff, spowers,
  icoeff, ipowers, jcoeff, jpowers, kcoeff, kpowers, lcoeff, lpowers},
poly0 = y z;
scoeff = CoefficientList[poly0, s];
spowers = Take[listi, Length[scoeff]];

poly1 = scoeff.spowers; (*sum 1*)
icoeff = CoefficientList[poly1, i];
ipowers = Take[listj, Length[icoeff]] // Expand;

poly2 = icoeff.ipowers; (*sum 2*)
jcoeff = CoefficientList[poly2, j];
jpowers = Take[listk, Length[jcoeff]] // Expand;

poly3 = jcoeff.jpowers; (*sum 3*)
kcoeff = CoefficientList[poly3, k];
kpowers = Take[listl, Length[kcoeff]] // Expand;

poly4 = kcoeff.kpowers; (*sum 4*)
lcoeff = CoefficientList[poly4, l];
lpowers = Take[listM, Length[lcoeff]] // Expand;

lcoeff.lpowers // Expand
]

```

Basis polynomials

```

ClearAll[poly10, poly20, poly21, poly30, poly31, poly32, poly40, poly41, poly42,
  poly43, poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62,
  poly63, poly64, poly65, poly70, poly71, poly72, poly73, poly74, poly75,
  poly76, poly80, poly81, poly82, poly83, poly84, poly85, poly86, poly87];
poly10 = sum5[1, p[5, 1, s]]; poly20 = sum5[1, p[5, 2, s]];
poly21 = sum5[p[5, 1, s], p[5, 2, s]]; poly30 = sum5[1, p[5, 3, s]];
poly31 = sum5[p[5, 1, s], p[5, 3, s]];
poly32 = sum5[p[5, 2, s], p[5, 3, s]]; poly40 = sum5[1, p[5, 4, s]];
poly41 = sum5[p[5, 1, s], p[5, 4, s]]; poly42 = sum5[p[5, 2, s], p[5, 4, s]];
poly43 = sum5[p[5, 3, s], p[5, 4, s]]; poly50 = sum5[1, p[5, 5, s]];
poly51 = sum5[p[5, 1, s], p[5, 5, s]]; poly52 = sum5[p[5, 2, s], p[5, 5, s]];
poly53 = sum5[p[5, 3, s], p[5, 5, s]]; poly54 = sum5[p[5, 4, s], p[5, 5, s]];
poly60 = sum5[1, p[5, 6, s]]; poly61 = sum5[p[5, 1, s], p[5, 6, s]];
poly62 = sum5[p[5, 2, s], p[5, 6, s]]; poly63 = sum5[p[5, 3, s], p[5, 6, s]];
poly64 = sum5[p[5, 4, s], p[5, 6, s]];
poly65 = sum5[p[5, 5, s], p[5, 6, s]];

```


The coefficients as a function of M

```
rules5 = Solve[{poly10, poly20, poly21,
  poly30, poly31, poly32, poly40, poly41, poly42, poly43,
  poly50, poly51, poly52, poly53, poly54, poly60, poly61, poly62,
  poly63, poly64, poly65} = 0, {a[5, 1, 0], a[5, 2, 1], a[5, 2, 0],
  a[5, 3, 2], a[5, 3, 1], a[5, 3, 0], a[5, 4, 3], a[5, 4, 2], a[5, 4, 1],
  a[5, 4, 0], a[5, 5, 4], a[5, 5, 3], a[5, 5, 2], a[5, 5, 1], a[5, 5, 0],
  a[5, 6, 5], a[5, 6, 4], a[5, 6, 3], a[5, 6, 2], a[5, 6, 1], a[5, 6, 0]}][[1]]
```

$$\{a[5, 1, 0] \rightarrow \frac{1}{6} (a[5, 1, 1] - M a[5, 1, 1]),$$

$$a[5, 2, 1] \rightarrow -\frac{1}{2} M a[5, 2, 2], a[5, 2, 0] \rightarrow \frac{1}{28} (2 - 3M + M^2) a[5, 2, 2],$$

$$a[5, 3, 2] \rightarrow -\frac{3}{10} (a[5, 3, 3] + 3M a[5, 3, 3]), a[5, 3, 1] \rightarrow \frac{1}{10} (5 - M + 2M^2) a[5, 3, 3],$$

$$a[5, 3, 0] \rightarrow -\frac{1}{120} (-6 + 11M - 6M^2 + M^3) a[5, 3, 3],$$

$$a[5, 4, 3] \rightarrow -\frac{2}{3} (a[5, 4, 4] + 2M a[5, 4, 4]),$$

$$a[5, 4, 2] \rightarrow \frac{1}{11} (17 + 2M + 6M^2) a[5, 4, 4],$$

$$a[5, 4, 1] \rightarrow -\frac{2}{165} (11 + 61M - 9M^2 + 6M^3) a[5, 4, 4],$$

$$a[5, 4, 0] \rightarrow \frac{1}{495} (24 - 50M + 35M^2 - 10M^3 + M^4) a[5, 4, 4],$$

$$a[5, 5, 4] \rightarrow -\frac{5}{14} (3 a[5, 5, 5] + 5M a[5, 5, 5]),$$

$$a[5, 5, 3] \rightarrow \frac{5}{91} (62 + 15M + 20M^2) a[5, 5, 5],$$

$$a[5, 5, 2] \rightarrow -\frac{25}{182} (9 + 21M + 2M^3) a[5, 5, 5],$$

$$a[5, 5, 1] \rightarrow \frac{(899 + 675M^2 - 75M^3 + 25M^4) a[5, 5, 5]}{1001},$$

$$a[5, 5, 0] \rightarrow -\frac{(-120 + 274M - 225M^2 + 85M^3 - 15M^4 + M^5) a[5, 5, 5]}{2002},$$

$$a[5, 6, 5] \rightarrow -\frac{3}{4} (2 a[5, 6, 6] + 3M a[5, 6, 6]),$$

$$a[5, 6, 4] \rightarrow \frac{5}{8} (10 + 3M + 3M^2) a[5, 6, 6],$$

$$a[5, 6, 3] \rightarrow -\frac{5}{28} (24 + 44M + 3M^2 + 4M^3) a[5, 6, 6],$$

$$a[5, 6, 2] \rightarrow \frac{1}{728} (3722 + 1395M + 2325M^2 - 60M^3 + 90M^4) a[5, 6, 6],$$

$$a[5, 6, 1] \rightarrow -\frac{1}{364} (234 + 833M - 90M^2 + 175M^3 - 15M^4 + 3M^5) a[5, 6, 6],$$

$$a[5, 6, 0] \rightarrow \frac{1}{8008} (720 - 1764M + 1624M^2 - 735M^3 + 175M^4 - 21M^5 + M^6) a[5, 6, 6]\}$$

Norm squares (for $m = 5$ up to degree 7)

```

m5normsquare = {sum5[1, 1],
  sum5[p[5, 1, s], p[5, 1, s]],
  sum5[p[5, 2, s], p[5, 2, s]],
  sum5[p[5, 3, s], p[5, 3, s]],
  sum5[p[5, 4, s], p[5, 4, s]],
  sum5[p[5, 5, s], p[5, 5, s]],
  sum5[p[5, 6, s], p[5, 6, s]]} /. rules5 /. Maincoeff // Factor
{
  1
  --- M (1 + M) (2 + M) (3 + M) (4 + M),  $\frac{(-1 + M) M (2 + M) (3 + M) (4 + M) (5 + M)}{1512 (1 + M)}$ ,
  120
  ---  $\frac{(-2 + M) (-1 + M) M (3 + M) (4 + M) (5 + M) (6 + M)}{4704 (1 + M) (2 + M)}$ ,
  4704 (1 + M) (2 + M)
  ---  $\frac{(-3 + M) (-2 + M) (-1 + M) M (4 + M) (5 + M) (6 + M) (7 + M)}{9504 (1 + M) (2 + M) (3 + M)}$ ,
  9504 (1 + M) (2 + M) (3 + M)
  ---  $(49 (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (5 + M) (6 + M) (7 + M) (8 + M)) /$ 
  (764 478 (1 + M) (2 + M) (3 + M) (4 + M)),
  (9 (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (6 + M) (7 + M) (8 + M) (9 + M)) /
  (204 490 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M)),
  (3 (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M (7 + M) (8 + M) (9 + M) (10 + M)) /
  (91 936 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M)) }

```

Norm squares

