Stochastics
Problem sheet 1 - Basic probability
Fall 2022

1. We roll two fair dice. Let A denote the event that the sum of the two rolls is 6. Let B denote the event that the first roll is 4. Show that A and B are not independent. Let C denote the event that the sum of the two rolls is 7. Is B independent from C?

2. We flip a fair coin three times. Let A denote the event that the first flip is heads. Let B denote the event that there are more heads than tails from the three flips. Calculate $\mathbf{P}(B)$ and $\mathbf{P}(B|A)$.

3. Dennis has 2 identically-looking dice, one of which is fair (it gives the numbers 1, 2, 3, 4, 5 and 6 with probability $\frac{1}{6} - \frac{1}{6}$ each), but the other one is loaded: 6 has a probability of $\frac{1}{2}$. Dennis picks one of them at random and rolls it twice. What is the probability that he rolls two sixes? What is the conditional probability of the event that he picked the loaded die, assuming he rolls two sixes?

4. A test for a certain disease works the following way: if the subject has the disease, it will be positive all the time; however, if the subject does not have the disease, the test will still be positive with probability 1%. In the entire population, 1 in 10000 people have this disease. What is the conditional probability of somebody actually having the disease assuming that his test was positive?

5. (a) We know the Smith family has two children, but we do not know how many of them are boys or girls. Assuming that at least one of their children is a girl, what is the probability that both are girls?

   (b) We know the Smith family has two children, but we do not know how many of them are boys or girls. After knocking on their door, a girl opens the door. What is the probability that the other child is a girl as well?

6. Twenty-three randomly chosen people are asked which month and day they have their birthdays. (There are 365 days in a year.) What is the probability that among the 23 people, we find two having the same birthday?

7. We flip a coin twice. Let A, B, C be the events that the {first flip is tail}, {the second flip is tail}, {the two flips coincide}, respectively. Show that the events are pairwise independent, however the three events together are not independent!

8. Assume that an urn contains 5 red balls, 3 white balls and 7 green balls. Draw 3 balls from the urn without replacing the already drawn balls to the urn. Find the probability that "the first and the second draws are red and the third one is green".

9. A hat contains 3 slips of paper marked with 1, 2 slips of paper marked with 2 and 1 slip of paper with 3. We draw one slip of paper from the hat. Let X denote the result. E(X)=?, Var(X)=?

10. A miner gets lost in the mine. He is in a chamber with 5 doors. Door 1 leads to a tunnel to the exit after 2 hours of walking. Door 2 leads to a tunnel to Door 3 after 1 hours of walking. Door 4 leads to a tunnel to Door 5 after 3 hours of walking. The miner picks a door at random, goes through the tunnel, but whenever he gets back to the chamber, he forgets his previous choices, and picks one of the five doors at random again.

    X denotes the total time it takes him to get to the exit. Calculate $\mathbf{E}X$. (Hint: let B_2 denote the event that he picks Door 2 first. Argue that $\mathbf{E}(X|B_2) = \mathbf{E}(X) + 1$. Use total probability.)

11. Assume that $X = \sqrt{RND}$. $P(X < \frac{1}{2} | X > \frac{1}{3}) = ?$, $E(X) = ?$, $Var(X) = ?$

12. At a sports competition, participants have to throw a ball as far as possible. Let X denote the result of a single throw of Jane (in meters). X has the following probability density function:

$$f(x) = \begin{cases} \frac{75}{x^2} & 30 \leq x \leq 50 \\ 0 & \text{otherwise} \end{cases}$$

    (a) Calculate the probability that Jane throws further away than 45 meters.

    (b) Calculate the cumulative distribution function of X.

    (c) Calculate $\mathbf{E}X$.

    (d) Each participant can throw the ball 3 times, and their score is the maximum of the 3 throws. Calculate the distribution of Jane's score (we assume that different throws are independent).

13. A thermometer works the following way: if the real temperature is x degrees, then the thermometer will display a uniform random value between $x - 1$ and x. To counteract this, the temperature is measured 5 times, then the largest value is used. What is the probability that the obtained measurement differs from the real temperature by more than 0.2 degrees?

14. We throw a fair coin 5 times. What is the probability of getting two heads?

15. We start rolling a regular 6-sided die. Let X denote the total number of rolls until we get a 6, including the 6. Calculate the distribution of X. Let Y denote the total number of rolls until we get a 6, not including the 6. Calculate the distribution of Y.

16. Let X denote the total number of rolls needed to get a 6 with a regular 6-sided die. What is the distribution of X? Assuming the first roll is not a 6, what is the conditional distribution of the additional number of rolls needed to get a 6? (This is called the memoryless property of the geometric distribution.)

17. A test has 20 yes or no questions. For each question, we know the correct answer with probability $\frac{5}{7}$, we are convinced of the wrong answer with probability $\frac{1}{7}$. If we don't know the answer, we guess yes or no with probability $\frac{1}{2}$–$\frac{1}{2}$. What is the probability of giving a correct answer for the first question? What is the distribution of the number of correct answers? What is the probability of giving at least 18 correct answers?

18. Assume that there are 200 seats on an air-plane, and 202 tickets are sold for a flight on the airplane. Let us assume that each passenger may miss the flight independently of the others with a probability $p = 0.03$. What is the probability that more than 200 passengers appear?

19. There is an average of 2.3 shark attacks registered at the beaches of Florida each year. What is the probability that in a given year, at most 1 attack occurs?

20. A book with 500 pages contains 1000 typos. What is the probability that on a random page there are at least 2 typos? (We assume that each typo appears on every page with the same probability, and independently from other typos.)

21. In the nearest forest a running contest were organized. We know that 300 contestants found 1 tick, 75 contestants found 2 ticks on their bodies. Can we guess the total number of contestants in the contest?

22. Assume that a web server has on average 5 arrivals per minute. What is the probability that during a 30 second interval, there are at least 3 arrivals?

23. Assume that the age of a light bulb X (measured in 100 hours) has an exponential distribution such that $\mathbf{P}(X > 10) = 0.8$. Calculate the parameter of the exponential distribution and the mean of X.

24. In a given population, the height of a randomly chosen woman follow normal distribution. The expected value and the standard deviation are 170 cm and deviation 6 cm, respectively. What is the probability that a woman picked at random has height over 182 cm?

25. In a class of 120 students, Stochastics and Calculus marks are as follows:

$C\backslash S$	1	2	3	4	5
1	1	2	2	1	4
2	2	4	4	8	2
3	4	8	8	12	8
4	5	4	6	9	6
5	0	6	4	6	4

We pick a student at random; let X denote his Stochastics mark and Y his Calculus mark.

(a) $\mathbf{P}$(the student failed from at least one of the courses) $=?$

(b) $\mathbf{E}(X) =?$

(c) $\mathbf{E}(X|Y \geq 4) =?$

(d) Are X and Y independent?

(e) $\text{cov}(X, Y) =?$

HW1. (Deadline: 27 Sept.) A shooting gallery has 6 guns. Three of them are such that we hit the target with probability 0.5, with one, the probability of hitting the target is 0.7 and with two, the probability of hitting the target is 0.8. We pick a gun at random, then shoot. What is the probability of hitting the target? What is the conditional probability of choosing a 0.8 gun assuming that we hit the target?