```

m1normsquare
{M,  $\frac{(-1 + M) M}{3 (1 + M)}$ ,  $\frac{(-2 + M) (-1 + M) M}{5 (1 + M) (2 + M)}$ ,  $\frac{(-3 + M) (-2 + M) (-1 + M) M}{7 (1 + M) (2 + M) (3 + M)}$ ,
   $\frac{(-4 + M) (-3 + M) (-2 + M) (-1 + M) M}{9 (1 + M) (2 + M) (3 + M) (4 + M)}$ ,  $\frac{(-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M}{11 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M)}$ ,
   $\frac{(-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M}{13 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M)}$ ,
  ---  $(-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M /$ 
  (15 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M)),
  ---  $(-8 + M) (-7 + M) (-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M) M /$ 
  (17 (1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M) (8 + M)) }

```

m2normsquare

$$\left\{ \frac{1}{2} M (1+M), \frac{(-1+M) M (2+M)}{9 (1+M)}, \frac{3 (-2+M) (-1+M) M (3+M)}{50 (1+M) (2+M)}, \right.$$

$$\frac{2 (-3+M) (-2+M) (-1+M) M (4+M)}{49 (1+M) (2+M) (3+M)}, \frac{5 (-4+M) (-3+M) (-2+M) (-1+M) M (5+M)}{162 (1+M) (2+M) (3+M) (4+M)},$$

$$\frac{3 (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (6+M)}{121 (1+M) (2+M) (3+M) (4+M) (5+M)},$$

$$(7 (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (7+M)) /$$

$$(338 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M)),$$

$$(4 (-7+M) (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (8+M)) /$$

$$(225 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M)) \left. \right\}$$

m3normsquare

$$\left\{ \frac{1}{6} M (1+M) (2+M), \frac{(-1+M) M (2+M) (3+M)}{40 (1+M)}, \right.$$

$$\frac{2 (-2+M) (-1+M) M (3+M) (4+M)}{175 (1+M) (2+M)}, \frac{25 (-3+M) (-2+M) (-1+M) M (4+M) (5+M)}{3528 (1+M) (2+M) (3+M)},$$

$$\frac{(-4+M) (-3+M) (-2+M) (-1+M) M (5+M) (6+M)}{198 (1+M) (2+M) (3+M) (4+M)},$$

$$(49 (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (6+M) (7+M)) /$$

$$(12584 (1+M) (2+M) (3+M) (4+M) (5+M)),$$

$$(8 (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (7+M) (8+M)) /$$

$$(2535 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M)),$$

$$(9 (-7+M) (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (8+M) (9+M)) /$$

$$(3400 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M) (7+M)) \left. \right\}$$

m4normsquare

$$\left\{ \frac{1}{24} M (1+M) (2+M) (3+M), \frac{(-1+M) M (2+M) (3+M) (4+M)}{225 (1+M)}, \right.$$

$$\frac{(-2+M) (-1+M) M (3+M) (4+M) (5+M)}{588 (1+M) (2+M)},$$

$$\frac{5 (-3+M) (-2+M) (-1+M) M (4+M) (5+M) (6+M)}{5292 (1+M) (2+M) (3+M)},$$

$$\frac{49 (-4+M) (-3+M) (-2+M) (-1+M) M (5+M) (6+M) (7+M)}{78408 (1+M) (2+M) (3+M) (4+M)},$$

$$(28 (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (6+M) (7+M) (8+M)) /$$

$$(61347 (1+M) (2+M) (3+M) (4+M) (5+M)),$$

$$(3 (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (7+M) (8+M) (9+M)) /$$

$$(8450 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M)) \left. \right\}$$

m5normsquare

$$\left\{ \frac{1}{120} M (1+M) (2+M) (3+M) (4+M), \frac{(-1+M) M (2+M) (3+M) (4+M) (5+M)}{1512 (1+M)}, \right. \\ \left. \frac{(-2+M) (-1+M) M (3+M) (4+M) (5+M) (6+M)}{4704 (1+M) (2+M)}, \right. \\ \left. \frac{(-3+M) (-2+M) (-1+M) M (4+M) (5+M) (6+M) (7+M)}{9504 (1+M) (2+M) (3+M)}, \right. \\ (49 (-4+M) (-3+M) (-2+M) (-1+M) M (5+M) (6+M) (7+M) (8+M)) / \\ (764 478 (1+M) (2+M) (3+M) (4+M)), \\ (9 (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (6+M) (7+M) (8+M) (9+M)) / \\ (204 490 (1+M) (2+M) (3+M) (4+M) (5+M)), \\ (3 (-6+M) (-5+M) (-4+M) (-3+M) (-2+M) (-1+M) M (7+M) (8+M) (9+M) (10+M)) / \\ (91 936 (1+M) (2+M) (3+M) (4+M) (5+M) (6+M)) \left. \right\}$$

3. The $p_{m,j}$ polynomials

$m = 1, j = 0, \dots, 5$

Table[p[1, i, x] /. rules1 /. Maincoeff, {i, 0, 5}]

$$\left\{ 1, \frac{2 \left(\frac{1-M}{2} + x \right)}{1+M}, \frac{6 \left(\frac{1}{6} (2 - 3M + M^2) + (1-M) x + x^2 \right)}{(1+M) (2+M)}, \right. \\ \left. \left(20 \left(\frac{1}{20} (6 - 11M + 6M^2 - M^3) + \frac{1}{10} (11 - 15M + 6M^2) x - \frac{3}{2} (-1+M) x^2 + x^3 \right) \right) / \right. \\ \left. ((1+M) (2+M) (3+M)), \right. \\ \left(70 \left(\frac{1}{70} (24 - 50M + 35M^2 - 10M^3 + M^4) - \frac{1}{7} (-10 + 17M - 9M^2 + 2M^3) x + \right. \right. \\ \left. \left. \frac{1}{7} (17 - 21M + 9M^2) x^2 - 2 (-1+M) x^3 + x^4 \right) \right) / ((1+M) (2+M) (3+M) (4+M)), \\ \left(252 \left(\frac{1}{252} (120 - 274M + 225M^2 - 85M^3 + 15M^4 - M^5) + \right. \right. \\ \left. \frac{1}{126} (274 - 525M + 365M^2 - 105M^3 + 15M^4) x - \frac{5}{6} (-5 + 8M - 4M^2 + M^3) x^2 + \right. \\ \left. \left. \frac{5}{9} (8 - 9M + 4M^2) x^3 - \frac{5}{2} (-1+M) x^4 + x^5 \right) \right) / ((1+M) (2+M) (3+M) (4+M) (5+M)) \left. \right\}$$

$m = 2, j = 0, \dots, 5$

Table[p[2, i, x] /. rules2 /. Maincoeff, {i, 0, 5}]

$$\left\{ 1, \frac{2 \left(\frac{1-M}{3} + x \right)}{1+M}, \frac{6 \left(\frac{1}{10} (2-3M+M^2) + \frac{1}{5} (3-4M)x + x^2 \right)}{(1+M)(2+M)}, \right. \\ \left. \left(20 \left(\frac{1}{35} (6-11M+6M^2-M^3) + \frac{1}{7} (5-6M+3M^2)x - \frac{3}{7} (-2+3M)x^2 + x^3 \right) \right) / \right. \\ \left. ((1+M)(2+M)(3+M)), \right. \\ \left. \left(70 \left(\frac{1}{126} (24-50M+35M^2-10M^3+M^4) - \frac{1}{63} (-50+95M-45M^2+12M^3)x + \right. \right. \right. \\ \left. \left. \frac{1}{3} (5-5M+3M^2)x^2 - \frac{2}{9} (-5+8M)x^3 + x^4 \right) \right) / ((1+M)(2+M)(3+M)(4+M)), \\ \left. \left(252 \left(\frac{1}{462} (120-274M+225M^2-85M^3+15M^4-M^5) + \frac{1}{66} (84-150M+115M^2-30M^3+5M^4) \right. \right. \right. \\ \left. \left. \left. x - \frac{5}{33} (-15+29M-12M^2+4M^3)x^2 + \frac{5}{11} (7-6M+4M^2)x^3 - \frac{5}{11} (-3+5M)x^4 + x^5 \right) \right) / \right. \\ \left. ((1+M)(2+M)(3+M)(4+M)(5+M)) \right\}$$

$m = 3, j = 0, \dots, 5$

Table[p[3, i, x] /. rules3 /. Maincoeff, {i, 0, 5}]

$$\left\{ 1, \frac{2 \left(\frac{1-M}{4} + x \right)}{1+M}, \frac{6 \left(\frac{1}{15} (2-3M+M^2) + \frac{1}{3} (1-2M)x + x^2 \right)}{(1+M)(2+M)}, \right. \\ \left. \left(20 \left(\frac{1}{56} (6-11M+6M^2-M^3) + \frac{1}{56} (31-27M+18M^2)x - \frac{3}{8} (-1+3M)x^2 + x^3 \right) \right) / \right. \\ \left. ((1+M)(2+M)(3+M)), \right. \\ \left. \left(70 \left(\frac{1}{210} (24-50M+35M^2-10M^3+M^4) - \frac{2}{15} (-3+8M-3M^2+M^3)x + \right. \right. \right. \\ \left. \left. \frac{1}{5} (7-4M+4M^2)x^2 - \frac{2}{5} (-1+4M)x^3 + x^4 \right) \right) / ((1+M)(2+M)(3+M)(4+M)), \\ \left. \left(252 \left(\frac{1}{792} (120-274M+225M^2-85M^3+15M^4-M^5) + \right. \right. \right. \\ \left. \left. \frac{1}{198} (177-235M+230M^2-50M^3+10M^4)x - \frac{5}{132} (-25+89M-24M^2+12M^3)x^2 + \right. \right. \\ \left. \left. \frac{5}{66} (37-15M+20M^2)x^3 - \frac{5}{12} (-1+5M)x^4 + x^5 \right) \right) / \right. \\ \left. ((1+M)(2+M)(3+M)(4+M)(5+M)) \right\}$$

4. The Ξ_2 matrix

G1 and G2 respectively denote the matrices $G(1, v_1)$ and $G(2, v_2)$ in Remark 1. The Ξ_2 matrix is given in list form and matrix form.

ClearAll[r, q]

r[M_, m_, i_] := m! Binomial[M-1+m-i, m];

q[M_, m_, j_, i_] := r[M, m-1, i] p[m, j, i] /. rules1 /. rules2 /. Maincoeff

G1 =

$$\begin{aligned}
& \{\text{CoefficientList}[p[1, 0, x] /. \text{rules1} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0, 0, 0, 0, 0\}, \\
& \text{CoefficientList}[p[1, 1, x] /. \text{rules1} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0, 0, 0, 0\}, \\
& \text{CoefficientList}[p[1, 2, x] /. \text{rules1} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0, 0, 0\}, \\
& \text{CoefficientList}[p[1, 3, x] /. \text{rules1} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0, 0\}, \\
& \text{CoefficientList}[p[1, 4, x] /. \text{rules1} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0\}, \\
& \text{CoefficientList}[p[1, 5, x] /. \text{rules1} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0\}, \\
& \text{CoefficientList}[p[1, 6, x] /. \text{rules1} /. \text{Maincoeff}, x] \} \\
& \{ \{1, 0, 0, 0, 0, 0, 0\}, \left\{ \frac{1-M}{1+M}, \frac{2}{1+M}, 0, 0, 0, 0, 0 \right\}, \\
& \left\{ \frac{2-3M+M^2}{(1+M)(2+M)}, \frac{6(1-M)}{(1+M)(2+M)}, \frac{6}{(1+M)(2+M)}, 0, 0, 0, 0 \right\}, \\
& \left\{ \frac{6-11M+6M^2-M^3}{(1+M)(2+M)(3+M)}, \frac{2(11-15M+6M^2)}{(1+M)(2+M)(3+M)}, -\frac{30(-1+M)}{(1+M)(2+M)(3+M)}, \right. \\
& \left. \frac{20}{(1+M)(2+M)(3+M)}, 0, 0, 0 \right\}, \left\{ \frac{24-50M+35M^2-10M^3+M^4}{(1+M)(2+M)(3+M)(4+M)}, \right. \\
& -\frac{10(-10+17M-9M^2+2M^3)}{(1+M)(2+M)(3+M)(4+M)}, \frac{10(17-21M+9M^2)}{(1+M)(2+M)(3+M)(4+M)}, \\
& -\frac{140(-1+M)}{(1+M)(2+M)(3+M)(4+M)}, \frac{70}{(1+M)(2+M)(3+M)(4+M)}, 0, 0 \left. \right\}, \\
& \left\{ \frac{120-274M+225M^2-85M^3+15M^4-M^5}{(1+M)(2+M)(3+M)(4+M)(5+M)}, \frac{2(274-525M+365M^2-105M^3+15M^4)}{(1+M)(2+M)(3+M)(4+M)(5+M)}, \right. \\
& -\frac{210(-5+8M-4M^2+M^3)}{(1+M)(2+M)(3+M)(4+M)(5+M)}, \frac{140(8-9M+4M^2)}{(1+M)(2+M)(3+M)(4+M)(5+M)}, \\
& -\frac{630(-1+M)}{(1+M)(2+M)(3+M)(4+M)(5+M)}, \frac{252}{(1+M)(2+M)(3+M)(4+M)(5+M)}, 0 \left. \right\}, \\
& \left\{ \frac{720-1764M+1624M^2-735M^3+175M^4-21M^5+M^6}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}, \right. \\
& -\frac{42(-84+178M-140M^2+55M^3-10M^4+M^5)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}, \\
& \frac{42(178-315M+215M^2-60M^3+10M^4)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}, \\
& -\frac{420(-21+32M-15M^2+4M^3)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}, \\
& \frac{210(32-33M+15M^2)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}, \\
& -\frac{2772(-1+M)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}, \\
& \left. \frac{924}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)} \right\} \}
\end{aligned}$$

G2 =

$$\begin{aligned}
 & \{ \text{CoefficientList}[q[M, 2, 0, x] /. \text{rules2} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0, 0, 0, 0\}, \\
 & \text{CoefficientList}[q[M, 2, 1, x] /. \text{rules2} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0, 0, 0\}, \\
 & \text{CoefficientList}[q[M, 2, 2, x] /. \text{rules2} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0, 0\}, \\
 & \text{CoefficientList}[q[M, 2, 3, x] /. \text{rules2} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0, 0\}, \\
 & \text{CoefficientList}[q[M, 2, 4, x] /. \text{rules2} /. \text{Maincoeff}, x] \sim \text{Join} \sim \{0\}, \\
 & \text{CoefficientList}[q[M, 2, 5, x] /. \text{rules2} /. \text{Maincoeff}, x] // \text{Factor} \\
 & \{ \{M, -1, 0, 0, 0, 0, 0\}, \left\{ -\frac{2(-1+M)M}{3(1+M)}, \frac{2(-1+4M)}{3(1+M)}, -\frac{2}{1+M}, 0, 0, 0, 0 \right\}, \\
 & \left\{ \frac{3(-2+M)(-1+M)M}{5(1+M)(2+M)}, -\frac{3(-2+3M)(-1+3M)}{5(1+M)(2+M)}, \frac{18(-1+3M)}{5(1+M)(2+M)}, -\frac{6}{(1+M)(2+M)}, \right. \\
 & 0, 0, 0 \left. \right\}, \left\{ -\frac{4(-3+M)(-2+M)(-1+M)M}{7(1+M)(2+M)(3+M)}, \frac{8(-3+18M-18M^2+8M^3)}{7(1+M)(2+M)(3+M)}, \right. \\
 & -\frac{20(5-12M+12M^2)}{7(1+M)(2+M)(3+M)}, \frac{40(-3+8M)}{7(1+M)(2+M)(3+M)}, -\frac{20}{(1+M)(2+M)(3+M)}, 0, 0 \left. \right\}, \\
 & \left\{ \frac{5(-4+M)(-3+M)(-2+M)(-1+M)M}{9(1+M)(2+M)(3+M)(4+M)}, -\frac{5(24-150M+225M^2-100M^3+25M^4)}{9(1+M)(2+M)(3+M)(4+M)}, \right. \\
 & \frac{250(-2+8M-6M^2+3M^3)}{9(1+M)(2+M)(3+M)(4+M)}, -\frac{350(3-5M+5M^2)}{9(1+M)(2+M)(3+M)(4+M)}, \\
 & \frac{9(1+M)(2+M)(3+M)(4+M)}{350(-2+5M)}, -\frac{70}{(1+M)(2+M)(3+M)(4+M)}, 0 \left. \right\}, \\
 & \left\{ -\frac{6(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)M}{11(1+M)(2+M)(3+M)(4+M)(5+M)}, \right. \\
 & \frac{6(-120+862M-1275M^2+890M^3-225M^4+36M^5)}{11(1+M)(2+M)(3+M)(4+M)(5+M)}, \\
 & -\frac{126(28-100M+135M^2-50M^3+15M^4)}{11(1+M)(2+M)(3+M)(4+M)(5+M)}, \\
 & \frac{420(-15+50M-30M^2+16M^3)}{11(1+M)(2+M)(3+M)(4+M)(5+M)}, -\frac{1260(7-9M+9M^2)}{11(1+M)(2+M)(3+M)(4+M)(5+M)}, \\
 & \frac{756(-5+12M)}{11(1+M)(2+M)(3+M)(4+M)(5+M)}, -\frac{252}{(1+M)(2+M)(3+M)(4+M)(5+M)} \left. \right\} \}
 \end{aligned}$$

G2.Inverse[G1] // Factor

$$\begin{aligned}
 & \left\{ \left\{ \frac{1+M}{2}, \frac{1}{2}(-1-M), 0, 0, 0, 0, 0 \right\}, \left\{ 0, \frac{2+M}{3}, \frac{1}{3}(-2-M), 0, 0, 0, 0 \right\}, \right. \\
 & \left\{ 0, 0, \frac{3(3+M)}{10}, -\frac{3}{10}(3+M), 0, 0, 0 \right\}, \left\{ 0, 0, 0, \frac{2(4+M)}{7}, -\frac{2}{7}(4+M), 0, 0 \right\}, \\
 & \left. \left\{ 0, 0, 0, 0, \frac{5(5+M)}{18}, -\frac{5}{18}(5+M), 0 \right\}, \left\{ 0, 0, 0, 0, 0, \frac{3(6+M)}{11}, -\frac{3}{11}(6+M) \right\} \right\}
 \end{aligned}$$

G2.Inverse[G1] // Factor // MatrixForm

$$\begin{pmatrix}
 \frac{1+M}{2} & \frac{1}{2}(-1-M) & 0 & 0 & 0 & 0 & 0 \\
 0 & \frac{2+M}{3} & \frac{1}{3}(-2-M) & 0 & 0 & 0 & 0 \\
 0 & 0 & \frac{3(3+M)}{10} & -\frac{3}{10}(3+M) & 0 & 0 & 0 \\
 0 & 0 & 0 & \frac{2(4+M)}{7} & -\frac{2}{7}(4+M) & 0 & 0 \\
 0 & 0 & 0 & 0 & \frac{5(5+M)}{18} & -\frac{5}{18}(5+M) & 0 \\
 0 & 0 & 0 & 0 & 0 & \frac{3(6+M)}{11} & -\frac{3}{11}(6+M)
 \end{pmatrix}$$

5. Lambda matrix

Case $m = 1$

Creating the lambda matrix

```

c[1, 1_, 1] :=
  p[1, 1, M - 1] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /. Maincoeff
c[1, 1_, 0] :=
  -p[1, 1, -1] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /. Maincoeff
szumma[1_] := Sum[L[1, s] p[1, s, j], {s, 0, 1}] /. rules1 /. rules2 /. rules3 /.
  rules4 /. rules5 /. Maincoeff

```

$l = 0$

```

CoefficientList[szumma[0], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /.
  Maincoeff
{L[0, 0]}
lambda0 = {1, -1, 1, 0, 0, 0, 0, 0, 0, 0}
{1, -1, 1, 0, 0, 0, 0, 0, 0, 0}

```

$l = 1$

```

CoefficientList[szumma[1], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /.
  Maincoeff
{L[1, 0] +  $\frac{(1-M) L[1, 1]}{1+M}$ ,  $\frac{2 L[1, 1]}{1+M}$ }
righthandside =
  CoefficientList[szumma[1], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /.
  Maincoeff
lefthandside = CoefficientList[p[1, 1, j - 1], j] /. rules1 /. rules2 /. rules3 /.
  rules4 /. rules5 /. Maincoeff
soll = Solve[righthandside == lefthandside, {L[1, 0], L[1, 1]}]
{L[1, 0] +  $\frac{(1-M) L[1, 1]}{1+M}$ ,  $\frac{2 L[1, 1]}{1+M}$ }
{- $\frac{2}{1+M}$  +  $\frac{1-M}{1+M}$ ,  $\frac{2}{1+M}$ }
{{L[1, 0] → - $\frac{2}{1+M}$ , L[1, 1] → 1}}

```


lambda1 =

$$\{c[1, 1, 1], c[1, 1, 0], L[1, 0], L[1, 1], 0, 0, 0, 0, 0, 0\} /. sol1[[1]] // Factor$$

$$\left\{ \frac{-1+M}{1+M}, 1, -\frac{2}{1+M}, 1, 0, 0, 0, 0, 0, 0 \right\}$$

l = 2

s1 =

$$\text{CoefficientList}[szumma[2], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /. \text{Maincoeff}$$

$$s2 = \text{CoefficientList}[p[1, 2, j - 1], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /. \text{Maincoeff}$$

$$sol2 = \text{Solve}[s1 == s2, \{L[2, 0], L[2, 1], L[2, 2]\}]$$

$$\left\{ L[2, 0] + \frac{(1-M)L[2, 1]}{1+M} + \frac{(2-3M+M^2)L[2, 2]}{(1+M)(2+M)}, \right.$$

$$\left. \frac{2L[2, 1]}{1+M} + \frac{6(1-M)L[2, 2]}{(1+M)(2+M)}, \frac{6L[2, 2]}{(1+M)(2+M)} \right\}$$

$$\left\{ \frac{6}{(1+M)(2+M)} - \frac{6(1-M)}{(1+M)(2+M)} + \frac{2-3M+M^2}{(1+M)(2+M)}, \right.$$

$$\left. -\frac{12}{(1+M)(2+M)} + \frac{6(1-M)}{(1+M)(2+M)}, \frac{6}{(1+M)(2+M)} \right\}$$

$$\left\{ \left\{ L[2, 0] \rightarrow \frac{6}{(1+M)(2+M)}, L[2, 1] \rightarrow -\frac{6}{2+M}, L[2, 2] \rightarrow 1 \right\} \right\}$$

lambda2 =

$$\{c[1, 2, 1], c[1, 2, 0], L[2, 0], L[2, 1], L[2, 2], 0, 0, 0, 0, 0\} /. sol2[[1]] // Factor$$

$$\left\{ \frac{(-2+M)(-1+M)}{(1+M)(2+M)}, -1, \frac{6}{(1+M)(2+M)}, -\frac{6}{2+M}, 1, 0, 0, 0, 0, 0 \right\}$$

/ = 3

s1 =

CoefficientList[szumma[3], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /.
Maincoeff

s2 = CoefficientList[p[1, 3, j - 1], j] /. rules1 /. rules2 /. rules3 /. rules4 /.
rules5 /. Maincoeff

sol3 = Solve[s1 == s2, {L[3, 0], L[3, 1], L[3, 2], L[3, 3]}]

$$\left\{ \begin{aligned} & L[3, 0] + \frac{(1-M)L[3, 1]}{1+M} + \frac{(2-3M+M^2)L[3, 2]}{(1+M)(2+M)} + \frac{(6-11M+6M^2-M^3)L[3, 3]}{(1+M)(2+M)(3+M)}, \\ & \frac{2L[3, 1]}{1+M} + \frac{6(1-M)L[3, 2]}{(1+M)(2+M)} + \frac{2(11-15M+6M^2)L[3, 3]}{(1+M)(2+M)(3+M)}, \\ & \frac{6L[3, 2]}{(1+M)(2+M)} - \frac{30(-1+M)L[3, 3]}{(1+M)(2+M)(3+M)}, \frac{20L[3, 3]}{(1+M)(2+M)(3+M)} \end{aligned} \right\}$$

$$\left\{ \begin{aligned} & -\frac{20}{(1+M)(2+M)(3+M)} - \frac{30(-1+M)}{(1+M)(2+M)(3+M)} - \\ & \frac{2(11-15M+6M^2)}{(1+M)(2+M)(3+M)} - \frac{-6+11M-6M^2+M^3}{(1+M)(2+M)(3+M)}, \\ & \frac{60}{(1+M)(2+M)(3+M)} + \frac{60(-1+M)}{(1+M)(2+M)(3+M)} + \frac{2(11-15M+6M^2)}{(1+M)(2+M)(3+M)}, \\ & -\frac{60}{(1+M)(2+M)(3+M)} - \frac{30(-1+M)}{(1+M)(2+M)(3+M)}, \frac{20}{(1+M)(2+M)(3+M)} \end{aligned} \right\}$$

$$\left\{ \left\{ L[3, 0] \rightarrow -\frac{2(11+M^2)}{(1+M)(2+M)(3+M)}, \right. \right.$$

$$\left. \left. L[3, 1] \rightarrow \frac{30}{(2+M)(3+M)}, L[3, 2] \rightarrow -\frac{10}{3+M}, L[3, 3] \rightarrow 1 \right\} \right\}$$

lambda3 =

{c[1, 3, 1], c[1, 3, 0], L[3, 0], L[3, 1], L[3, 2], L[3, 3], 0, 0, 0, 0} /. sol3[[1]] //

Factor

$$\left\{ \frac{(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)}, 1, \right.$$

$$\left. -\frac{2(11+M^2)}{(1+M)(2+M)(3+M)}, \frac{30}{(2+M)(3+M)}, -\frac{10}{3+M}, 1, 0, 0, 0, 0 \right\}$$

$l = 4$

s1 =

**CoefficientList[szumma[4], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /.
Maincoeff;**

**s2 = CoefficientList[p[1, 4, j - 1], j] /. rules1 /. rules2 /. rules3 /. rules4 /.
rules5 /. Maincoeff;**

sol4 = Solve[s1 == s2, {L[4, 0], L[4, 1], L[4, 2], L[4, 3], L[4, 4]}]

$$\left\{ \left\{ L[4, 0] \rightarrow \frac{20 (5 + M^2)}{(1 + M) (2 + M) (3 + M) (4 + M)}, L[4, 1] \rightarrow -\frac{6 (26 + M^2)}{(2 + M) (3 + M) (4 + M)}, \right. \right.$$

$$\left. L[4, 2] \rightarrow \frac{70}{(3 + M) (4 + M)}, L[4, 3] \rightarrow -\frac{14}{4 + M}, L[4, 4] \rightarrow 1 \right\}$$

lambda4 = {c[1, 4, 1], c[1, 4, 0], L[4, 0],

L[4, 1], L[4, 2], L[4, 3], L[4, 4], 0, 0, 0} /. sol4[[1]] // Factor

$$\left\{ \frac{(-4 + M) (-3 + M) (-2 + M) (-1 + M)}{(1 + M) (2 + M) (3 + M) (4 + M)}, -1, \frac{20 (5 + M^2)}{(1 + M) (2 + M) (3 + M) (4 + M)}, \right.$$

$$\left. -\frac{6 (26 + M^2)}{(2 + M) (3 + M) (4 + M)}, \frac{70}{(3 + M) (4 + M)}, -\frac{14}{4 + M}, 1, 0, 0, 0 \right\}$$

$l = 5$

s1 =

**CoefficientList[szumma[5], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /.
Maincoeff;**

**s2 = CoefficientList[p[1, 5, j - 1], j] /. rules1 /. rules2 /. rules3 /. rules4 /.
rules5 /. Maincoeff;**

sol5 = Solve[s1 == s2, {L[5, 0], L[5, 1], L[5, 2], L[5, 3], L[5, 4], L[5, 5]}]

$$\left\{ \left\{ L[5, 0] \rightarrow -\frac{2 (274 + 85 M^2 + M^4)}{(1 + M) (2 + M) (3 + M) (4 + M) (5 + M)}, \right. \right.$$

$$L[5, 1] \rightarrow \frac{84 (11 + M^2)}{(2 + M) (3 + M) (4 + M) (5 + M)}, L[5, 2] \rightarrow -\frac{10 (47 + M^2)}{(3 + M) (4 + M) (5 + M)},$$

$$\left. L[5, 3] \rightarrow \frac{126}{(4 + M) (5 + M)}, L[5, 4] \rightarrow -\frac{18}{5 + M}, L[5, 5] \rightarrow 1 \right\}$$

lambda5 = {c[1, 5, 1], c[1, 5, 0], L[5, 0], L[5, 1],

L[5, 2], L[5, 3], L[5, 4], L[5, 5], 0, 0} /. sol5[[1]] // Factor

$$\left\{ \frac{(-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M)}{(1 + M) (2 + M) (3 + M) (4 + M) (5 + M)}, 1, -\frac{2 (274 + 85 M^2 + M^4)}{(1 + M) (2 + M) (3 + M) (4 + M) (5 + M)}, \right.$$

$$\left. \frac{84 (11 + M^2)}{(2 + M) (3 + M) (4 + M) (5 + M)}, -\frac{10 (47 + M^2)}{(3 + M) (4 + M) (5 + M)}, \frac{126}{(4 + M) (5 + M)}, -\frac{18}{5 + M}, 1, 0, 0 \right\}$$

l = 6

s1 =

**CoefficientList[szumma[6], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /.
Maincoeff;**

**s2 = CoefficientList[p[1, 6, j - 1], j] /. rules1 /. rules2 /. rules3 /. rules4 /.
rules5 /. Maincoeff;**

sol6 = Solve[s1 == s2, {L[6, 0], L[6, 1], L[6, 2], L[6, 3], L[6, 4], L[6, 5], L[6, 6]}]

$$\left\{ \left\{ L[6, 0] \rightarrow \frac{42 (84 + 35 M^2 + M^4)}{(1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M)}, \right. \right.$$

$$L[6, 1] \rightarrow -\frac{6 (1044 + 155 M^2 + M^4)}{(2 + M) (3 + M) (4 + M) (5 + M) (6 + M)},$$

$$L[6, 2] \rightarrow \frac{180 (19 + M^2)}{(3 + M) (4 + M) (5 + M) (6 + M)}, L[6, 3] \rightarrow -\frac{14 (74 + M^2)}{(4 + M) (5 + M) (6 + M)},$$

$$\left. L[6, 4] \rightarrow \frac{198}{(5 + M) (6 + M)}, L[6, 5] \rightarrow -\frac{22}{6 + M}, L[6, 6] \rightarrow 1 \right\}$$

lambda6 = {c[1, 6, 1], c[1, 6, 0], L[6, 0], L[6, 1],

L[6, 2], L[6, 3], L[6, 4], L[6, 5], L[6, 6], 0} /. sol6[[1]] // Factor

$$\left\{ \frac{(-6 + M) (-5 + M) (-4 + M) (-3 + M) (-2 + M) (-1 + M)}{(1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M)}, -1, \right.$$

$$\frac{42 (84 + 35 M^2 + M^4)}{(1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M)}, -\frac{6 (1044 + 155 M^2 + M^4)}{(2 + M) (3 + M) (4 + M) (5 + M) (6 + M)},$$

$$\frac{180 (19 + M^2)}{(3 + M) (4 + M) (5 + M) (6 + M)}, -\frac{14 (74 + M^2)}{(4 + M) (5 + M) (6 + M)}, \frac{198}{(5 + M) (6 + M)}, -\frac{22}{6 + M}, 1, 0 \left. \right\}$$

l = 7

s1 =

**CoefficientList[szumma[7], j] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5 /.
Maincoeff;**

**s2 = CoefficientList[p[1, 7, j - 1], j] /. rules1 /. rules2 /. rules3 /. rules4 /.
rules5 /. Maincoeff;**

sol7 = Solve[s1 == s2, {L[7, 0], L[7, 1], L[7, 2], L[7, 3],

L[7, 4], L[7, 5], L[7, 6], L[7, 7]}]

$$\left\{ \left\{ L[7, 0] \rightarrow -\frac{2 (13068 + 6769 M^2 + 322 M^4 + M^6)}{(1 + M) (2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M)}, \right. \right.$$

$$L[7, 1] \rightarrow \frac{54 (892 + 185 M^2 + 3 M^4)}{(2 + M) (3 + M) (4 + M) (5 + M) (6 + M) (7 + M)},$$

$$L[7, 2] \rightarrow -\frac{10 (2754 + 245 M^2 + M^4)}{(3 + M) (4 + M) (5 + M) (6 + M) (7 + M)},$$

$$L[7, 3] \rightarrow \frac{308 (29 + M^2)}{(4 + M) (5 + M) (6 + M) (7 + M)}, L[7, 4] \rightarrow -\frac{18 (107 + M^2)}{(5 + M) (6 + M) (7 + M)},$$

$$\left. L[7, 5] \rightarrow \frac{286}{(6 + M) (7 + M)}, L[7, 6] \rightarrow -\frac{26}{7 + M}, L[7, 7] \rightarrow 1 \right\}$$

$$\begin{aligned} \text{lambda7} = & \{c[1, 7, 1], c[1, 7, 0], L[7, 0], L[7, 1], L[7, 2], \\ & L[7, 3], L[7, 4], L[7, 5], L[7, 6], L[7, 7]\} /. \text{sol7}[[1]] // \text{Factor} \\ & \{((-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)) / \\ & ((1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)), \\ & 1, -\frac{2(13068+6769M^2+322M^4+M^6)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)}, \\ & \frac{54(892+185M^2+3M^4)}{(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)}, -\frac{10(2754+245M^2+M^4)}{(3+M)(4+M)(5+M)(6+M)(7+M)}, \\ & \frac{308(29+M^2)}{(4+M)(5+M)(6+M)(7+M)}, -\frac{18(107+M^2)}{(5+M)(6+M)(7+M)}, \frac{286}{(6+M)(7+M)}, -\frac{26}{7+M}, 1\} \end{aligned}$$

The lambda matrix in list form:

$$\begin{aligned} & \{\text{lambda0}, \text{lambda1}, \text{lambda2}, \text{lambda3}, \text{lambda4}, \text{lambda5}, \text{lambda6}, \text{lambda7}\} \\ & \{ \{1, -1, 1, 0, 0, 0, 0, 0, 0, 0\}, \left\{ \frac{-1+M}{1+M}, 1, -\frac{2}{1+M}, 1, 0, 0, 0, 0, 0, 0 \right\}, \\ & \left\{ \frac{(-2+M)(-1+M)}{(1+M)(2+M)}, -1, \frac{6}{(1+M)(2+M)}, -\frac{6}{2+M}, 1, 0, 0, 0, 0, 0 \right\}, \\ & \left\{ \frac{(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)}, 1, -\frac{2(11+M^2)}{(1+M)(2+M)(3+M)}, \frac{30}{(2+M)(3+M)}, -\frac{10}{3+M}, 1, 0, \right. \\ & \left. 0, 0, 0 \right\}, \left\{ \frac{(-4+M)(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)(4+M)}, -1, \frac{20(5+M^2)}{(1+M)(2+M)(3+M)(4+M)}, \right. \\ & \left. -\frac{6(26+M^2)}{(2+M)(3+M)(4+M)}, \frac{70}{(3+M)(4+M)}, -\frac{14}{4+M}, 1, 0, 0, 0 \right\}, \\ & \left\{ \frac{(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)(4+M)(5+M)}, 1, -\frac{2(274+85M^2+M^4)}{(1+M)(2+M)(3+M)(4+M)(5+M)}, \right. \\ & \left. \frac{84(11+M^2)}{(2+M)(3+M)(4+M)(5+M)}, -\frac{10(47+M^2)}{(3+M)(4+M)(5+M)}, \frac{126}{(4+M)(5+M)}, -\frac{18}{5+M}, 1, 0, 0 \right\}, \\ & \left\{ \frac{(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}, -1, \right. \\ & \left. \frac{42(84+35M^2+M^4)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)}, -\frac{6(1044+155M^2+M^4)}{(2+M)(3+M)(4+M)(5+M)(6+M)}, \right. \\ & \left. \frac{180(19+M^2)}{(3+M)(4+M)(5+M)(6+M)}, -\frac{14(74+M^2)}{(4+M)(5+M)(6+M)}, \frac{198}{(5+M)(6+M)}, -\frac{22}{6+M}, 1, 0 \right\}, \\ & \{((-7+M)(-6+M)(-5+M)(-4+M)(-3+M)(-2+M)(-1+M)) / \\ & ((1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)), \\ & 1, -\frac{2(13068+6769M^2+322M^4+M^6)}{(1+M)(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)}, \\ & \frac{54(892+185M^2+3M^4)}{(2+M)(3+M)(4+M)(5+M)(6+M)(7+M)}, -\frac{10(2754+245M^2+M^4)}{(3+M)(4+M)(5+M)(6+M)(7+M)}, \\ & \frac{308(29+M^2)}{(4+M)(5+M)(6+M)(7+M)}, -\frac{18(107+M^2)}{(5+M)(6+M)(7+M)}, \frac{286}{(6+M)(7+M)}, -\frac{26}{7+M}, 1\} \end{aligned}$$

6. Z_m matrices

Notations: In formula (3) $r_{N,m-1}(i)$ is denoted by $r[M, m, i]$. In formula (28) $\tilde{q}_{N,m+j-2}(i)$ is denoted

by `qq[M_, m_, j_, i_]`. In formula (29) $\zeta_{j,i}^m$ is denoted by `xi[m, j, 1]` and ζ_j^m is denoted by `XI[m, j]`.

$m \geq 1$

```

ClearAll[r, qq, M, c]
r[M_, m_, i_] := m! Binomial[M - 1 + m - i, m];
qq[M_, m_, j_, i_] :=
  r[M, m - 1, i - 1] p[m, j, i - 1] - r[M, m - 1, i] p[m, j, i] /. rules1 /. rules2 /.
  rules3 /. rules4 /. rules5 /. Maincoeff

ClearAll[szumma2];
szumma2[m_, j_, i_] :=
  Sum[xi[m, j, 1] p[1, 1, i], {1, 0, m + j - 2}] /. rules1 /. rules2 /. rules3 /.
  rules4 /. rules5 /. Maincoeff;

ClearAll[c, M];
c[m_, j_, 1] :=
  r[M, m - 1, M - 1] p[m, j, M - 1] /. rules1 /. rules2 /. rules3 /. rules4 /. rules5;
c[m_, j_, 0] := -r[M, m - 1, -1] p[m, j, -1] /. rules1 /. rules2 /. rules3 /. rules4 /.
  rules5;

```

Substitution rules for the coefficients $\zeta_{j,i}^m$ for $m = 1, \dots, 5$.

```

ClearAll[l, m];
Zrule1[m_, l_] :=
  Solve[CoefficientList[qq[M, m, l, j], j] == CoefficientList[szomma2[m, l, j], j],
    Table[xi[m, l, i], {i, 0, l-1}]]

ClearAll[l, m];
Zrule2[m_, l_] :=
  Solve[CoefficientList[qq[M, m, l, j], j] == CoefficientList[szomma2[m, l, j], j],
    Table[xi[m, l, i], {i, 0, l}]]

ClearAll[l, m];
Zrule3[m_, l_] :=
  Solve[CoefficientList[qq[M, m, l, j], j] == CoefficientList[szomma2[m, l, j], j],
    Table[xi[m, l, i], {i, 0, l+1}]]

ClearAll[l, m];
Zrule4[m_, l_] :=
  Solve[CoefficientList[qq[M, m, l, j], j] == CoefficientList[szomma2[m, l, j], j],
    Table[xi[m, l, i], {i, 0, l+2}]]

ClearAll[l, m];
Zrule5[m_, l_] :=
  Solve[CoefficientList[qq[M, m, l, j], j] == CoefficientList[szomma2[m, l, j], j],
    Table[xi[m, l, i], {i, 0, l+3}]]

```

$m = 1$

Calculation of the row vector $\underline{\zeta}_j^1$:

```

XI[1, 0] = {c[1, 0, 1], c[1, 0, 0], 0, 0, 0, 0} /. Zrule1[1, 0] /. Maincoeff // Factor
{{1, -1, 0, 0, 0, 0}}

```

```

XI[1, 1] =
  {c[1, 1, 1], c[1, 1, 0], xi[1, 1, 0], 0, 0, 0} /. Zrule1[1, 1] /. Maincoeff // Factor
{{(-1+M)/(1+M), 1, -2/(1+M), 0, 0, 0}}

```

```

XI[1, 2] = {c[1, 2, 1], c[1, 2, 0], xi[1, 2, 0], xi[1, 2, 1], 0, 0} /. Zrule1[1, 2] /.
  Maincoeff // Factor
{{(-2+M)(-1+M)/(1+M)(2+M), -1, 6/(1+M)(2+M), -6/(2+M), 0, 0}}

```

XI[1, 3] =

**{c[1, 3, 1], c[1, 3, 0], xi[1, 3, 0], xi[1, 3, 1], xi[1, 3, 2], 0} /. Zrule1[1, 3] /.
Maincoeff // Factor**

$$\left\{ \left\{ \frac{(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)}, 1, -\frac{2(11+M^2)}{(1+M)(2+M)(3+M)}, \frac{30}{(2+M)(3+M)}, -\frac{10}{3+M}, 0 \right\} \right\}$$

XI[1, 4] =

**{c[1, 4, 1], c[1, 4, 0], xi[1, 4, 0], xi[1, 4, 1], xi[1, 4, 2], xi[1, 4, 3]} /.
Zrule1[1, 4] /. Maincoeff // Factor**

$$\left\{ \left\{ \frac{(-4+M)(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)(4+M)}, -1, \frac{20(5+M^2)}{(1+M)(2+M)(3+M)(4+M)}, -\frac{6(26+M^2)}{(2+M)(3+M)(4+M)}, \frac{70}{(3+M)(4+M)}, -\frac{14}{4+M} \right\} \right\}$$

The Z_1 matrix in list form and matrix form:

Table[XI[1, i][1], {i, 0, 4}] // Factor

$$\left\{ \{1, -1, 0, 0, 0, 0\}, \left\{ \frac{-1+M}{1+M}, 1, -\frac{2}{1+M}, 0, 0, 0 \right\}, \left\{ \frac{(-2+M)(-1+M)}{(1+M)(2+M)}, -1, \frac{6}{(1+M)(2+M)}, -\frac{6}{2+M}, 0, 0 \right\}, \left\{ \frac{(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)}, 1, -\frac{2(11+M^2)}{(1+M)(2+M)(3+M)}, \frac{30}{(2+M)(3+M)}, -\frac{10}{3+M}, 0 \right\}, \left\{ \frac{(-4+M)(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)(4+M)}, -1, \frac{20(5+M^2)}{(1+M)(2+M)(3+M)(4+M)}, -\frac{6(26+M^2)}{(2+M)(3+M)(4+M)}, \frac{70}{(3+M)(4+M)}, -\frac{14}{4+M} \right\} \right\}$$

Table[XI[1, i][1], {i, 0, 4}] // Factor // MatrixForm

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ \frac{-1+M}{1+M} & 1 & -\frac{2}{1+M} & 0 & 0 & 0 \\ \frac{(-2+M)(-1+M)}{(1+M)(2+M)} & -1 & \frac{6}{(1+M)(2+M)} & -\frac{6}{2+M} & 0 & 0 \\ \frac{(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)} & 1 & -\frac{2(11+M^2)}{(1+M)(2+M)(3+M)} & \frac{30}{(2+M)(3+M)} & -\frac{10}{3+M} & 0 \\ \frac{(-4+M)(-3+M)(-2+M)(-1+M)}{(1+M)(2+M)(3+M)(4+M)} & -1 & \frac{20(5+M^2)}{(1+M)(2+M)(3+M)(4+M)} & -\frac{6(26+M^2)}{(2+M)(3+M)(4+M)} & \frac{70}{(3+M)(4+M)} & -\frac{14}{4+M} \end{pmatrix}$$

$m = 2$

Calculation of the row vector $\underline{\zeta}_j^2$:

XI[2, 0] =

{c[2, 0, 1], c[2, 0, 0], xi[2, 0, 0], 0, 0, 0} /. Zrule2[2, 0] /. Maincoeff // Factor

$$\{1, -1-M, 1, 0, 0, 0\}$$

XI[2, 1] = {c[2, 1, 1], c[2, 1, 0], xi[2, 1, 0], xi[2, 1, 1], 0, 0} /. Zrule2[2, 1] /.

Maincoeff // Factor

$$\left\{ \left\{ \frac{4(-1+M)}{3(1+M)}, \frac{2(2+M)}{3}, -\frac{2(5+M)}{3(1+M)}, 2, 0, 0 \right\} \right\}$$

XI[2, 2] =

{c[2, 2, 1], c[2, 2, 0], xi[2, 2, 0], xi[2, 2, 1], xi[2, 2, 2], 0} /. Zrule2[2, 2] /.

Maincoeff // Factor

$$\left\{ \left\{ \frac{9(-2+M)(-1+M)}{5(1+M)(2+M)}, -\frac{3}{5}(3+M), \frac{3(20+3M+M^2)}{5(1+M)(2+M)}, -\frac{9(8+M)}{5(2+M)}, 3, 0 \right\} \right\}$$

XI[2, 3] =

{c[2, 3, 1], c[2, 3, 0], xi[2, 3, 0], xi[2, 3, 1], xi[2, 3, 2], xi[2, 3, 3]} /.

Zrule2[2, 3] /. Maincoeff // Factor

$$\left\{ \left\{ \frac{16(-3+M)(-2+M)(-1+M)}{7(1+M)(2+M)(3+M)}, \frac{4(4+M)}{7}, \right. \right. \\ \left. \left. -\frac{4(94+11M+14M^2+M^3)}{7(1+M)(2+M)(3+M)}, \frac{12(46+5M+M^2)}{7(2+M)(3+M)}, -\frac{20(11+M)}{7(3+M)}, 4 \right\} \right\}$$

The Z_2 matrix in list form and matrix form:

Table[XI[2, i][[1], {i, 0, 3}] // Factor

$$\left\{ \{1, -1-M, 1, 0, 0, 0\}, \left\{ \frac{4(-1+M)}{3(1+M)}, \frac{2(2+M)}{3}, -\frac{2(5+M)}{3(1+M)}, 2, 0, 0 \right\}, \right. \\ \left\{ \frac{9(-2+M)(-1+M)}{5(1+M)(2+M)}, -\frac{3}{5}(3+M), \frac{3(20+3M+M^2)}{5(1+M)(2+M)}, -\frac{9(8+M)}{5(2+M)}, 3, 0 \right\}, \\ \left. \left\{ \frac{16(-3+M)(-2+M)(-1+M)}{7(1+M)(2+M)(3+M)}, \frac{4(4+M)}{7}, \right. \right. \\ \left. \left. -\frac{4(94+11M+14M^2+M^3)}{7(1+M)(2+M)(3+M)}, \frac{12(46+5M+M^2)}{7(2+M)(3+M)}, -\frac{20(11+M)}{7(3+M)}, 4 \right\} \right\}$$

Table[XI[2, i][[1], {i, 0, 3}] // Factor // MatrixForm

$$\begin{pmatrix} 1 & -1-M & 1 & 0 & 0 & 0 \\ \frac{4(-1+M)}{3(1+M)} & \frac{2(2+M)}{3} & -\frac{2(5+M)}{3(1+M)} & 2 & 0 & 0 \\ \frac{9(-2+M)(-1+M)}{5(1+M)(2+M)} & -\frac{3}{5}(3+M) & \frac{3(20+3M+M^2)}{5(1+M)(2+M)} & -\frac{9(8+M)}{5(2+M)} & 3 & 0 \\ \frac{16(-3+M)(-2+M)(-1+M)}{7(1+M)(2+M)(3+M)} & \frac{4(4+M)}{7} & -\frac{4(94+11M+14M^2+M^3)}{7(1+M)(2+M)(3+M)} & \frac{12(46+5M+M^2)}{7(2+M)(3+M)} & -\frac{20(11+M)}{7(3+M)} & 4 \end{pmatrix}$$

$m = 3$

Calculation of the row vector $\underline{\zeta}_j^3$:

XI[3, 0] = {c[3, 0, 1], c[3, 0, 0], xi[3, 0, 0], xi[3, 0, 1], 0, 0} /. Zrule3[3, 0] /.

Maincoeff // Factor

$$\{ \{2, -(1+M)(2+M), 3+M, -1-M, 0, 0\} \}$$

XI[3, 1] =

**{c[3, 1, 1], c[3, 1, 0], xi[3, 1, 0], xi[3, 1, 1], xi[3, 1, 2], 0} /. Zrule3[3, 1] /.
Maincoeff // Factor**

$$\left\{ \left\{ \frac{3(-1+M)}{1+M}, \frac{1}{2}(2+M)(3+M), -\frac{17+6M+M^2}{2(1+M)}, \frac{3(5+M)}{2}, -2-M, 0 \right\} \right\}$$

XI[3, 2] =

**{c[3, 2, 1], c[3, 2, 0], xi[3, 2, 0], xi[3, 2, 1], xi[3, 2, 2], xi[3, 2, 3]} /.
Zrule3[3, 2] /. Maincoeff // Factor**

$$\left\{ \left\{ \frac{24(-2+M)(-1+M)}{5(1+M)(2+M)}, -\frac{2}{5}(3+M)(4+M), \right. \right. \\ \left. \left. \frac{2(86+23M+10M^2+M^3)}{5(1+M)(2+M)}, -\frac{6(38+9M+M^2)}{5(2+M)}, 2(7+M), -\frac{6}{5}(3+M) \right\} \right\}$$

The Z_3 matrix in list form and matrix form:

Table[XI[3, i][[1], {i, 0, 2}] // Factor

$$\left\{ \left\{ 2, -(1+M)(2+M), 3+M, -1-M, 0, 0 \right\}, \right. \\ \left\{ \frac{3(-1+M)}{1+M}, \frac{1}{2}(2+M)(3+M), -\frac{17+6M+M^2}{2(1+M)}, \frac{3(5+M)}{2}, -2-M, 0 \right\}, \\ \left\{ \frac{24(-2+M)(-1+M)}{5(1+M)(2+M)}, -\frac{2}{5}(3+M)(4+M), \right. \\ \left. \frac{2(86+23M+10M^2+M^3)}{5(1+M)(2+M)}, -\frac{6(38+9M+M^2)}{5(2+M)}, 2(7+M), -\frac{6}{5}(3+M) \right\} \left. \right\}$$

Table[XI[3, i][[1], {i, 0, 2}] // Factor // MatrixForm

$$\begin{pmatrix} 2 & -(1+M)(2+M) & 3+M & -1-M & 0 & 0 \\ \frac{3(-1+M)}{1+M} & \frac{1}{2}(2+M)(3+M) & -\frac{17+6M+M^2}{2(1+M)} & \frac{3(5+M)}{2} & -2-M & 0 \\ \frac{24(-2+M)(-1+M)}{5(1+M)(2+M)} & -\frac{2}{5}(3+M)(4+M) & \frac{2(86+23M+10M^2+M^3)}{5(1+M)(2+M)} & -\frac{6(38+9M+M^2)}{5(2+M)} & 2(7+M) & -\frac{6}{5}(3+M) \end{pmatrix}$$

$m = 4$

Calculation of the row vector $\underline{\zeta}_j^4$:

XI[4, 0] =

**{c[4, 0, 1], c[4, 0, 0], xi[4, 0, 0], xi[4, 0, 1], xi[4, 0, 2], 0} /. Zrule4[4, 0] /.
Maincoeff // Factor**

$$\left\{ \left\{ 6, -(1+M)(2+M)(3+M), 11+6M+M^2, -\frac{3}{2}(1+M)(4+M), \frac{1}{2}(1+M)(2+M), 0 \right\} \right\}$$

XI[4, 1] =

**{c[4, 1, 1], c[4, 1, 0], xi[4, 1, 0], xi[4, 1, 1], xi[4, 1, 2], xi[4, 1, 3]} /.
Zrule4[4, 1] /. Maincoeff // Factor**

$$\left\{ \left\{ \frac{48(-1+M)}{5(1+M)}, \frac{2}{5}(2+M)(3+M)(4+M), -\frac{2(74+35M+10M^2+M^3)}{5(1+M)}, \right. \right. \\ \left. \left. \frac{6}{5}(26+9M+M^2), -\frac{2}{5}(2+M)(17+3M), \frac{2}{5}(2+M)(3+M) \right\} \right\}$$

The Z_4 matrix in list form:

Table[XI[4, i][[1]], {i, 0, 1}] // Factor

$$\left\{ \left\{ 6, -(1+M)(2+M)(3+M), 11+6M+M^2, -\frac{3}{2}(1+M)(4+M), \frac{1}{2}(1+M)(2+M), 0 \right\}, \right. \\ \left. \left\{ \frac{48(-1+M)}{5(1+M)}, \frac{2}{5}(2+M)(3+M)(4+M), -\frac{2(74+35M+10M^2+M^3)}{5(1+M)}, \right. \right. \\ \left. \left. \frac{6}{5}(26+9M+M^2), -\frac{2}{5}(2+M)(17+3M), \frac{2}{5}(2+M)(3+M) \right\} \right\}$$

$m = 5$

The Z_5 matrix in list form:

XI[5, 0] =

**{c[5, 0, 1], c[5, 0, 0], xi[5, 0, 0], xi[5, 0, 1], xi[5, 0, 2], xi[5, 0, 3]} /.
Zrule5[5, 0] /. Maincoeff // Factor**

$$\left\{ \left\{ 24, -(1+M)(2+M)(3+M)(4+M), (5+M)(10+5M+M^2), \right. \right. \\ \left. \left. -\frac{3}{5}(1+M)(58+25M+3M^2), (1+M)(2+M)(5+M), -\frac{1}{5}(1+M)(2+M)(3+M) \right\} \right\}$$